

# Enumerating Locally Finite Vertex-Transitive Planar Graphs

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- Goal : study families of graphs possessing
  - (i) a planar embedding
  - (ii) many symmetriesin particular **vertex-transitive** planar graphs.
- Vertex-transitive graphs “look the same” from any vertex.
- Guideline : represent the graphs by their local invariants
- Objectives : vertex-transitive graphs provide appropriate structures for cellular automata [[Mazoyer, Delorme 98](#)] and routing networks [[Heydemann 97](#)].

# About the enumeration of the vertex-transitive graphs

- The finite planar case is well known [Fleischner, Imrich 79].
- The finite case is well known up to 26 vertices [McKay, Royle 90].

⇒ In contrast, we realize a general approach of the planar case, dealing both with finite and infinite graphs.

- 1 Properties of the embedding
- 2 Geometrical invariants
- 3 Labeling schemes
- 4 Applications

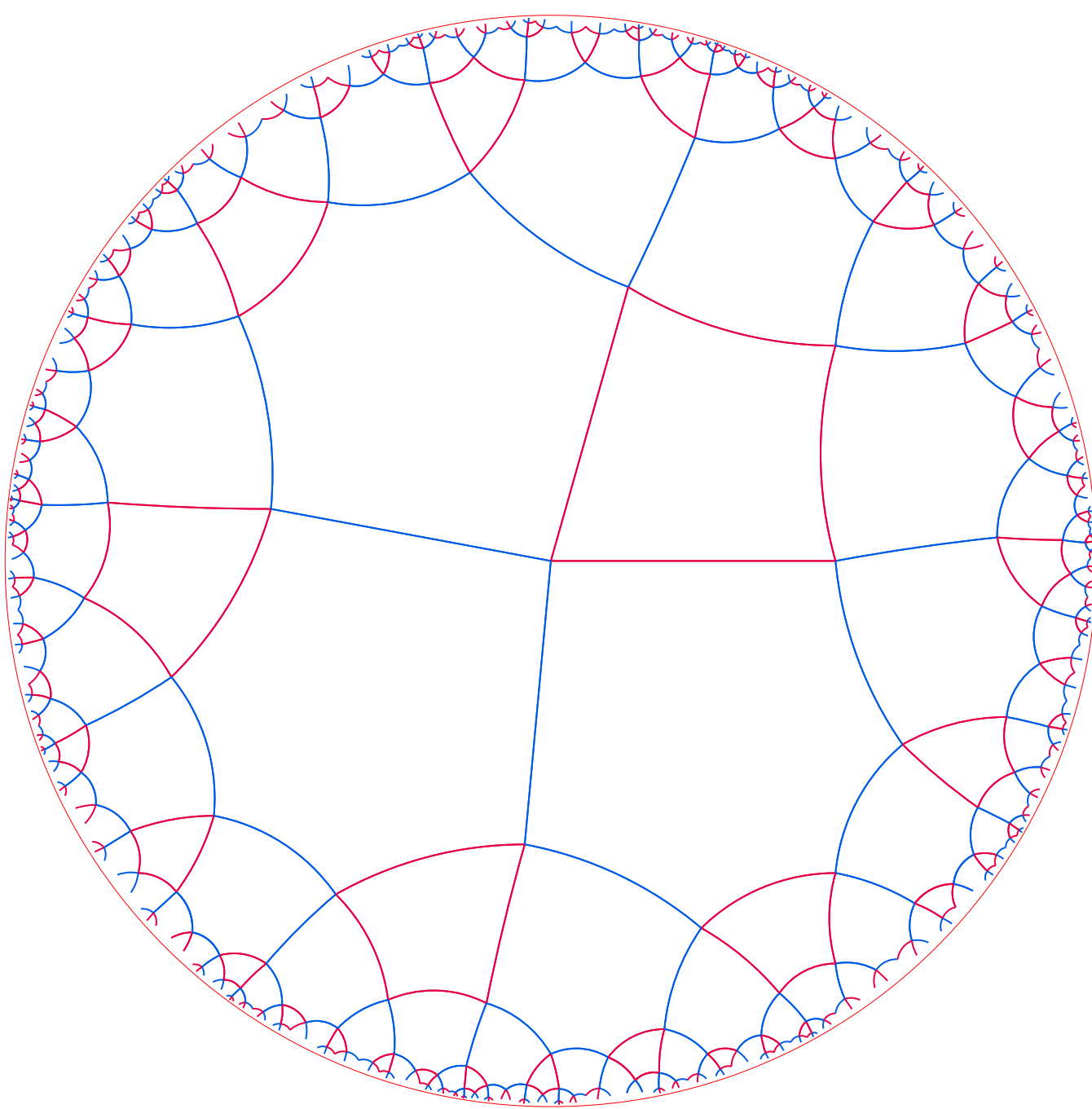
Let  $\Gamma$  be a planar graph.

### Vertex-transitive graph

Every vertex can be mapped by an automorphism of  $\Gamma$  on any other vertex.

### Locally finite graph

There exists a planar embedding of  $\Gamma$  such that there is no accumulation of the vertices in the plane.

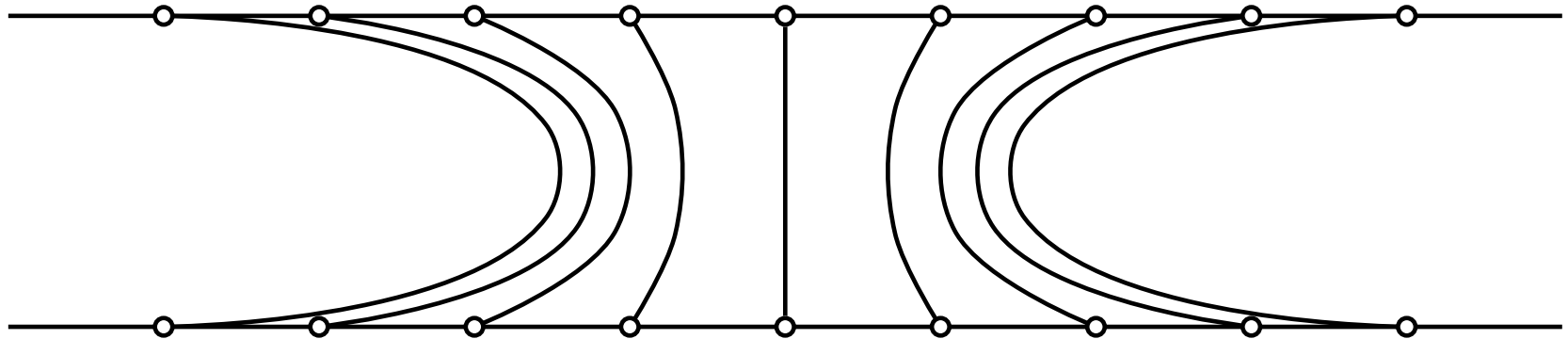


Example of a vertex-transitive locally finite graph drawn in the hyperbolic plane.

In the following,  $\Gamma$  is a locally finite vertex-transitive graph.

There exists a planar locally finite embedding of  $\Gamma$  such that :

- the set of edges is locally finite;

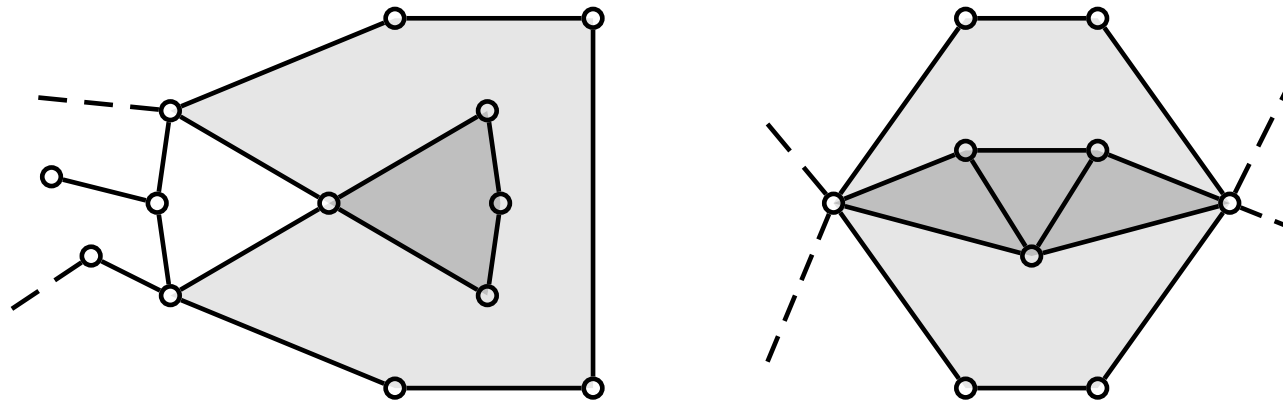


- finite 1- and 2-separations are trivial;
- the automorphisms map finite faces onto finite faces.

