Tutorial #1 - Type systems and derivations

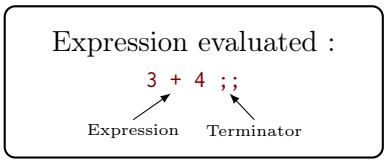
▷ OCaml (`https://ocaml.org`) is a functional programming language developed at INRIA, distributed with a compiler `ocamlc` and an interaction loop `ocaml`.

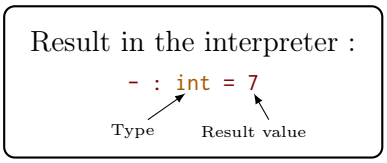
In order to write OCaml code, the most direct way consists in launching `emacs` on a `.ml` file, and then running `M-x tuareg-mode`. Then, start an interaction loop using the `C-c C-s` shortcut, and using the default interpreter name (`ocaml`).

Execute every expression in the interaction loop using the `C-x C-e` shortcut.

Exercice 1: Playing with OCaml

The evaluation of an expression in OCaml consists in :





Each of the following expression is meant to be executed with the OCaml interpreter and related to the expressions of the λ-calculus studied during the class. Afterwards, this list can be considered as a cheat sheet for finding the syntax of expressions.

```
true;;                    (* Booleans *)
false;;
0;;                       (* Integers *)
65535;;
if true then 7 else 14;;  (* Conditional *)
fun x → x + 1;;           (* Abstraction *)
(fun x → x + 1) 8;;       (* Application *)
"abcde";;                 (* Strings *)
"meta"^"for";;            (* String concatenation *)
```

▷ To bind a name to a value at the toplevel : `let x = 1;;`
To bind a name to a value as a local expression : `let y = "a" in y ^ y;;`

The following code is provided as an example of the infamous factorial function :

```
let rec fact x =          (* rec indicates that the function is recursive *)
  if (x <= 0) then 1
  else let y = x-1 in     (* declaration of a local binding *)
       x * fact y
```

Exercice 2: Algebraic data types

In this exercise, we introduce a particular sort of types named *algebraic data types*, also called *sum types* in OCaml. Such datatypes are defined using the following construction :

```
type ⟨type_name⟩ =
  | ⟨Constr_1⟩ of ⟨Type_1⟩
  | ⟨Constr_2⟩ of ⟨Type_2⟩
    ...
  | ⟨Constr_n⟩ of ⟨Type_n⟩
```

Example :

```
type value =
  | Red
  | Gray of int (* ints between 0 and 255 *)
  | RGB of (int * int * int);;
```

In order to manipulate sum types, OCaml contains a construct named *pattern-matching*, working as follows :

```
match ⟨expr⟩ with
  | ⟨p_1⟩ → ⟨expr_1⟩
  | ⟨p_2⟩ → ⟨expr_2⟩
    ...
  | ⟨p_n⟩ → ⟨expr_n⟩
```

Example :

```
let red_component c = match c with
  | Red          → 255
  | Gray g       → g
  | RGB (r,g,b) → r;;
```

The following lines explain how to construct and use such values :

```
let c1 = Gray 100;;
red_component c1 ;; (* → 100 *)
let c2 = RGB (50,150,250);;
red_component c2 ;; (* → 50 *)
```

For further reading on the subject, one can find at the address `https://caml.inria.fr/pub/docs/oreilly-book/html/book-ora016.html` some documentation and examples about the definition of types and the pattern-matching in OCaml.

1. Write a sum type `weight` for the manipulation of weights in kilos, in pounds and in carats.
2. Write the translation function that can convert any element of type `weight` in kilos. (*Cultural hint :* 1kg = 2.205 lbs = 5000 carats)

Exercice 3: Simple untyped λ-calculus

Consider the following definition of a language based on the untyped λ-calculus discussed during the class :

Syntax			Evaluation rules
$t ::=$		*expressions*	$\dfrac{t_1 \to_\beta t_1'}{(t_1\ t_2) \to_\beta (t_1'\ t_2)}$
	x	*variable*	
	$\lambda x.t$	*abstraction*	
	$(t\ t)$	*application*	$\dfrac{t \to_\beta t'}{(v\ t) \to_\beta (v\ t')}$
$v ::=$		*values*	
	$\lambda x.t$	*abstraction value*	$(\lambda x.t_1\ t_2) \to_\beta [x \mapsto t_2]t_1$

Let us translate this definition into OCaml.

▷ First, retrieve the code given in the sources, *compile it* with `make` and load the `main.ml` file in your editor. Start the interaction loop (`C-c C-s`) with the command : `ocaml syntax.cma` (you'll need to give the absolute path to `syntax.cma` if the file is not in the same directory you started `emacs` with).

Execute (`C-x C-e`) the three first lines (`open Syntax;;` and its siblings). After that, it becomes possible to copy-paste the examples (*not* the type declarations) into the buffer and test them.

