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ON DEEPLY CRITICAL ORIENTED GRAPHS

Oleg V. Borodin[1], **Dimitri Fon-Der-Flaass**[2]
Institute of Mathematics, 630090 Novosibirsk, Russia

Alexander V. Kostochka[3]
Institute of Mathematics, 630090 Novosibirsk, Russia, and University of Illinois at Urbana-Champaign, Urbana, IL, 61801, USA.

André Raspaud and Éric Sopena[4]
LaBRI, Université Bordeaux I, 33405 Talence Cedex, France

Abstract. For every positive integer k, we present an oriented graph G_k such that deleting any vertex of G_k decreases its oriented chromatic number by at least k and deleting any arc decreases the oriented chromatic number of G_k by two.

Keywords. Oriented colorings, Critical graphs.

1 Introduction and results

Oriented graphs are directed graphs without opposite arcs. The *oriented chromatic number* $o(H)$ of an oriented graph H is defined to be the minimum order of an oriented graph H' such that H has a homomorphism to H'. In other words, $o(H)$ is the minimum positive integer m such that there exists a proper (in usual sense) colouring f of $V(H)$ with m colours with the additional property that

$$\forall v, w, u, z \in V(H) \ \ [vw \in E(H),\, uz \in E(H) \,\&\, f(v) = f(z)] \implies f(w) \neq f(u). \tag{1}$$

Several properties of oriented chromatic number differ from those of the (ordinary) chromatic number (see e.g. [1, 3, 5]). In this note, we observe one more distinction. Deleting a vertex or an edge from a graph decreases its chromatic number by at most one. For the oriented chromatic number, this is not true.

Observation 1 *(i) If for some oriented graph H and some $v \in V(H)$, $o(H - v) \leq k$, then $o(H) \leq 2k + 1$. On the other hand, for every positive integer k, there exist an oriented graph H_k and a vertex $v \in V(H_k)$ such that $o(H_k) = 2k + 1$ and $o(H_k - v) = k$.*
(ii) For every oriented graph H and $(v, u) \in E(H)$,

$$o(H - (v, u)) \geq o(H) - 2. \tag{2}$$

Studying colour-critical (w.r.t. usual chromatic number) graphs helped to understand some properties of chromatic number. In view of (2), we call an oriented graph H *deeply critical* if for every $(v, u) \in E(H)$,

$$o(H) - o(H - (v, u)) = 2. \tag{3}$$

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The oriented 5-cycle $\vec{C}_5$ is an example of a deeply critical graph: $o(\vec{C}_5) = 5$ and for every arc e in $\vec{C}_5$, $o(\vec{C}_5 - e) = 3$.

The main result of this note is that there are infinitely many deeply critical graphs.

Theorem 2 *For every positive integer k, there exists a deeply critical graph G_k such that $o(G_k) - o(G_k - v) \geq k$ for every $v \in V(G_k)$.*

2 Proofs

Proof of Observation 1. In order to prove (i), fix an oriented colouring f of $H - v$ with at most k colours, say, $1, \ldots, k$. Define the colouring f' of H as follows:

$$f'(x) = \begin{cases} f(x), & \text{if } x \in V(H) \setminus (N^+(v) \cup \{v\}, \\ k + f(x), & \text{if } x \in N^+(v), \\ 2k+1, & \text{if } x = v. \end{cases}$$

Observe that
(a) colour $2k+1$ is used only for v;
(b) the colours used for $N^+(v)$ are distinct from those used for $N^-(v)$;
(c) if any two vertices x and y have the same colour in f' then they have the same colour in f.
This implies that f' is an oriented colouring.

In order to show that the bound is sharp, consider the following construction described by Sopena in [5]. Let H' and H'' be disjoint oriented graphs and let the graph H be obtained by adding a new vertex v and arcs (w', v) for every $w' \in V(H')$ and (v, u'') for every $u'' \in V(H'')$. Then $o(H) = o(H') + o(H'') + 1$. This yields (i), if we take as H' and H'' two isomorphic oriented graphs with oriented chromatic number k.

To prove (ii), consider an oriented colouring f of $H - (u, v)$ with the minimum number (say, k) of colours. Then the colouring f' defined by

$$f'(x) = \begin{cases} f(x), & \text{if } x \neq u, v, \\ k+1, & \text{if } x = u, \\ k+2, & \text{if } x = v, \end{cases}$$

is an oriented colouring of H. This proves (ii). □

Proof of Theorem 2. It is enough to prove the theorem for some infinite sequence $k_1, k_2, \ldots$. We deliver the proof in a series of claims.

Let T_m be the set of ternary vectors of length m. Let also T'_m be the subset of vectors in T_m whose entries are zeros and ones and T''_m be the subset of vectors in T_m whose entries are zeros and two's. Clearly for every $t'' \in T''_m$, there exists exactly one $t' \in T'_m$ such that $t'' = t' + t'$.

The following two claims are evident.