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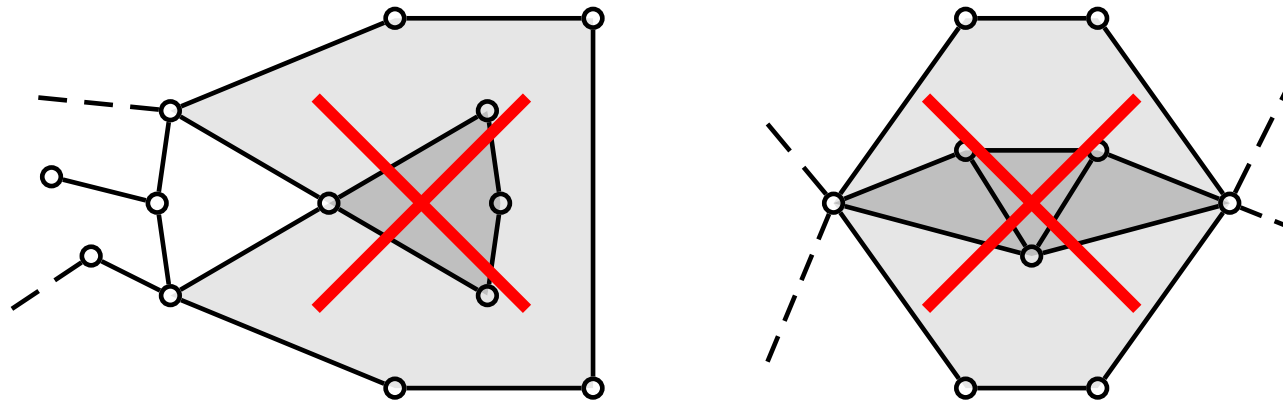
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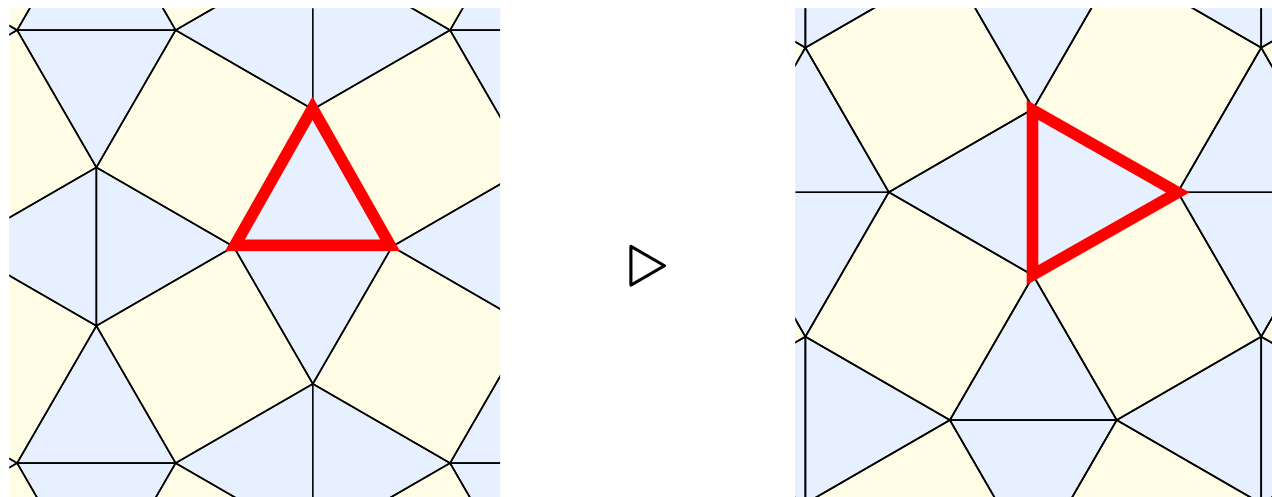


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# Coloring of the graph

Let  $\Gamma$  be a locally finite vertex-transitive graph.

## Coloring

There exists a coloring of the edges and the finite faces of  $\Gamma$  induced by the group of automorphisms of the graph.

The number of colors for the edges and the finite faces is finite.

$$\Rightarrow \begin{cases} E \text{ is the set of colors of the edges;} \\ F \text{ is the set of colors of the finite faces.} \end{cases}$$

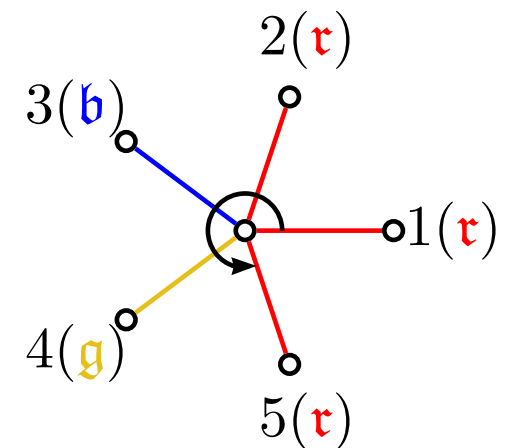
Let  $s$  be a vertex of  $\Gamma$ .

An **edge vector** around  $s$  describes the colors of the edges incident to  $s$  in a cyclic ordering :

$$[\mathbf{r}; \mathbf{r}; \mathbf{b}; \mathbf{g}; \mathbf{r}]$$

A **face vector** around  $s$  describes the colors of the faces incident to  $s$  in a cyclic ordering.

$$[o; v; g; o; v]$$



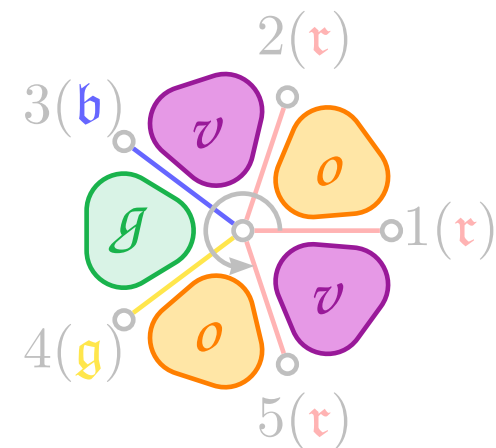
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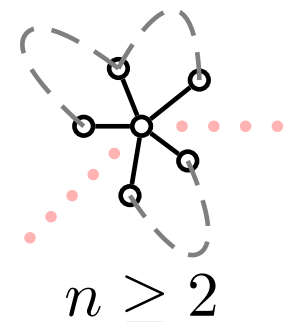
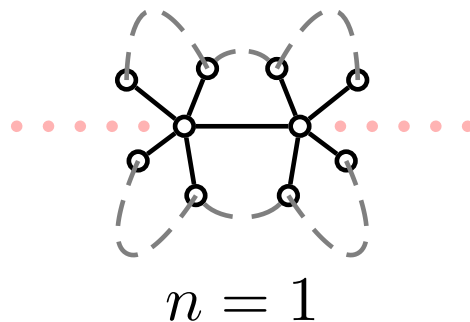
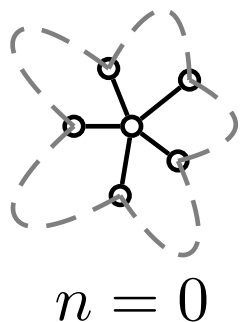


