Oldies: (which does not necessarily mean goldies :-) 
Pomset logic, proof-nets and coherence semantics

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Warning and apologies

- Old work, nineties (with little time to re-work it)
- Special thanks to Sylvain Pogodalla, who spent part of his thesis to try to prove with me yet unsolved questions
- Motivated by a possibility to solve open questions:
  - More fashionable sequent/term/rewrite calculus
  - Correspondence with BV
Coherence Semantics

- Formulae: (possibly infinite) graphs
- Proofs up to normalisation: cliques
- Morphisms, linear maps:
  - $F$ sends cliques to cliques
  - When a union is a clique:
    - Commute with union
    - Commute with intersection
Multiplicative coherence spaces
Girard’s remark

- Vertices: pairs of vertices
- Par: both
- Times: both
- One non commutative «< »:
  - A:
  - B:
- No other multiplicative.

\[
\begin{array}{c|c|c|c}
A \setminus B & 0 & = & 1 \\
\hline
0 & 1 & 1 & 1 \\
= & 1 & 1 & ? \\
\hline
? & ? & ? & ?
\end{array}
\]
Before

- Written <
  - Non commutative
  - Associative
  - Self-dual \((A < B) \downarrow \equiv (A^\perp < B^\perp)\)

Girard’s question:
what syntax for this calculus?
# Bicoloured proof nets

<table>
<thead>
<tr>
<th>Name</th>
<th>axiom-link</th>
<th>par-link</th>
<th>before-link</th>
<th>times-link</th>
</tr>
</thead>
<tbody>
<tr>
<td>Premises</td>
<td>none</td>
<td>$A$ and $B$</td>
<td>$A$ and $B$</td>
<td>$A$ and $B$</td>
</tr>
<tr>
<td>R&amp;B-graph</td>
<td>$a \perp a$</td>
<td>$A \varnothing B$</td>
<td>$A &lt; B$</td>
<td>$A \otimes B$</td>
</tr>
<tr>
<td>Conclusions</td>
<td>$a$ and $a^\perp$</td>
<td>$A \varnothing B$</td>
<td>$A &lt; B$</td>
<td>$A \otimes B$</td>
</tr>
</tbody>
</table>
Proof nets

- Extra-arc for denoting an order (preferably SP, definable) between conclusions
- Criterion no alternate elementary cycle
- Viewing cuts as \((\exists K)K \otimes K\) they take part in the order
Cut elimination preserves correctness

Cut on axiom
Cut elimination preserves correctness and order

Cut before/before

<table>
<thead>
<tr>
<th></th>
<th>Proof Net</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Proof Net Diagram" /></td>
<td><img src="image" alt="Order Diagram" /></td>
<td><img src="image" alt="Order Diagram" /></td>
</tr>
</tbody>
</table>
Cut elimination preserves correctness and order

Cut times/par
Interpreting proofs

- Choose a token for each axiom
- Collect the tuples: they are a clique of the coherence space associated with the partially ordered set of conclusions:

\[
\bar{x} \sim \bar{y}[\langle A_i \rangle_{i \in (I, \prec)}] \\
\iff \\
\exists i \ x_i \sim y_i \land (\forall j > i \ x_j = y_j)
\]
Interpreting proofs:

soundness and « completeness »

- Proof: would lead to an infinite alternate elementary path incoherent moving up, coherent moving down.

- Moreover the converse is true: if the proofnet is not correct, some interpretations are not cliques even in a single finite coherence space: \( \mathbb{N} \) (isomorphic to its orthogonal \( \mathbb{Z} \)).
Directed cographs

- Directed cographs for denoting formulae:
  - Containing the single vertex graphs
  - Closed under
    - Disjoint union
    - Undirected series composition
    - Directed series composition
    - (Hence under complementation if an undirected edge is viewed a pair of opposite directed edges)
Directed cographs

- Universal characterisation:
  - The directed part is an SP order
  - The undirected part is a cograph
  - Weak transitivity

\[(x, y) \in R \land (y, x) \notin R \land (y, z) \in R \Rightarrow (x, z) \in R\]

\[(x, y) \in R \land (y, z) \in R \land (z, y) \notin R \Rightarrow (x, z) \in R\]
Handsome proofnets

- Vertices: propositional variables and their negations
- A directed cograph (the formula)
- Plus a perfect matching (the axioms)
- Criterion:
  - Every alternate elementary cycle contains a chord
Uncorrect
Correct
Fold

\[ \text{II.} \]

\[ \begin{align*}
X & \quad Y \\
X & \quad Y \\
A_1 & \\
A_2 & \\
A_3 &
\end{align*} \]
Correct
Correct with a link
Correct with three links
Property

- Fold and unfold preserve the criterion that every alternate elementary cycle contains a chord.
- Observe that when there are only links, this means that there is no alternate elementary cycle at all.
Cut-elimination

- Works directly on axioms
- Also derives from the one on proof nets with links.
- Looks like Girard’s turbo cut-elimination
Rewriting
(black lollipop preserves correctness)

<table>
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<tr>
<th>Rule</th>
<th>Premise</th>
<th>Inference</th>
<th>Conclusion</th>
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<tr>
<td>$\otimes\varphi_1$</td>
<td>$(X \otimes Y) \otimes (U \otimes V)$</td>
<td>$\rightarrow$</td>
<td>$(X \otimes U) \otimes (Y \otimes V)$</td>
</tr>
<tr>
<td>$\otimes\varphi_2$</td>
<td>$Y \otimes U$</td>
<td>$\rightarrow$</td>
<td>$U \otimes Y$</td>
</tr>
<tr>
<td>$\otimes&lt;4$</td>
<td>$(X \ltimes Y) \otimes (U \ltimes V)$</td>
<td>$\rightarrow$</td>
<td>$(X \otimes U) \ltimes (Y \otimes V)$</td>
</tr>
<tr>
<td>$\otimes&lt;\varphi_3$</td>
<td>$Y \otimes (U \ltimes V)$</td>
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Conjecture

- All correct handsome proofnets are obtained by the correct rewriting from
  \[
  \bigotimes_i (a_i \otimes a_i^\perp)
  \]
  (True for MLL)
Sequent calculus?

- Times as usual
- Par as usual
- MIX introduces the order the restrictions of K to G and D should be I and J

\[ \frac{\frac{}{\Gamma[I]} \quad \frac{}{\Delta[J]}}{\Gamma,\Delta[K]} \]

- Yields all correct proof nets?
Alternative conjecture (would directly yield sequentialisation)

- Given a correct handsome proofnet, there exists a partition $A_1 \ A_2$ of the axiom links (hence a partition $V_1 \ V_2$ of the vertices, since they are a complete matching) such that:
  - All the crossing edges are undirected and define a complete bipartite graph $K(U_1,U_2)$ with $U_1$ included in $V_1$ and $U_2$ included in $V_2$
  - All the crossing edges are directed and they all go from $V_1$ to $V_2$ or they all go from $V_2$ to $V_1$. 
Old references

- 1993 Réseaux et séquents ordonnés PhD Thesis Paris 7
- 1997 Pomset logic a non commutative extension of classical linear logic. TLCA
- 1997 (with Bechet and de Groote) A complete axiomatisation of the inclusion a SP orders. RTA
- 1997 A semantic characterisation of the correctness of a proof nets. MSCS / INRIA Report
- 2003 Handsome proofnets: perfect matchings and cographs. TCS / INRIA Report