Claim 3 *Every $t \in T_m$ is a sum of two vectors in T'_m.*

Claim 4 *For every $t'' \in T''_m$, there is only one way to represent t'' as a sum of two vectors in T'_m (namely, $t'' = t' + t'$, where $t' = t''/2$).*

Let $p = 3^m$ and $q = 2p - 1$. Let $V_m = \{0, 1, \ldots, q-1\}$ and for $v, w \in V_m$, let $d(v, w) = \min\{v - w \pmod{q}, w - v \pmod{q}\}$. Then for any distinct $v, w \in V_m$, we have

$$1 \leq d(v, w) \leq p - 1. \tag{4}$$

To every integer v, $0 \leq v \leq p-1$, we can assign the m-dimensional ternary vector $t(v)$ whose entries are the first (from the right) m digits of the ternary expansion of the number v. (For example, if $v = 11$ and $m = 4$, then $t(v) = (0, 1, 0, 2)$.) This correspondence is one-to-one.

Let D_m be the oriented graph with $V(D_m) = V_m$ such that

$$(u, v) \in E(D_m) \quad \text{if and only if} \quad v - u \,(\bmod q) < p \text{ and } t(d(v, u)) \in T'_m. \tag{5}$$

Remark 1. If $t(d(u, v)) \in T'_m$, then $d(u, v) \leq 3^{m-1} + 3^{m-2} + \ldots + 1 = (p-1)/2 = (q-1)/4$.

Claim 5 $o(D_m) = q$.

Proof. Let u and v be arbitrary distinct vertices of D_m. By (4) and Claim 3, $t(d(u, v))$ is a sum of two vectors in T'_m. Hence, by (5), there is an oriented path of length at most two (of length one if $t(d(u, v)) \in T'_m$) either from u to v or from v to u. Therefore, in any oriented colouring of D_m the vertices u and v must have different colours. This proves the claim. □

Claim 6 *For every* $e \in E(D_m)$,

$$o(D_m - e) = q - 2.$$

Proof. Because of the symmetry, we may assume that $e = (0, v)$ where $t(v) \in T'_m$. Let $f(0) = f(2v) = 1$ and $f(v) = f(q - v) = 2$. Let all other vertices get different colours from 3 to $q - 2$.

Due to Remark 1 and the absence of e, $D_m - e$ has no arcs with the tail in $\{0, 2v\}$ and the head in $\{v, q - v\}$. By Claim 4 and Remark 1, $D_m - e$ has no oriented paths of length at most two between 0 and $2v$ and between $q - v$ and v. It follows that colouring f of $D_m - e$ satisfies (1) and thus $o(D_m - e) \leq q - 2$. The lower bound on $o(D_m - e)$ follows from Observation 1. □

Claim 7 *For every* $x \in V(D_m)$,

$$o(D_m - x) \leq q - \binom{m}{\lceil \frac{m+1}{2} \rceil}.$$

Proof. Because of the symmetry, we may assume that $x = 0$. Let T^0_m be the set of vectors in T'_m with exactly $\lceil \frac{m+1}{2} \rceil$ ones. Let $V^0 = \{v \in V_m \mid v < p \text{ and } t(v) \in T^0_m\}$ and $V^{-0} = \{q - v \mid v \in V^0\}$. For every $v \in V^0$, we define $f(v) = f(q - v) = c_v$ where c_v are distinct for distinct v. All other vertices are coloured with different colours. If f is a proper oriented colouring of $D_m - 0$, then $o(D_m) \leq q - \binom{m}{\lceil \frac{m+1}{2} \rceil}$ and the claim is proved. So, let us show that f is a proper oriented colouring of $D_m - 0$.

First we prove that $V^0 \cup V^{-0}$ is an independent set in $D_m - 0$. If some vertices $v, u \in V^0 \cup V^{-0}$ are adjacent, then u, v and 0 form a triangle in D_m which by Remark 1 is transitively oriented. Therefore by (5), T'_m contains three non-zero vectors t_1, t_2 and t_3 such that
(a) $t_1 + t_2 = t_3$ and
(b) at least two of t_1, t_2 and t_3 belong to T^0_m.
But for (a) to hold, we need that the ones in t_1 and t_2 be in disjoint coordinates and that the number of ones in t_3 be the sum of the numbers of ones in t_1 and t_2. The former condition violates the possibility that both t_1 and t_2 are in T^0_m (since every $w \in T^0_m$ has more than $m/2$ ones) and the latter condition violates the possibility that both t_1 and t_3 (or t_2 and t_3) are in T^0_m (since every $w \in T^0_m$ has the same number of ones).

Next we observe that for every $v \in V^0$, there is no oriented path of length two between $q - v$ and v. This follows from Claim 4 and Remark 1. Thus, f satisfies (1), and the claim holds. □

Claims 6 and 7 yield the theorem for $k_m = \binom{m}{\lceil \frac{m+1}{2} \rceil}$. By the first paragraph of the proof, we are done. □

3 Comments

The proof of Theorem 2 shows that for infinitely many integers q, there are deeply critical graphs with oriented chromatic number q such that deleting any vertex decreases the oriented chromatic number by at least $q^{\log_3 2}/\sqrt{\log q} = q^{0.6309\ldots}/\sqrt{\log q}$. The statement (i) of Observation 1 shows that sometimes we make the oriented chromatic number twice less by deleting a vertex. It would be interesting to find out if there are infinitely many (not necessarily deeply critical) graphs in which deleting any vertex decreases the oriented chromatic number by at least one percent.

Although we present deeply critical graphs only with very specific number of vertices (namely, with the number of the form $2 \cdot 3^m - 1$), experimenting on computer indicates that deeply critical graphs on q vertices with oriented chromatic number q might exist for all odd $q > 31$.

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