# Link with connectivity

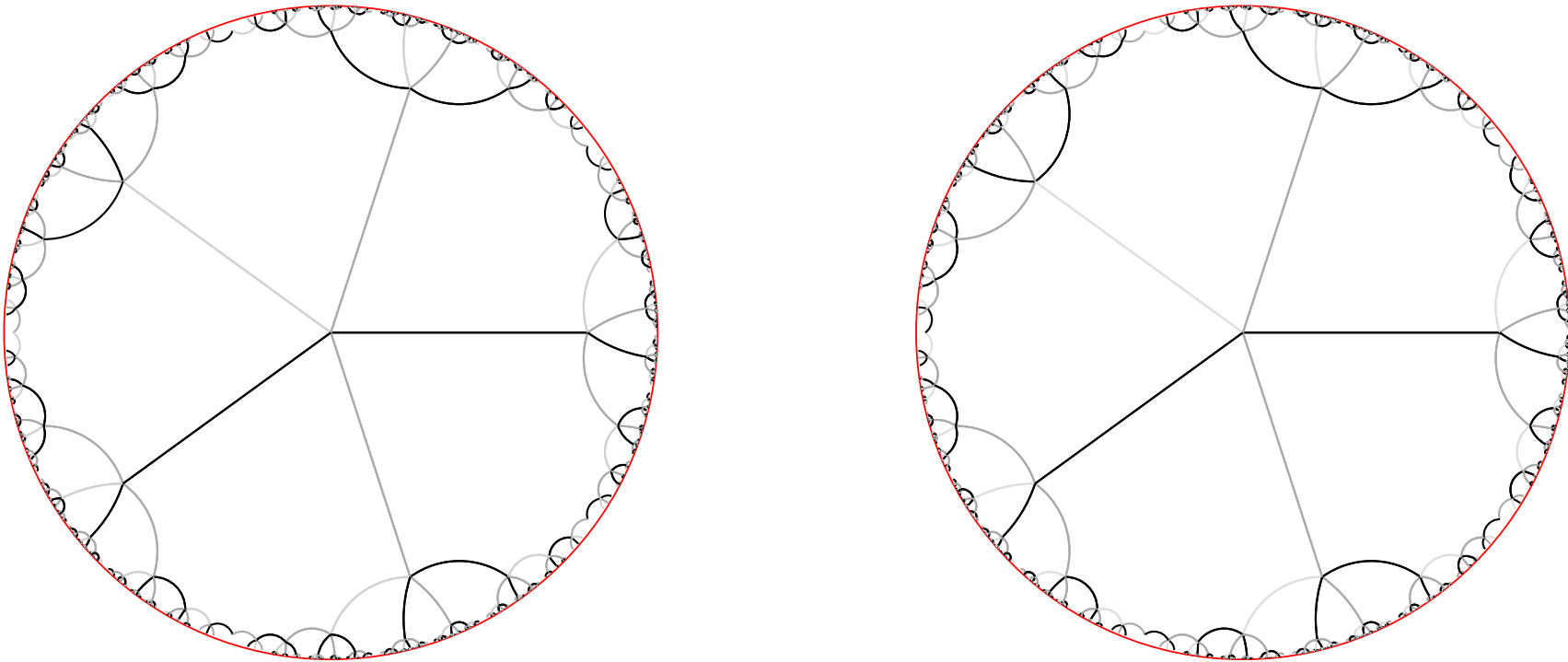
## Connectivity

Let  $n$  be the number of infinite faces appearing around a given vertex of  $\Gamma$ . Then, depending on the value of  $n$ :

- $n = 0 \Leftrightarrow \Gamma$  is 3-connected;
- $n = 1 \Leftrightarrow \Gamma$  is 2-separable;
- $n \geq 2 \Leftrightarrow \Gamma$  is 1-separable.



## A non-trivial example

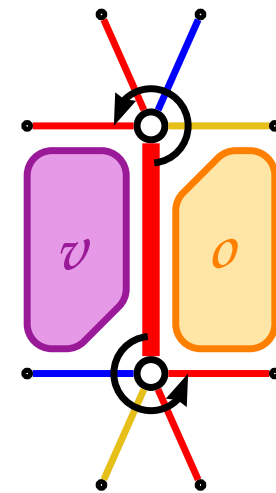


Non-isomorphic graphs having same edge vectors

Let  $e$  be an edge of  $\Gamma$ .

An **edge neighborhood**  $\eta$  around  $e$  describes :

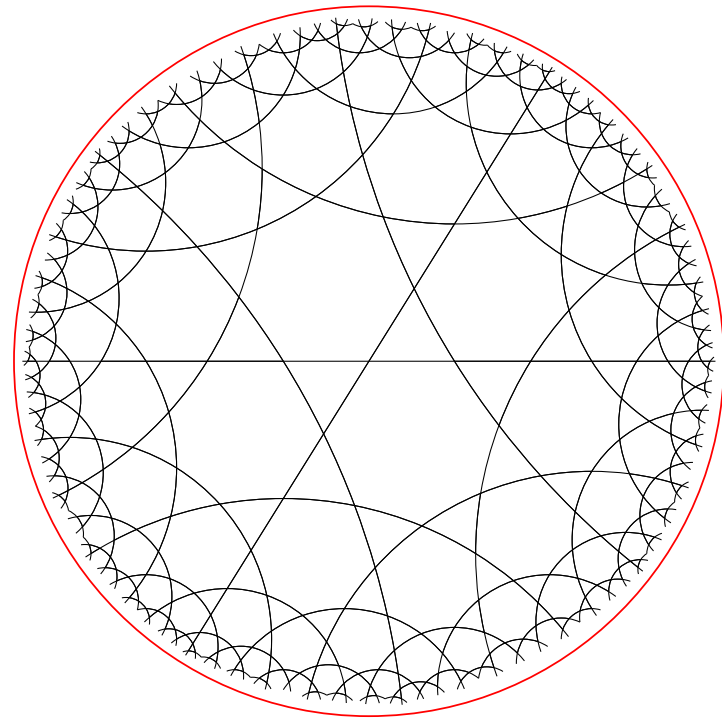
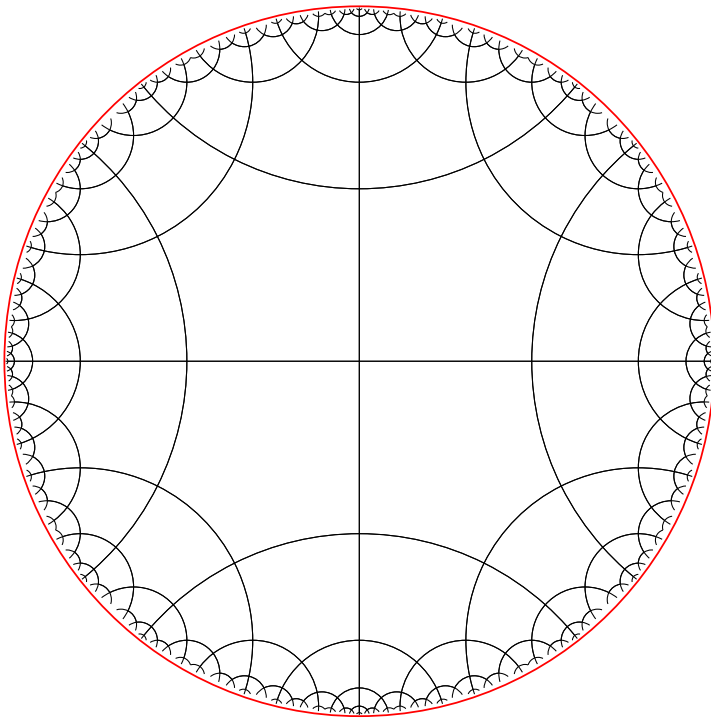
- the edge and face vectors around the extremities of  $e$ ;
- the colors of the faces separated by  $e$ .



The **color** of  $\eta$  is the color of the edge  $e$ .



## Another non-trivial example



Non-isomorphic graphs having same edge neighborhoods

Let  $\Gamma$  be a locally finite vertex-transitive graph.

### Lemma

The edge and face vectors of  $\Gamma$  are independent of the vertex.

### Lemma

The edge neighborhoods of  $\Gamma$  depend only on the color of the edges that they represent.

Let  $(E, F)$  be finite sets of colors of edges and faces.

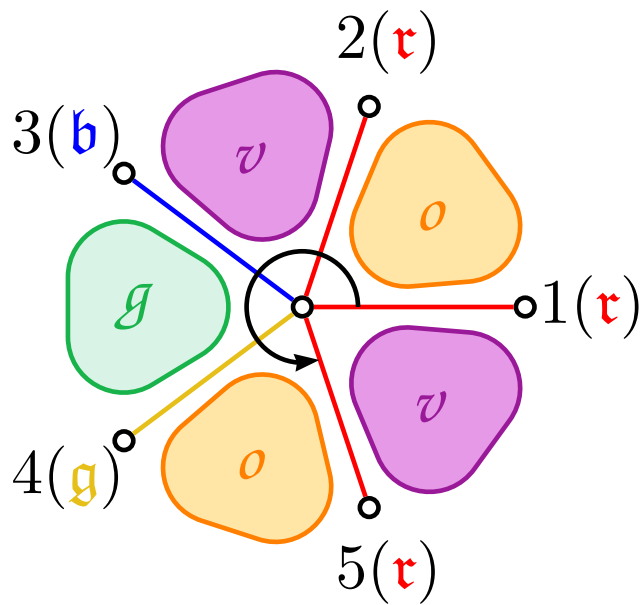
### Labeling scheme

A labeling scheme on  $(E, F)$  is a 3-uple  $(\xi, \phi, \eta)$  such that :

