Enumerating Locally Finite Vertex-Transitive Planar Graphs

Renault D.

Laboratoire Bordelais de Recherche en Informatique
Université Bordeaux 1, Talence, France

Sunday, June 13
Goal: study families of graphs possessing

(i) a planar embedding
(ii) many symmetries

in particular vertex-transitive planar graphs.

Vertex-transitive graphs “look the same” from any vertex.

Guideline: represent the graphs by their local invariants

Objectives: vertex-transitive graphs provide appropriate structures for cellular automata [Mazoyer, Delorme 98] and routing networks [Heydemann 97].
About the enumeration of the vertex-transitive graphs

- The finite planar case is well known [Fleischner, Imrich 79].
- The finite case is well known up to 26 vertices [McKay, Royle 90].

In contrast, we realize a general approach of the planar case, dealing both with finite and infinite graphs.
1 Properties of the embedding
2 Geometrical invariants
3 Labeling schemes
4 Applications
Let $\Gamma$ be a planar graph.

**Vertex-transitive graph**

Every vertex can be mapped by an automorphism of $\Gamma$ on any other vertex.

**Locally finite graph**

There exists a planar embedding of $\Gamma$ such that there is no accumulation of the vertices in the plane.
Example of a vertex-transitive locally finite graph drawn in the hyperbolic plane.
In the following, $\Gamma$ is a locally finite vertex-transitive graph.

There exists a planar locally finite embedding of $\Gamma$ such that:

- the set of edges is locally finite;

- finite 1- and 2-separations are trivial;

- the automorphisms map finite faces onto finite faces.
In the following, $\Gamma$ is a locally finite vertex-transitive graph.

There exists a planar locally finite embedding of $\Gamma$ such that:

- the set of edges is locally finite;
- finite 1- and 2-separations are trivial;
- the automorphisms map finite faces onto finite faces.
In the following, $\Gamma$ is a locally finite vertex-transitive graph.

There exists a planar locally finite embedding of $\Gamma$ such that:

- the set of edges is locally finite;
- finite 1- and 2-separations are trivial;
- the automorphisms map finite faces onto finite faces.
In the following, $\Gamma$ is a locally finite vertex-transitive graph.

There exists a planar locally finite embedding of $\Gamma$ such that:
- the set of edges is locally finite;
- finite 1- and 2-separations are trivial;
- the automorphisms map finite faces onto finite faces.
Coloring of the graph

Let $\Gamma$ be a locally finite vertex-transitive graph.

There exists a coloring of the edges and the finite faces of $\Gamma$ induced by the group of automorphisms of the graph.

The number of colors for the edges and the finite faces is finite.

\[ \Rightarrow \begin{cases} E \text{ is the set of colors of the edges; } \\ F \text{ is the set of colors of the finite faces.} \end{cases} \]
Let $s$ be a vertex of $\Gamma$.

An edge vector around $s$ describes the colors of the edges incident to $s$ in a cyclic ordering:

$$[r; r; b; g; r]$$

A face vector around $s$ describes the colors of the faces incident to $s$ in a cyclic ordering:

$$[o; v; g; o; v]$$
Let $s$ be a vertex of $\Gamma$.

An edge vector around $s$ describes the colors of the edges incident to $s$ in a cyclic ordering:

$$[r; r; b; g; r]$$

A face vector around $s$ describes the colors of the faces incident to $s$ in a cyclic ordering:

$$[o; v; g; o; v]$$
Let $n$ be the number of infinite faces appearing around a given vertex of $\Gamma$. Then, depending on the value of $n$:

- $n = 0 \iff \Gamma$ is 3-connected;
- $n = 1 \iff \Gamma$ is 2-separable;
- $n \geq 2 \iff \Gamma$ is 1-separable.
A non-trivial example

Non-isomorphic graphs having same edge vectors
Let $e$ be an edge of $\Gamma$.

An **edge neighborhood** $\eta$ around $e$ describes:
- the edge and face vectors around the extremities of $e$;
- the colors of the faces separated by $e$.

The **color** of $\eta$ is the color of the edge $e$. 
Another non-trivial example

Non-isomorphic graphs having same edge neighborhoods
Let $\Gamma$ be a locally finite vertex-transitive graph.

Lemma

The edge and face vectors of $\Gamma$ are independent of the vertex.

Lemma

The edge neighborhoods of $\Gamma$ depend only on the color of the edges that they represent.
Let \((E, F)\) be finite sets of colors of edges and faces.

### Labeling scheme

A labeling scheme on \((E, F)\) is a 3-uple \((\xi, \phi, \eta)\) such that:

- an edge vector \(\xi \in E^d\),
- a face vector \(\phi \in F^d\),
- for each color \(\epsilon\) in \(\xi\), an edge neighborhood \(\eta_\epsilon \in \eta\) of the same color.
Example of labeling scheme

Edge and Face vectors:

Edge neighborhoods:

LaBRI Renault D.  
Enumerating Locally Finite Vertex-Transitive Graphs
Example of a reconstruction

Consider the previous labeling scheme.
Let us compute the border of an orange face:
Let $(\xi, \phi, \eta)$ be a labeling scheme.

**Lemma**

Given a color of a face in $F$, the sequence of edges appearing in the border is recognized by a finite state automaton.

$\implies$ Construction of the border automaton associated with $(\xi, \phi, \eta)$
Example of a border automaton

Follow alternatively black and gray edges to compute a border.
Example of graph built with the preceding automaton
Given a number $d \geq 2$, it is possible to enumerate all planar locally finite transitive graphs having internal degree $d$.

→ Classification of the graphs depending on:
- their connectivity
- their geometry
## Results of enumeration

<table>
<thead>
<tr>
<th>Degree</th>
<th>V.T.</th>
<th>Cayley</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>n.s.</td>
<td>n.s.</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>52</td>
<td>24</td>
</tr>
<tr>
<td>5</td>
<td>174</td>
<td>52</td>
</tr>
<tr>
<td>6</td>
<td>775</td>
<td>185</td>
</tr>
</tbody>
</table>
Recognition

Given a locally finite vertex-transitive planar graph, it is possible to decide whether it is a Cayley graph of a group or not.

→ Enumeration of the possible presentations;
→ Generation of vertex-transitive non Cayley graphs.
Perspectives

- Extend the enumeration to families of graphs with less symmetries, known as cofinite graphs.
- Study the quotients of the graphs onto compact surfaces (the torus for example).
- Evaluate the case of planar graphs in general (i.e. not locally finite).
Appendix
Scheme n°2

Return
Another non-trivial example
Another non-trivial example
Example of a reconstruction

Consider the previous labeling scheme.
Let us compute the border of an orange face:
Example of a reconstruction

Consider the previous labeling scheme.
Let us compute the border of an orange face:

\[ \text{Return} \]
Example of a reconstruction

Consider the previous labeling scheme.
Let us compute the border of an orange face:
Example of a reconstruction

Consider the previous labeling scheme.
Let us compute the border of an orange face:

\[
\begin{array}{c}
\begin{array}{c}
\text{v} \\
\text{o} \\
\text{g}
\end{array}
\end{array}
\]