QUANTIFICATION
IN ORDINARY LANGUAGE
from a critic
of set-theoretic approaches
to a proof-theoretic proposal

Vito Michele Abrusci
(Università di Roma tre)
Christian Retoré
(Université de Bordeaux, INRIA, LaBRI-CNRS)
Initially (lexical semantics in type theory)
- *I put all the books in the cellar*, (physical object)
  *indeed, i already read them all*. (information content)
- There can be several occurrences of the “same” book.

Standard quantification (history, linguistic data)

Models, generalized quantifiers

Second order and individual concepts

What is a quantifier (in proof theory)?
- Generic elements (Hilbert)
- Cut-elimination

Conclusion
USUAL QUANTIFICATION

Some, a, there is,...
All, each, any, every,...
Aristotle, & Scholastics (Avicenna, Scott, Ockham)

- A and B are terms
  (« term » is vague: middle-age distinction between terms, « suppositionnes », eg. Ockham)
  1. All A are B
  2. Some A are B
  3. No A are B
  4. Not all A are B
- Rules, syllogisms
- Remarks:
  - Little about models or truth condition
  - Always a restriction (sorts, kinds,?)
  - « not all » is not lexicalized and some A are not B has a different focus.
Frege and Analytic Philosophy

- Attempt of a deductive system
- A single universe where variables « vary »:
  - All A are B
  - $\forall x (A(x) \rightarrow B(x))$
- Deduction, proofs (Hilbert) using a generic element
- Models, truth condition (Tarski)
- Adequation proofs-models: completeness theorem (Gödel, Herbrand, ~1930)
  - Whatever is provable is true in any model.
  - What is true in every model is provable.
- Extensions:
  - Logical extensions are possible (intuitionistic, modal,...)
  - No satisfying extension to higher order
  - No proper deductive system for generalized quantifiers
HOW DOES ONE ASSERT, USE OR REFUTE USUAL QUANTIFIED SENTENCES

• « For all » introduction rule
  • (how to prove $\forall$ as a conclusion)
  • Derive $\forall x P(x)$, from $P(a)$ for an object $a$ without any particular property, i.e. a generic object $a$.
  • If the domain is known, $\forall x P(x)$ can be inferred from a proof of $P(a)$ for each object $a$ of the domain.
    The domain has to be finite to keep proofs finite. The Omega rule of Gentzen is an exception.

• « For all » elimination rule
  • (how to use $\forall$ as an assumption)
  • From $\forall x P(x)$, one can conclude $P(a)$ for any object $a$. 
HOW DOES ONE ASSERT, USE OR REFUTE USUAL QUANTIFIED SENTENCES

« Exists » introduction rule
- (how to prove $\exists$ as a conclusion):
  - if for some object $a$ $P(a)$ is proved, then we may infer $\exists x \ P(x)$

« Exists » elimination rule
- (how to use $\exists$ as an assumption):
  - If $C$ holds under the assumption $P(a)$, with $a$ only appearing in $P(a)$, and if we know that $\exists x P(x)$, we may infer $C$ without the assumption $P(a)$. 

REFUTATIONS

- \( \exists x P(x) \): little can be done apart from proving that all do not have the property.
- \( \forall x P(x) \): *Any dog may bite.*
  this can be refuted in at least two ways:
  - Displaying an object not satisfying P
    *Rex would never bite.*
  - Asserting that a subset does not satisfy P,
    thus remainig with generic elements:
    *Basset hounds do not bite.*

(ideas around Avicenna) a property is always asserted of a term as part of a class
(distinction homogenous/heterogenous predicate)
different sorts rather than a single Fregean universe
Existential are highly common: they even are used to structure a discourse as in Discourse Representation Theory.

- Generally with restriction, possibly implicit: human beings, things, events, ... 
  - There's a tramp sittin' on my doorstep 
  - Some girls give me money 
  - Something happened to me yesterday 

- Focus is difficult to account for: 
  - Some politicians are crooks. 
  - ? Some crooks are politicians.
Usual quantification in ordinary language universals

- Less common but present.
- With or without restriction:
  - Everyone, everything, anyone, anything,…
  - Every, all, each,…
- Generic (proofs), distributive (models)
  - Whoever, every,…
  - All, each,…
- Sometimes ranges over potentially infinite sets:
  - Each star in the sky is an enormous glowing ball of gas.
  - All groups of stars are held together by gravitational forces.
  - He believes whatever he is told.
  - Maths
**Usual quantification in ordinary language**

**Universal negative**

- With or without restriction:
  - No one, nothing, not any, ...
  - No,...
- Generic or distributive:
  - Because no planet's orbit is perfectly circular, the distance of each varies over the course of its year.
  - Porterfield went where no colleague had gone previously this season, realising three figures.
  - I got no expectations.
  - Nothing's gonna change my world.
Usual quantification in ordinary language
Existential negative

- Not lexicalised (in every human language?):
  - Not all, not every, ...
  - Alternative formulation (different focus):
    some ... are not ... / some ... do not ...