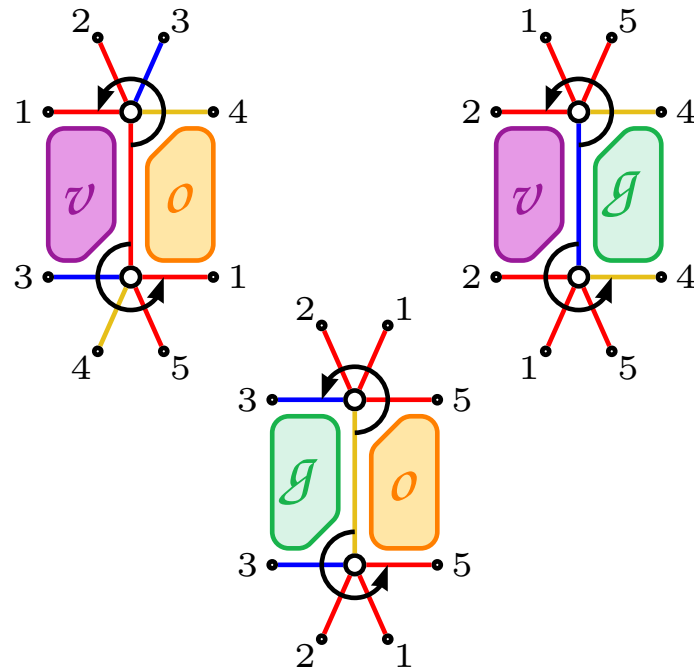
- an edge vector  $\xi \in E^d$ ,
- a face vector  $\phi \in F^d$ ,
- for each color  $e$  in  $\xi$ , an edge neighborhood  $\eta_e \in \eta$  of the same color.

# Example of labeling scheme

Edge and Face vectors :

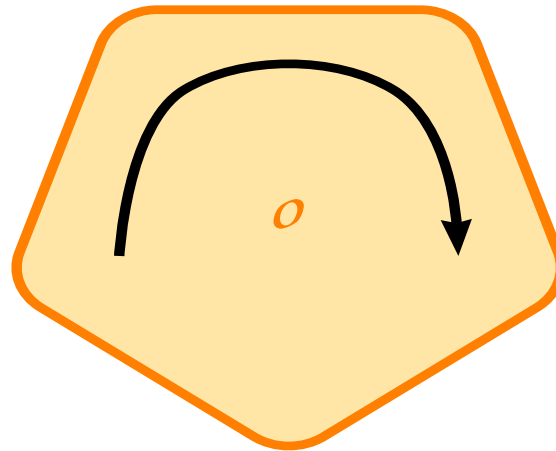
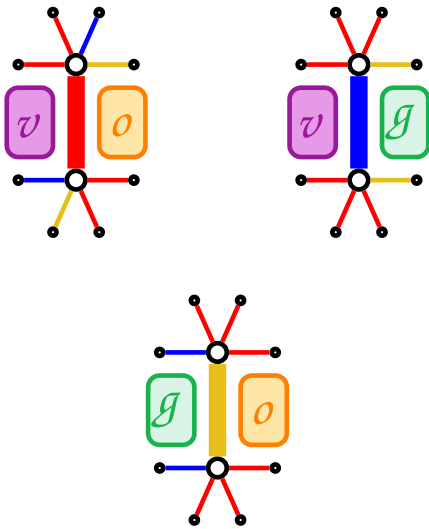


Edge neighborhoods :



# Example of a reconstruction

Consider the previous labeling scheme.  
Let us compute the border of an orange face :



▶ Example

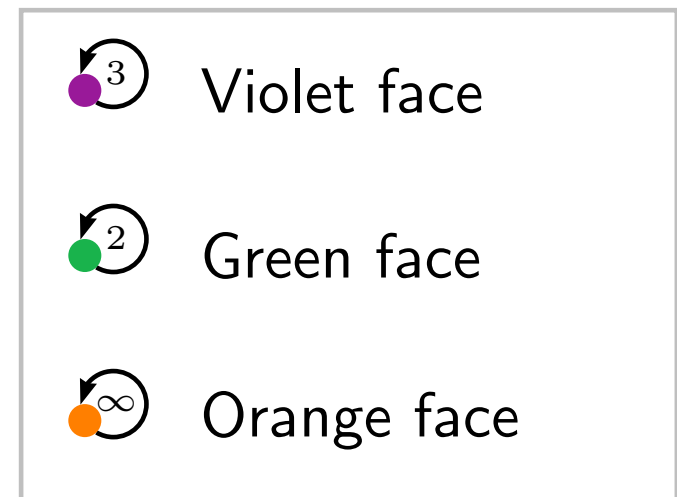
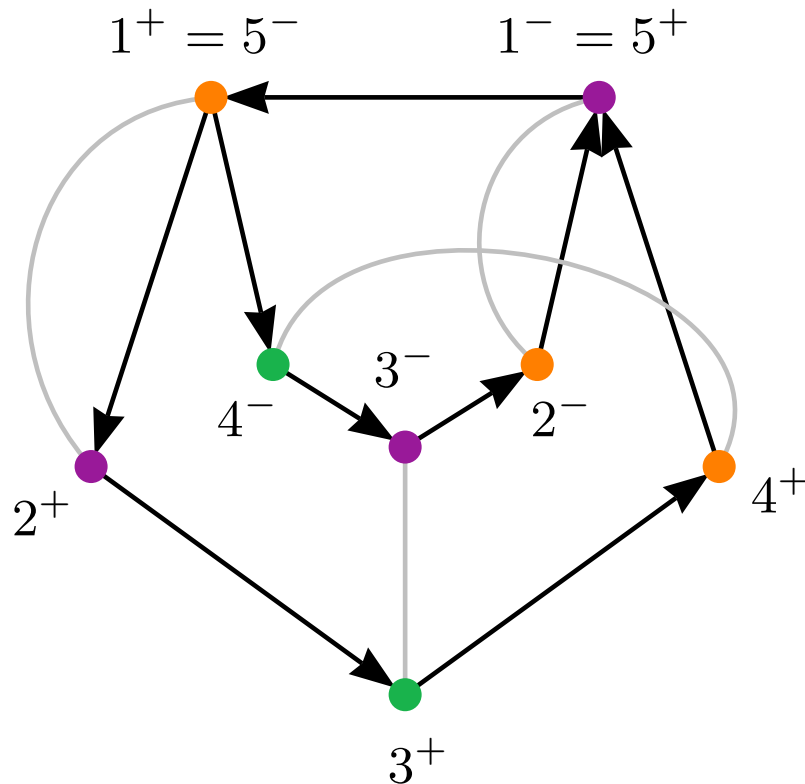
Let  $(\xi, \phi, \eta)$  be a labeling scheme.

### Lemma

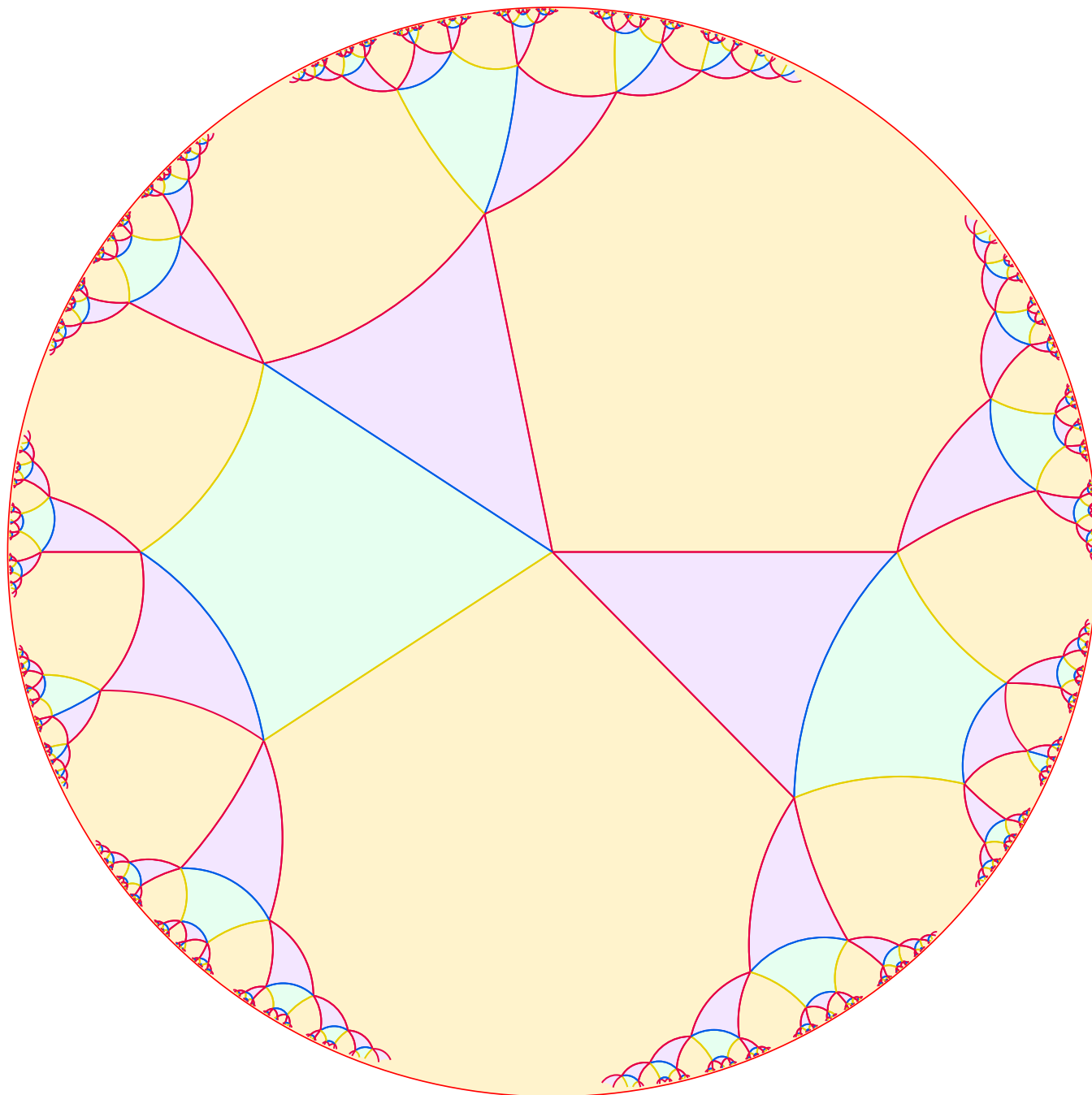
Given a color of a face in  $F$ , the sequence of edges appearing in the border is recognized by a finite state automaton.