Consider the following definition in the file `syntax.ml`, that corresponds to a sum type describing the grammar of the language :

```
type id = string                    (* Identifiers *)      (* do NOT copy-paste in the interpreter *)

type term =
  | TmVar  of id                    (* Variable *)
  | TmAbs  of id * term             (* Abstraction *)
  | TmApp  of term * term           (* Application *)
```

This code defines a type and a set of constructors for building λ-expressions : `TmVar`, `TmAbs` and `TmApp`. The following are examples of λ-expressions in OCaml :

```
TmAbs ("x", TmVar "x");;                    (* fun x -> x          *)
TmAbs ("y", TmAbs ("z", TmVar "z"));;       (* fun y -> fun z -> z *)
TmApp (TmAbs ("x", TmVar "x"), TmVar "z");; (* ((fun x -> x) z)    *)
```

The sources also provide a parser to simplify the writing of complex expressions, given with the `parse` function :

```
parse "fun x -> x";;                    (* -> TmAbs ("x", TmVar "x") *)
```

The syntax of the language is (as much as possible) the same as in OCaml.
To understand this code, it is possible to read the function `term_to_string` that transforms a λ-expression into a string acceptable by LaTeX :

```
let rec term_to_string t =
  match t with
  | TmVar v      -> "$"^(blue_string v)^"$"
  | TmAbs (v,e)  -> "$\\lambda "^(blue_string v)^"$ . "^(term_to_string e)
  | TmApp (a,b)  -> (term_to_string a)^" "^(term_to_string b)
```

Finally, the `syntax.ml` file contains functions for the substitution (`substitute`), the α-renaming (`rename`) and the β-reduction (`reduce_one` and `reduce`) of expressions, but most of these functions are not written.

1. Write the code for `substitute` and `rename`. The definition for these functions appears on the slides of the course.
2. Write the code for `reduce_one`. This function should return a λ-expression where only one β-reduction step has been applied.
3. Test your code with different examples. In particular, write a term whose reduction is not finite.

```
substitute "x" (TmVar "y") (parse "(fun x -> x) x");;       (* -> ((fun x -> x) y) *)
rename     (parse "(fun x -> x) (fun y -> y)");;            (* -> ((fun x1 -> x1) (fun x2 -> x2)) *)
reduce_one (parse "(fun x -> x x) (fun y -> y)") (Ctx []);; (* -> ((fun y -> y) (fun y -> y))*)
reduce     (parse "(fun x -> x x) (fun y -> y)") (Ctx []);; (* -> (fun y -> y) *)
```

Exercice 4: Simple extensions

In this exercise, we propose to extend the language so as to contain rules for booleans and integers. For the record, these rules are the following :

Syntax			Evaluation rules
t ::=	...	*expressions*	$\dfrac{\texttt{t}_1 \rightarrow_\beta \texttt{t}_1'}{\texttt{if t}_1\texttt{ then t}_2\texttt{ else t}_3 \rightarrow_\beta \texttt{if t}_1'\texttt{ then t}_2\texttt{ else t}_3}$
	true, false	*booleans*	
	zero, succ t	*naturals*	
	if t then t else t	*if-then-else*	$\texttt{if true then t}_2\texttt{ else t}_3 \rightarrow_\beta \texttt{t}_2$
	iszero t	*zero-equality*	$\texttt{if false then t}_2\texttt{ else t}_3 \rightarrow_\beta \texttt{t}_3$
v ::=	...	*values*	$\dfrac{\texttt{t} \rightarrow_\beta \texttt{t'}}{\texttt{iszero t} \rightarrow_\beta \texttt{iszero t'}}$
	true, false	*boolean value*	
	nv	*numeric value*	$\texttt{iszero zero} \rightarrow_\beta \texttt{true}$
nv ::=		*numeric values*	$\texttt{iszero (succ t)} \rightarrow_\beta \texttt{false}$
	zero	*zero value*	
	succ nv	*successor value*	

The first objective consists in extending the language with the boolean expressions (`true`, `false` and `if .. then .. else`). For the sake of safety, copy your entire code in a new directory.

1. Extend the OCaml grammar for `term` with boolean expressions (4 constructors).
2. Modify the parser file `parser.mly` so as to use these constructors (this mostly consists in adapting the file comments to your code).
3. Extend the printer `term_to_string` and the matcher `is_value`.
4. Extend in order : `substitute`, `rename` and `reduce_one` (the `reduce` function should continue to work as before).

The goal now is to keep this version of your code safe (possibly by making a copy into a new directory), and apply the same operations to handle the natural numbers.

Syntax			Evaluation rules
t ::=	...	*expression*	$\dfrac{\texttt{t} \rightarrow_\beta \texttt{t'}}{\texttt{iszero t} \rightarrow_\beta \texttt{iszero t'}}$
	zero, succ t	*naturals*	
	iszero t	*zero-equality*	
v ::=	...	*values*	$\texttt{iszero zero} \rightarrow_\beta \texttt{true}$
	nv	*numeric value*	$\texttt{iszero (succ t)} \rightarrow_\beta \texttt{false}$
nv ::=		*numeric values*	
	zero	*zero value*	
	succ nv	*successor value*	

5. Extend the OCaml grammar for `term` with integer expressions (2 constructors).
6. Modify the parser file `parser.mly` so as to use these constructors (this mostly consists in adapting the file comments to your code), `term_to_string` and `is_value`.
7. Extend in order : `substitute`, `rename` and `reduce_one` (the `reduce` function should continue to work as before).