- Harder to grasp (psycholinguistic tests),
  frequent misunderstandings (→ nothing, no one)

- Rather generic reading:
  - Not Every Picture Tells a Story
  - Everyone is entitled to an opinion, but not every opinion is entitled to student government funding.

- Alternative formulation (different focus):
  - Some Students Do Not Participate In Group Experiments Or Projects.
INDIVIDUAL CONCEPTS

Alternative view of individuals and quantification
**Motivation for Individual Concepts**

- Usual semantics with possible worlds: It is impossible to believe that Tullius ≠ Cicero with rigid designators.
- To come back to the notion of TERM
  - Individuals are particular cases of predicates.
- Quantification is a property of predicates.
FIRST ORDER IN SECOND ORDER: PROOFS

- P is an individual concept whenever IC(P):
  - ∀x ∀ y(P(x) ∧ P(y) → x=y)
  - Exists x P(x)

- First order quantification from second order quantification:
  - ΠP IC(P) → X(P)
  - ΣP IC(P) & X(P)

- As far as proofs are concerned, this is equivalent to first order quantification – if emptiness is allowed implications only (Lacroix & Ciardelli)
Models?

- Natural (aka principal models): no completeness
- Henkin models:
  completeness and compactness but unnatural,
  e.g. one satisfies all the following formulae:
  - $F_0$: every injective map is a bijection (Dedekind finite)
  - $F_n$, $n \geq 1$: there are at least $n$ elements
GENERALIZED QUANTIFIERS

Quite common in natural language
Central topic in analytic philosophy (models)
Proofs and refutations?
**DEFINITION**

- Generalized quantifiers are operators that gives a proposition from two properties (two unary predicates):
  - A restriction
  - A predicate
- Some are definable from usual first order logic:
  - At most two,
  - Exactly three
- And some are not (from compactness):
  - The majority of...
  - Few / a few ...
  - Most of... (strong majority + vague)
- Observe that Frege’s reduction cannot apply:
  - Most students go out on Thursday evening.
  - For most people, if they are student then they go out on Thursday evening
There are many studies about the models, the properties of such quantifiers, in particular monotony w.r.t. the restriction or the predicate.

Formalisation with cardinality are wrong:
- Most of the majority of
- Most numbers are not prime. Can be found in maths textbooks.
- Test on “average” people:
  - most number are prime (no)
  - most number are not prime (yes)
- No cardinality but measure, and what would be the corresponding generic element? An object enjoying most of the properties?

Little is known about the proofs (tableaux methods without specific rules, but taking the intended model into account).
« THE MAJORITY OF » ATTEMPT (PROOF VS. REFUTATION)

- Two ways of refuting the majority of (meaning at least 50%) the A have the property P:
  - Only a minority (less than) of the A has the property P
  - There is another property Q which holds for the majority of the A with no A satisfying P and Q.
  - What would be a generic majority element?
DEFINE JOINTLY RULES FOR:
1) THE MAJORITY OF
2) A MINORITY OF

- « For all » entails the « majority of »
- If any property Q which is true of the majority of A meets P, then P holds for the majority of A (impredicative definition, needs further study)
- A minority of A is NOT P should be equivalent to The majority of A is P
- The majority of does not entail a minority of
- Forall => majority of
- Only a minority => Exists

- A linguistic remark why do we say « The majority » but « A minority »?
What should be the shape of quantifier rules?

Proof-theoretical view: to allow cut-elimination.
In proofs, for all is not a large conjunction

- Existential rule keep the finiteness of proofs: one is enough, from $P(b)$ infer $\exists x\, P(x)$.

- Universal rule requires either:
  - A known domain $D$ (what is the status of constants)
    - Finite
    - Infinite (loss of the finiteness, recursive descriptions,...) $\Rightarrow$ infinite sequents if multiplicative conjunctions
  - Infer $\forall x\, P(x)$ when $P(x)$ is true of all (each) $x$ in $D$ (Gentzen Omega Rule)
  - A generic element (already in Pythagore)
COMMUNICATION (INTERACTION) BETWEEN PROOFS: CUT RULE

- Cut-rule: two proofs \( \pi \) and \( \rho \) may communicate (interact) by means of a formula \( A \), i.e. when
  - \( \pi \) ends with a formula \( A \) and other formulas \( \Gamma \)
  - \( \rho \) ends with the negation \( \sim A \) and other formulas \( \Lambda \)
- The communication (interaction) between such a pair of proofs produces a proof which ends with the formulas \( \Gamma \) and the formulas \( \Lambda \)
- Cut-elimination procedure is the development of such a communication (interaction)
A SPECIAL CASE OF COMMUNICATION, LEADING TO QUANTIFIERS RULES.