$\implies$  Construction of the **border automaton** associated with  $(\xi, \phi, \eta)$

# Example of a border automaton



Follow alternatively black and gray edges to compute a border.



Example of graph built with the preceding automaton



## Enumeration

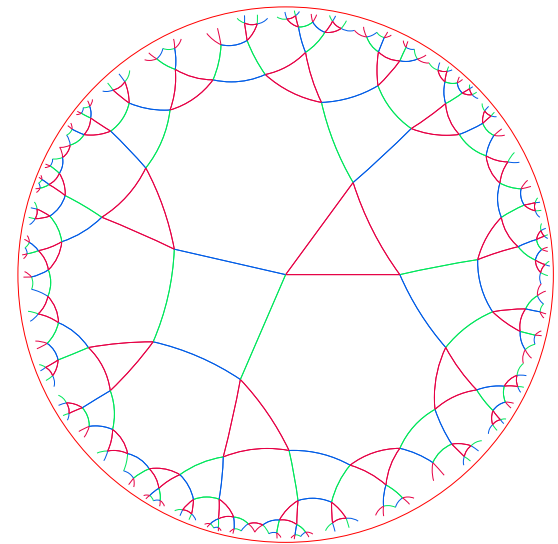
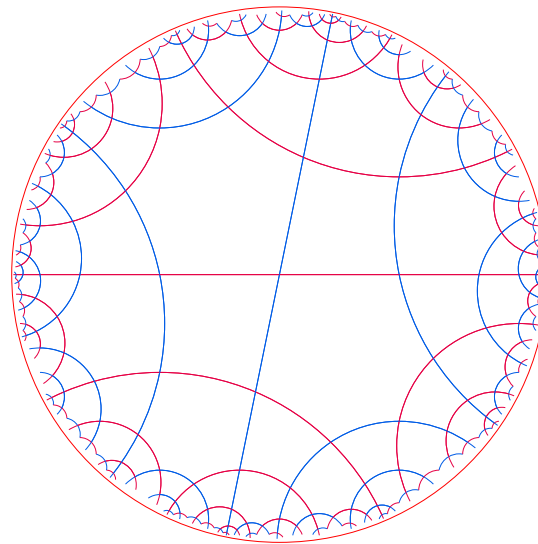
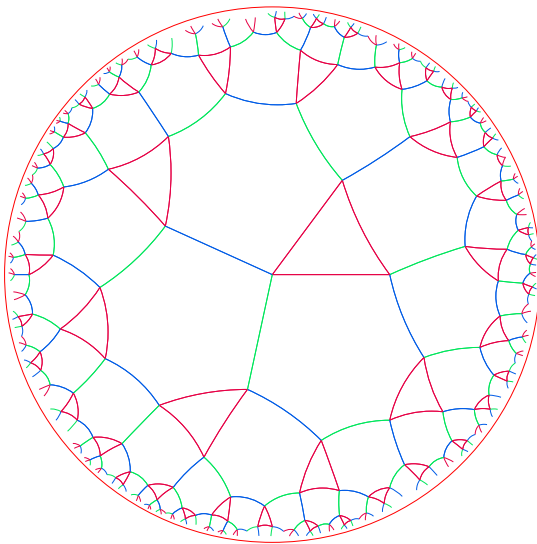
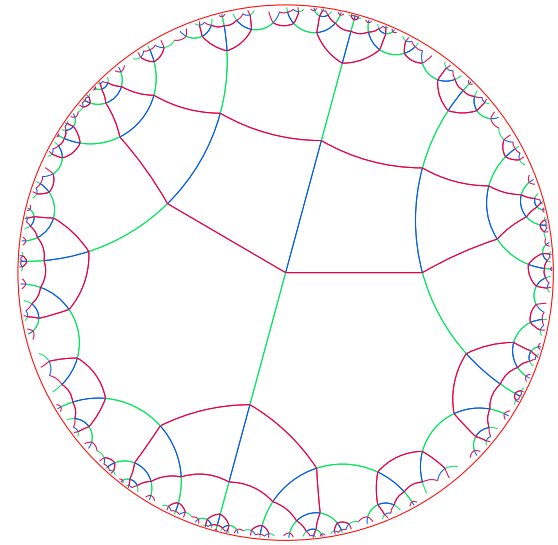
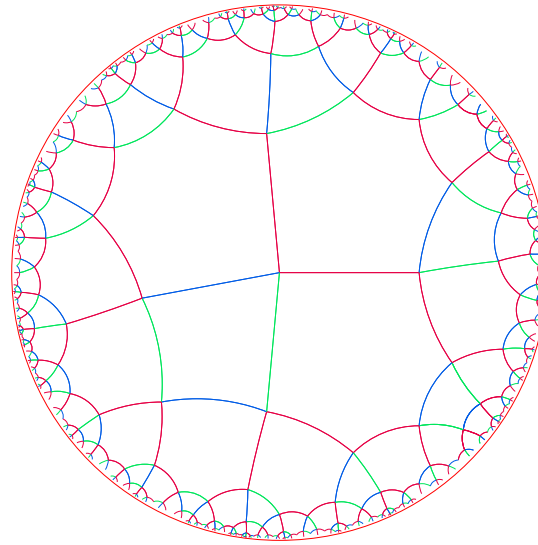
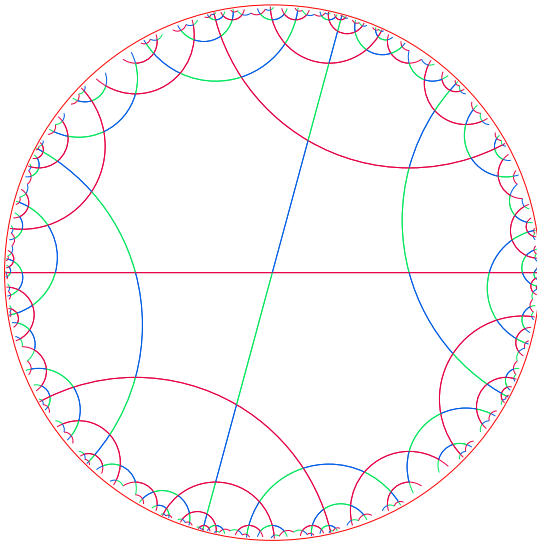
Given a number  $d \geq 2$ , it is possible to enumerate all planar locally finite transitive graphs having internal degree  $d$ .

