

Acyclic k -choosability of planar graphs

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February 26, 2010



Acyclic Coloring

Proper Coloring

A *k*-proper coloring of the vertices of a graph G is a mapping $c : V(G) \rightarrow \{1, \dots, k\}$ such that $\forall uv \in E(G), c(u) \neq c(v)$.

Acyclic Coloring (1)

A proper vertex coloring of a graph is *acyclic* if there is no bicolored cycle in G .



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Acyclic Coloring (2)

A proper vertex coloring of a graph is *acyclic* if the graph induced by the union of every two color classes is a forest.



Acyclic Chromatic number

The *acyclic chromatic number*, denoted by $\chi_a(G)$, of a graph G is the smallest integer k such that G has an acyclic k -coloring.

The acyclic coloring of graphs was introduced by Grünbaum in '73.



An example of Petersen graph

Conjecture on acyclic coloring

Conjecture (Grünbaum, IJM, 1973)

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- Borodin, 1979, $\chi_a(\mathcal{P}) \leq 5$.



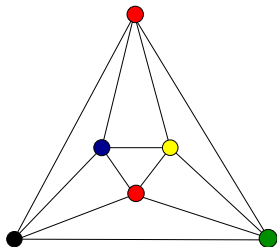
Theorem - [Borodin'79]

For any planar graph G , $\chi_a(G) \leq 5$



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A planar graph G with $\chi_a(G) = 5$



Acyclic L -coloring

- L is a **list assignment** of a graph G if it assigns **a list $L(v)$ of possible colors** to each vertex $v \in V$.
Denoted by $L = \{L(v) : v \in V\}$.



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- If G is acyclically L -list colorable for any list assignment L with $|L(v)| \geq k$ **for all $v \in V$** , then G is **acyclically k -choosable**.



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- If G is acyclically L -list colorable for any list assignment L with $|L(v)| \geq k$ for all $v \in V$, then G is **acyclically k -choosable**.
- The **acyclic list chromatic number** of G , denoted by $\chi_a^l(G)$, is **the smallest integer k** such that G is acyclically k -choosable.



Conjecture on acyclic L -coloring

♠ **Conjecture***: **Every planar graph is acyclically 5-choosable.**

⇒ Borodin's acyclic 5-color theorem (1979) and Thomassen's 5-choosability theorem (1994)

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Theorem

Every planar graph is acyclically 7-choosable.

Theorem (Wang and C., JGT, 2009)

*Every planar graph without **4-cycles** is acyclically 6-choosable.*



Known results - Acyclic 5-choosability

Theorem (Montassier, Raspaud, Wang, JGT, 2007)

Every planar graph either without $\{4, 5\}$ -cycles or without $\{4, 6\}$ -cycles is acyclically 5-choosable.



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Every planar graph without 4-cycles and without two 3-cycles at distance less than 3 is acyclically 5-choosable.



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Theorem (Zhang, Xu, DM, 2009)

Every planar graph having neither **4-cycles** nor **chordal 6-cycles** is acyclically 5-choosable.

Known results - Acyclic 4-choosability

Theorem

Planar graphs without $\{4, i, j\}$ -cycles with $5 \leq i < j \leq 8$ are acyclically 4-choosable.

4	5	6	7	8	
x	x	x			Montassier, R., Wang, 2006
x	x		x		Montassier, R., Wang, 2006
x	x			x	Chen, R., 2009
x		x	x		Chen, R., Wang, 2009
x		x		x	Chen, R., Wang, 2009
x			x	x	Chen, R., Roussel, Zhu, 2009



Definition: The *girth* $g(G)$ of a graph G is the length of *a shortest cycle in G* .

Theorem (Borodin, Kostochka, Woodall, JLM, 1999)

Let G be a planar graph.

(1) If $g(G) \geq 7$ then $\chi_a(G) \leq 3$.

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These two results are, respectively, improved by the following:

Theorem (Borodin, C., Ivanova, Raspaud, 2009)

If G is a planar graph with $g(G) \geq 7$, then $\chi_a^l(G) \leq 3$.

Theorem (Montassier, 2006)

If G is a planar graph with $g(G) \geq 5$, then $\chi_a^l(G) \leq 4$.

Theorem (Hocquard, Montassier, IPL, 2009)

*Every planar graph without cycles of **lengths 4 to 12** is acyclically 3-choosable.*



Maximum average degree

Definition (Maximum average degree)

$$\text{Mad}(G) = \max\left\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\right\}.$$

Observation

If G is a planar graph with girth g , then $\text{Mad}(G) < \frac{2 \cdot g}{g-2}$.



Theorem (Montassier, Ochem, Raspaud, JGT, 2005)

- (1) Every graph G with $Mad(G) < \frac{8}{3}$ is acyclically 3-choosable;
- (2) Every graph G with $Mad(G) < \frac{19}{6}$ is acyclically 4-choosable;
- (3) Every graph G with $Mad(G) < \frac{24}{7}$ is acyclically 5-choosable.



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By using relationship $Mad(G) < \frac{2 \cdot g}{g-2}$, then

Corollary

- (1) Every planar graph G with $g(G) \geq 8$ is acyclically 3-choosable;
- (2) Every planar graph G with $g(G) \geq 6$ is acyclically 4-choosable;
- (3) Every planar graph G with $g(G) \geq 5$ is acyclically 5-choosable.