- A proof $\pi$ of $A(b)$ under assumptions $\Gamma$
- A proof $\rho$ of $\neg A(d)$ under assumptions $\Lambda$
- These proofs may be composed (cut) when one of the following cases holds:
  - The object $b$ is the same as the object $d$ (indeed, replace $b$ by $d$ in $A(b)$, or replace $d$ by $b$ in $\neg A(d)$)
  - The object $b$ is generic in $\pi$ (i.e. it does not occur in the formulas $\Gamma$) (indeed, replace $b$ by $d$ in $A(b)$)
  - The object $d$ is generic in $\rho$ (i.e. it does not occur in the formulas $\Lambda$) (indeed, replace $d$ by $b$ in $\neg A(d)$)
GENERIC OBJECTS: HILBERT’S APPROACH

- Rules for $\tau x$:
  - *when $\tau x A(x)$ has the property $A$, every object has.*
  - From $A(b)$ with $b$ generic, infer $A(\tau x A(x)) [\forall x A(x)]$
  - From $\neg A(d)$, infer $\neg A(\tau x A(x)) [\neg \forall x A(x)]$
  - So, one reduces to general case of cut rule
  - The development of cut rule is: replace $\tau x A(x)$ by $d$

- Rules for $\varepsilon x$:
  - *when an object has the property $A$, $\varepsilon x A(x)$ has property $A$.
  - From $A(b)$ with $b$ generic, infer $A(\varepsilon x \neg A(x)) [\neg \exists x \neg A(x)]$
  - From $\neg A(d)$, infer $\neg A(\varepsilon x \neg A(x)) [\exists x \neg A(x)]$
  - So, one reduces to general case of cut rule
  - The development of cut rule is: replace $\varepsilon x \neg A(x)$ by $d$

- $A(\tau x A(x)) \leftrightarrow A(\varepsilon x \neg A(x)) [\forall x A(x)]$
- $A(\tau x \neg A(x)) \leftrightarrow A(\varepsilon x A(x)) [\exists x A(x)]$
HILBERT FUNCTIONS 
& USUAL FREGEAN RULES 
ARE EQUIVALENT

- The following equivalences hold:
  - $\forall x A(x) \leftrightarrow A(\tau x A(x))$
  - $\forall x A(x) \leftrightarrow A(\epsilon x \sim A(x))$
  - “Universal quantification”

- The following equivalence hold:
  - $\exists x A(x) \leftrightarrow A(\epsilon x A(x))$
  - $\exists x A(x) \leftrightarrow A(\tau x \sim A(x))$
  - “Existential quantification”
The two definitions are not equivalent for generalized quantifiers

- Observe that the Fregean definition of quantifiers with a single universe is not possible with generalized quantifiers. Need of quantifiers operating on two predicates:
  1. Most student go out on Thursday nights.
  2. For most people if they are students then they go out on Thursday nights.
    - 1 \rightarrow 2
- But still we can ask whether it is possible to introduce other quantifiers, in this proof-theoretical way.
NEW QUANTIFIERS? (IN PROOF-THEORY)

- Introduce a pair of quantifiers, a variant $\forall^*$ of $\forall$, and a variant $\exists^*$ of $\exists$.
- Decide one of the following two possibilities:
  - $\forall^*xA(x)$ implies $\forall x A(x)$ and so $\exists x A(x)$ implies $\exists^* x A(x)$
  - $\exists^* x A(x)$ implies $\exists x A(x)$ and so $\forall x A(x)$ implies $\forall^* x A(x)$
  - (the second one is more natural...)

- May we define in this way the quantifiers “the majority of $x$” or “most $x$ have the property $A$” ... in accordance with the “rules” suggested earlier?
CONCLUSION
Of this preliminary work
RULES FOR (GENERALIZED) QUANTIFIERS

- Which properties of quantifier rules guarantee that they behave properly in proofs and interaction?
- Is it possible to define a proof system for some generalized quantifiers?
  - Percentage?
  - Vague quantifiers?
  - ...
- What are the corresponding notions of generic elements?
**Predication, sorts and quantification**

- How do we take into account the sorts, what linguists call the restriction of the quantifier (in a typed system, a kind of ontology)?

- To avoid a paradox of the Fregean single sort:
  - Garance is tall (for a two year old girl).
  - Garance is not tall (as a person, e.g. for opening the fridge).

- One quantifier per type or a general quantifier which specializes? In type theory it would be a single constant of the system F:
  - ForAll/Exists: $\forall X (X \rightarrow t) \rightarrow t$
« If all roads lead to Rome, most segments of the transportation system lead to Roma Termini! »

Blog ˝Ron in Rome˝