⇒ Classification of the graphs depending on :

- their connectivity
- their geometry

# Results of enumeration

Degree	V.T.	Cayley
1	n.s.	n.s.
2	1	1
3	16	8
4	52	24
5	174	52
6	775	185



## Recognition

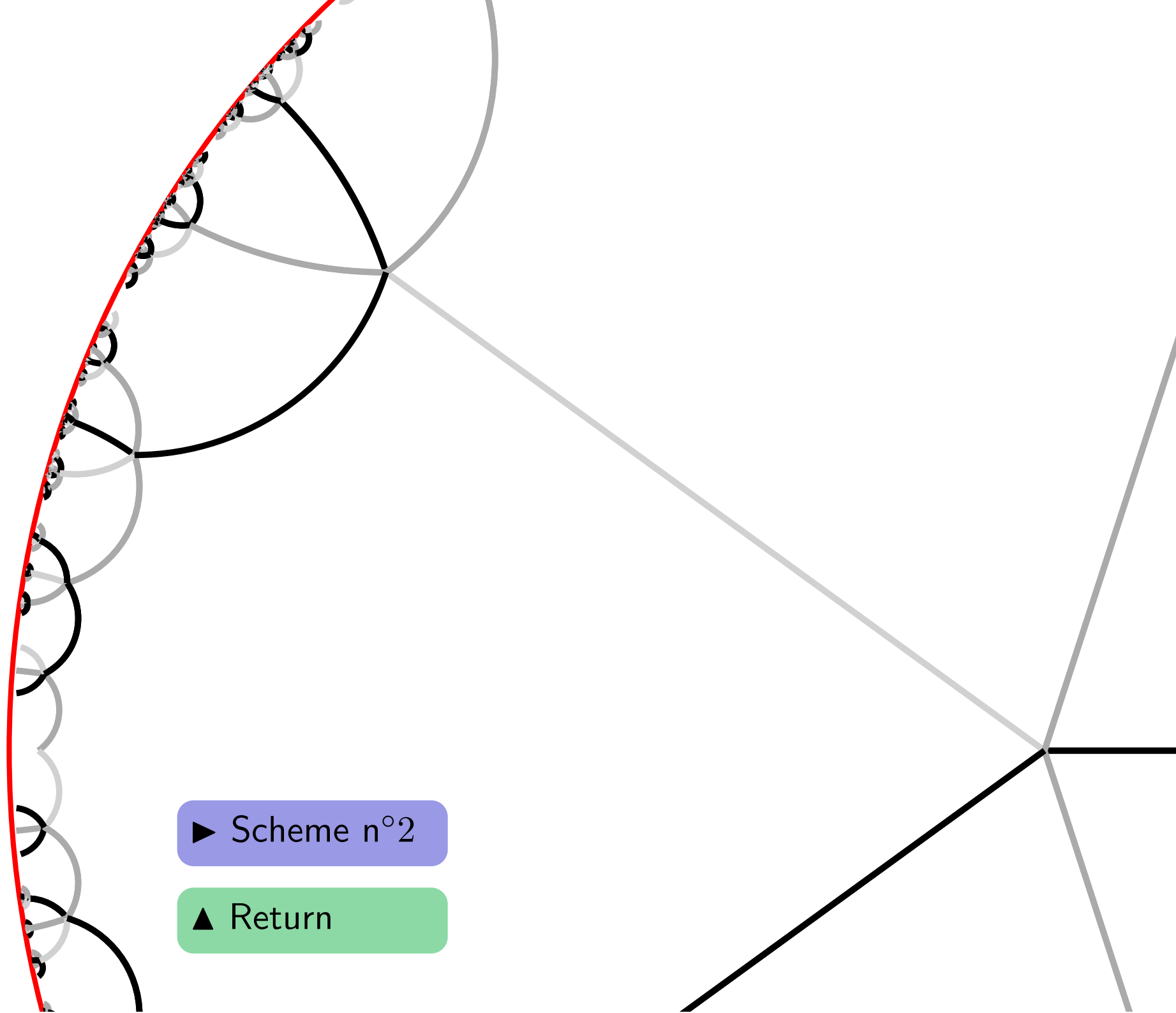
Given a locally finite vertex-transitive planar graph, it is possible to decide whether it is a Cayley graph of a group or not.

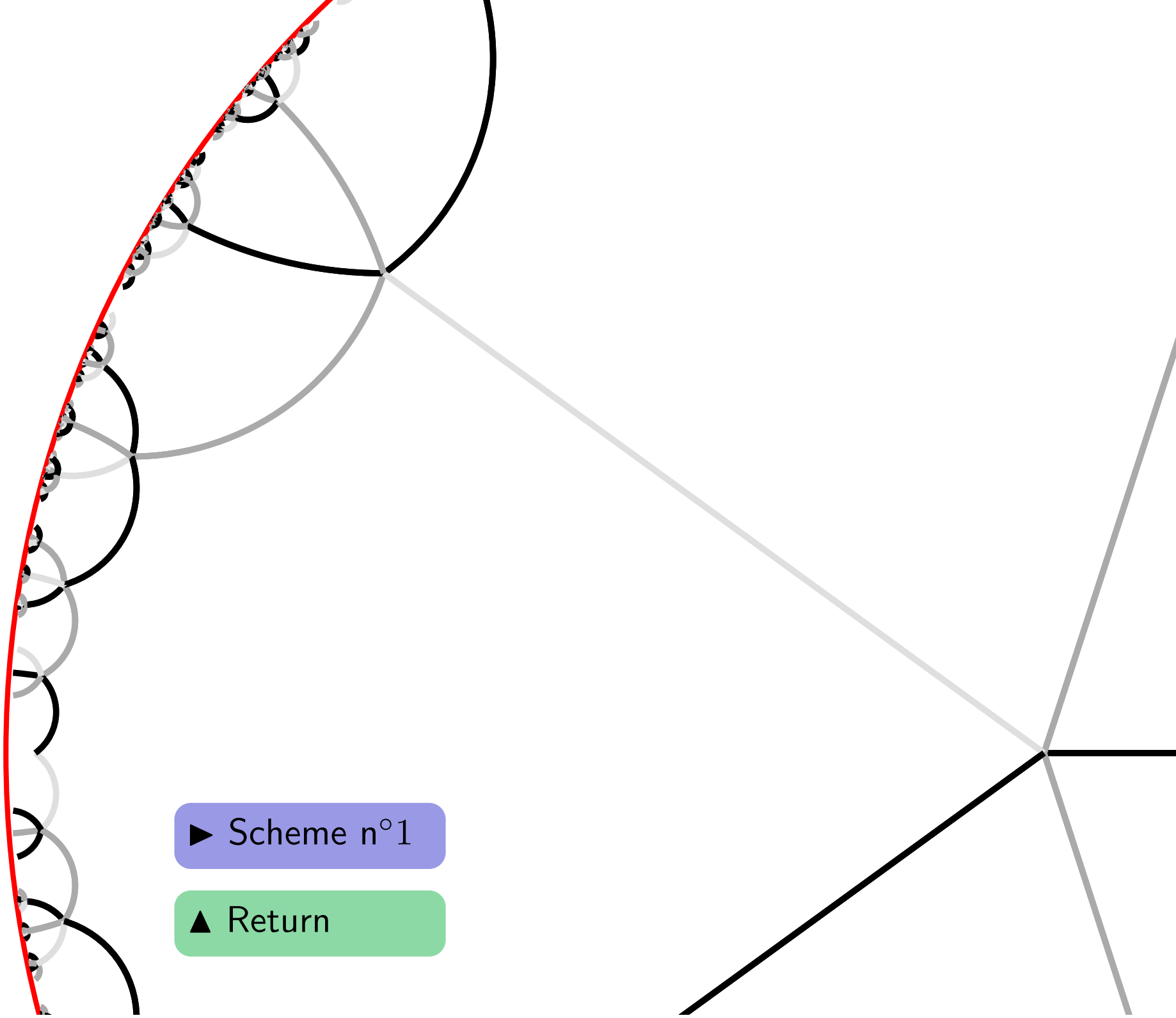
- ⇒ Enumeration of the possible presentations;
- ⇒ Generation of vertex-transitive non Cayley graphs.

# Perspectives

- Extend the enumeration to families of graphs with less symmetries, known as **cofinite graphs**.
- Study the quotients of the graphs onto compact surfaces (the torus for example).
- Evaluate the case of planar graphs in general (i.e. not locally finite).

# Appendix



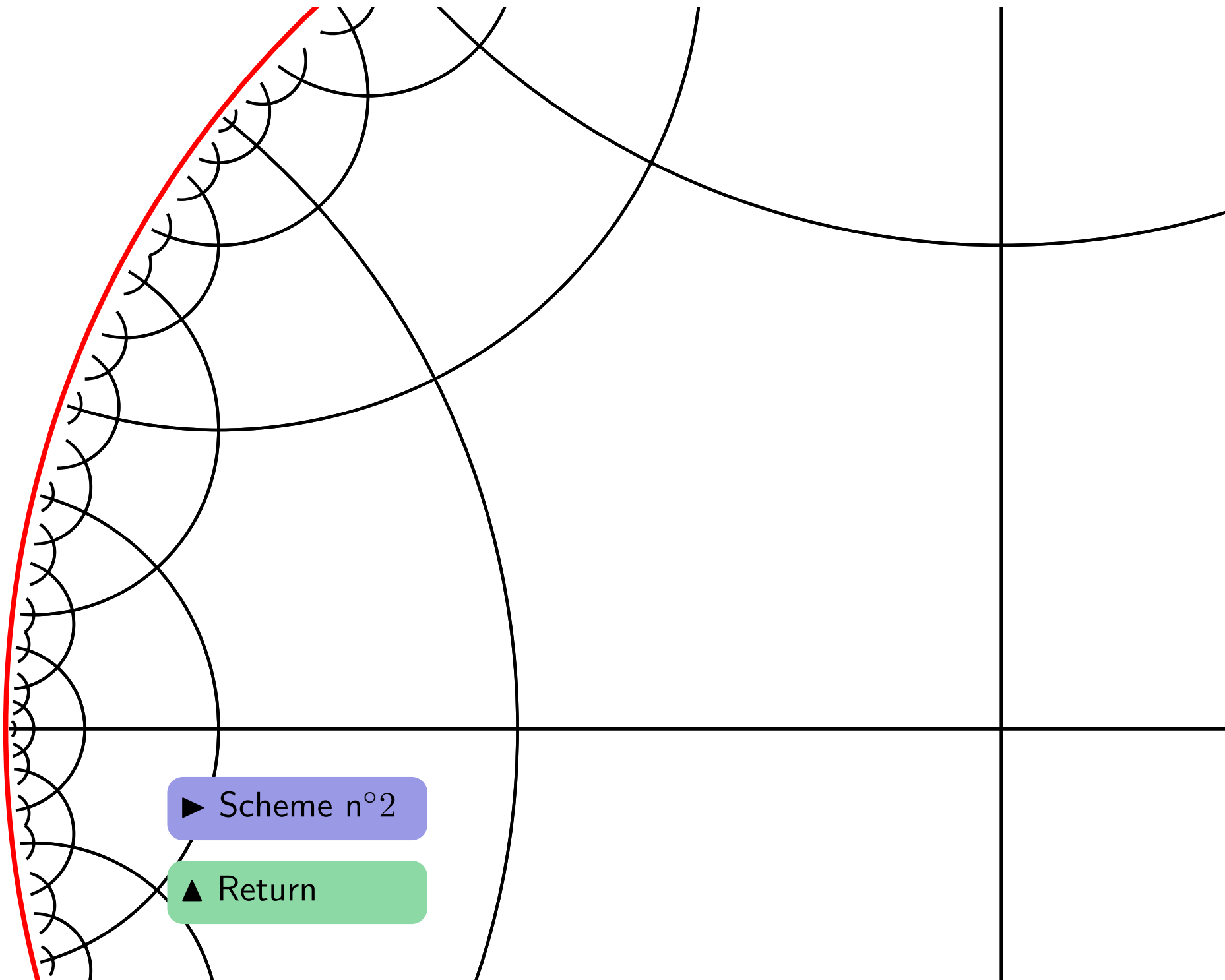


▶ Scheme n°1

▲ Return



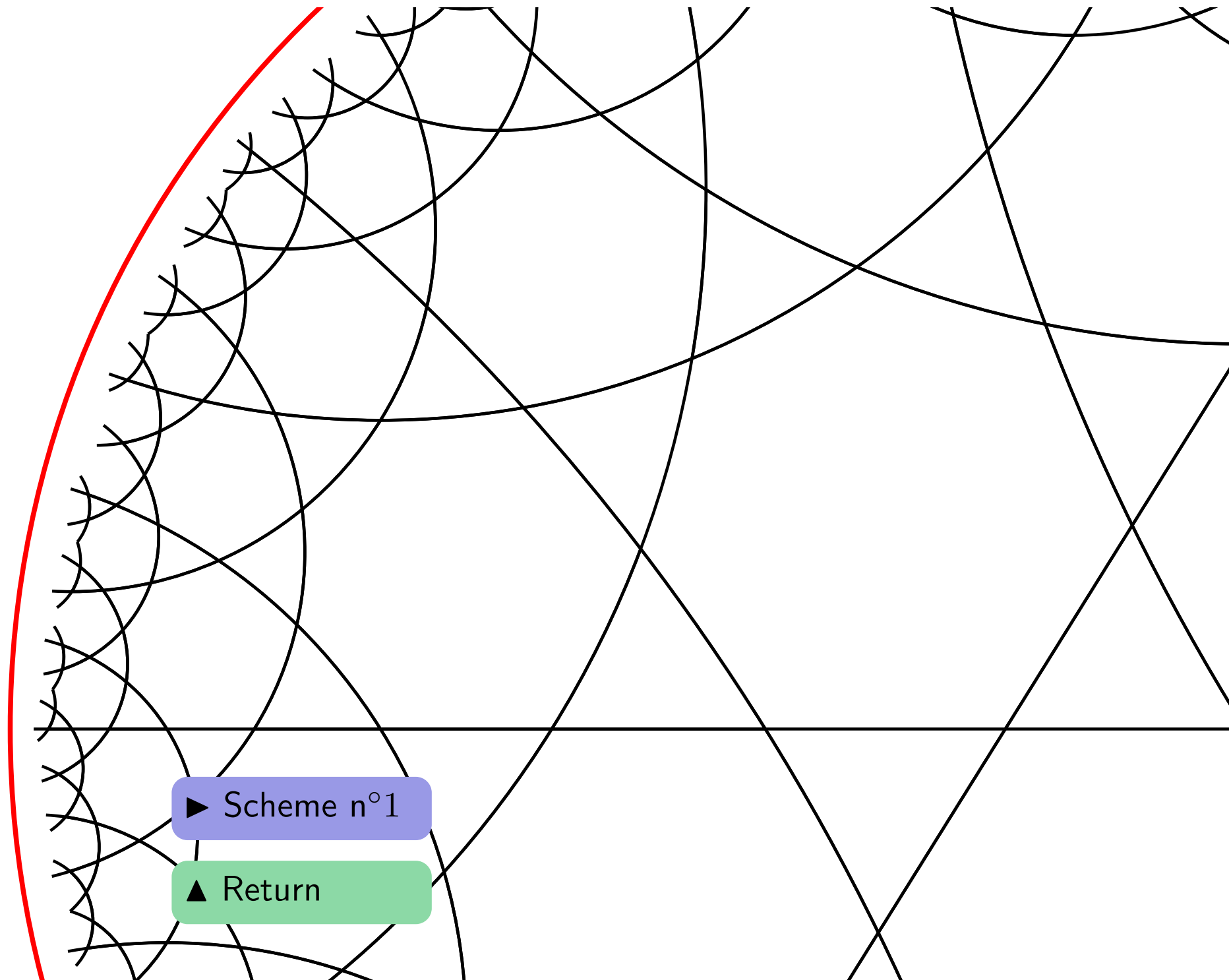
# Another non-trivial example



▶ Scheme n°2

▲ Return

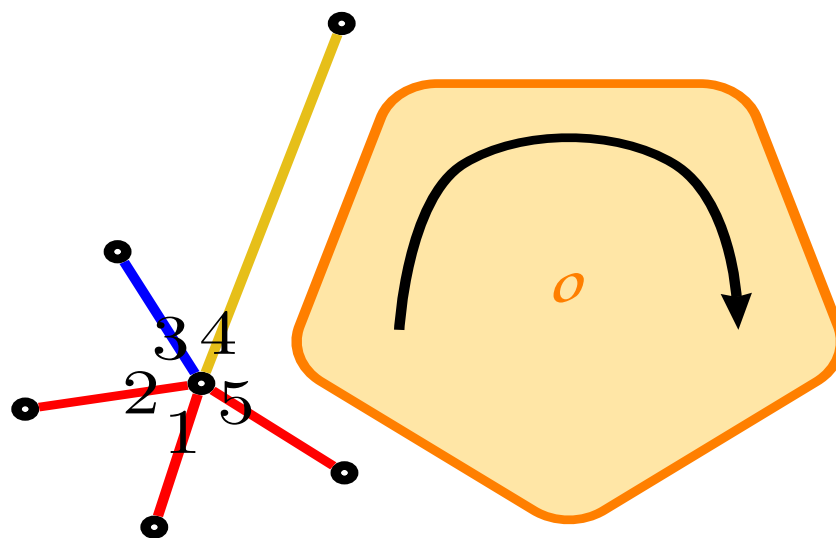
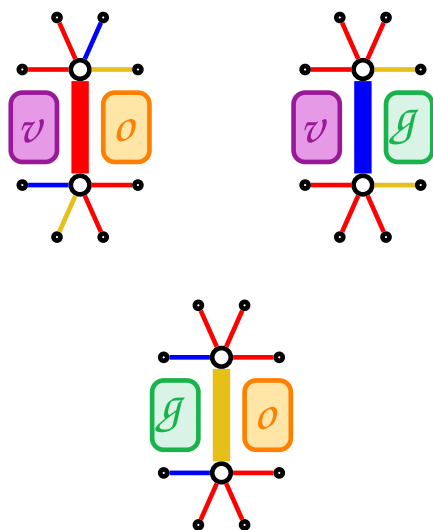
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# Example of a reconstruction

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Let us compute the border of an orange face :

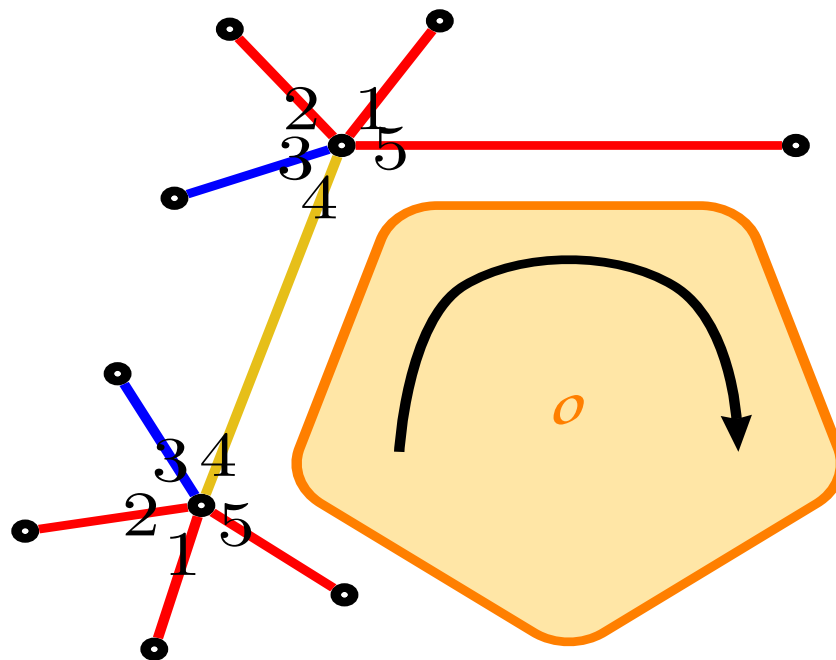
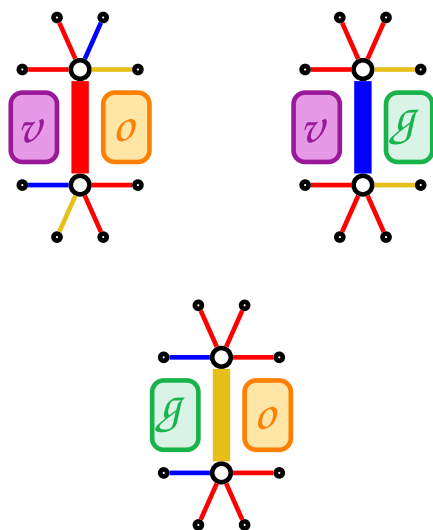


▶ Return

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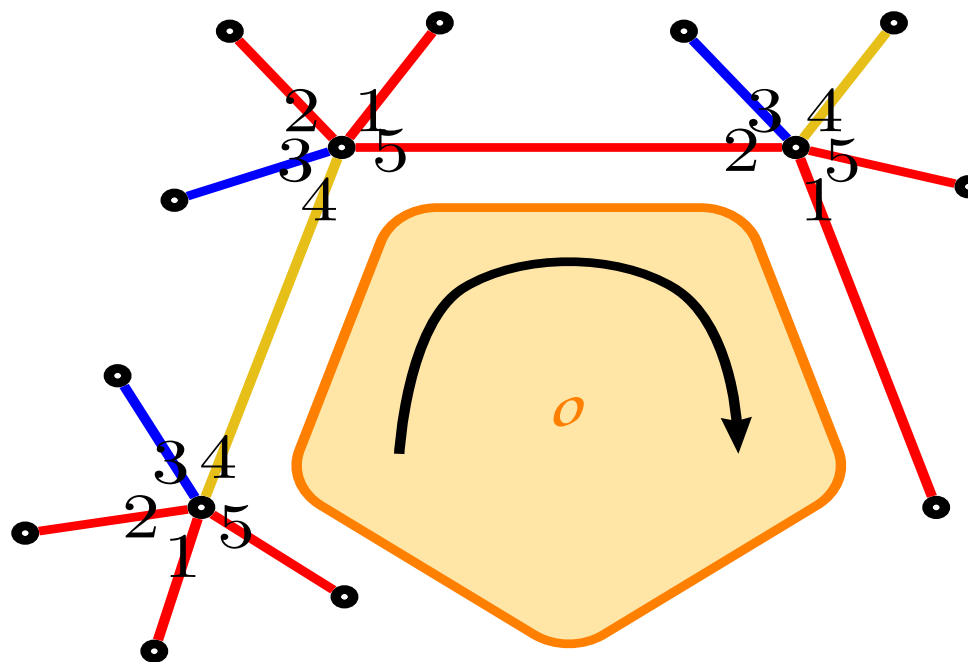
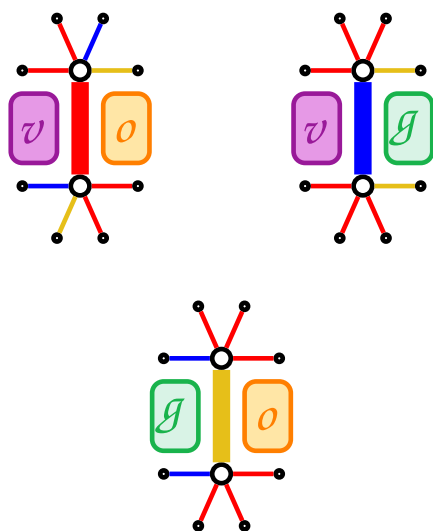


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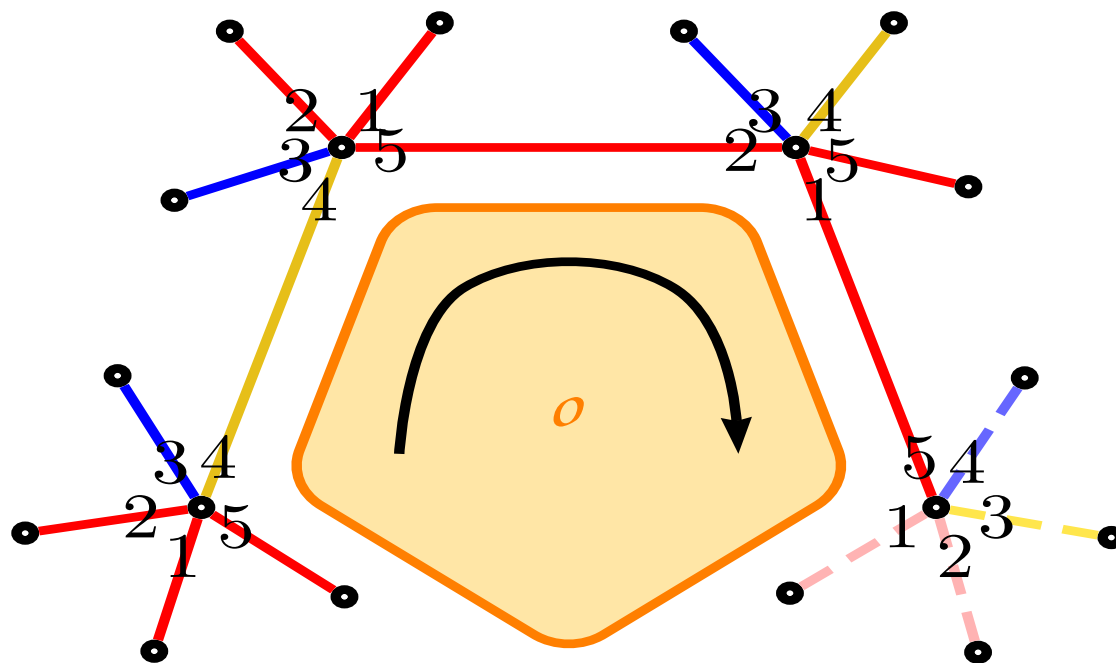
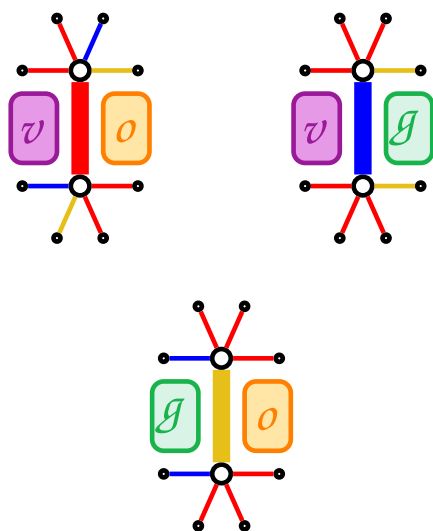


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