ACYCLIC LIST 7-COLORING OF PLANAR GRAPHS

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Abstract. The acyclic list chromatic number of every 1-planar graph is proved to be at most 7 and is conjectured to be at most 5.

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1 Introduction

We denote by $V(G)$ the set of vertices of a graph G and by $E(G)$ its set of edges. A (proper) *k-coloring* of G is a mapping $f : V(G) \longrightarrow \{1, 2, \dots, k\}$ such that $f(x) \neq f(y)$ whenever x and y are adjacent in G.

A proper vertex coloring of a graph is *acyclic* if every cycle uses at least three colors (Grünbaum [4]). Borodin [1] proved Grünbaum's conjecture that every planar graph is acyclically 5-colorable. This bound is best possible. Moreover, there are bipartite 2-degenerate planar graphs which are not acyclically 4-colorable (Kostochka and Mel'nikov, [6]). Acyclic colorings turned out to be useful for obtaining results about other types of colorings; for a survey see [5, 8].

Now suppose each vertex v of a graph G is given a list $L(v)$ of colors. The G-list L is *choosable* if there is a proper vertex coloring of G such that the color of each vertex v belongs to $L(v)$. A graph G is said to be *list k-colorable* if every G-list L is choosable provided that $|L(v)| \geq k$ for each $v \in V(G)$.

It is trivial that each planar graph is list 6-colorable, because every of its subgraphs has a vertex of degree at most 5. Thomassen [9] proved that each planar graph is list 5-colorable, and Voigt [10] showed that this bound is the best possible.

Our main result is

Theorem 1 *Every planar graph is acyclic list 7-colorable.*

This means that if each vertex v of a planar graph G has a list $L(v)$ of at least seven admissible colors, then we can choose a color $c(v)$ from $L(v)$ so that the resulting coloring of G is acyclic. We believe that the bound above is not sharp, i.e. that the following extension of Borodin's result in [1], which strengthens Thomassen's result in [9], is true:

Conjecture 2 *Every planar graph is acyclic list 5-colorable.*

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Denote the degree of a vertex v by $d(v)$. A k-vertex is that of degree k. We write a $\leq k$-vertex for that of degree at most k, etc. By *minor* vertices we mean those of degree at most 5.

The proof of Theorem 1 is based on a structural property of the plane triangulations (loops and multiple edges are allowed), which is of interest by itself. The *weight*, $w(f)$, of a face f in a triangulation is the degree sum of its boundary vertices. Kotzig [7] conjectured that each plane triangulation with the minimum degree 5 has a face of weight at most 17, which was proved by Borodin [2]. The bound 17 is sharp, as follows from the $(5,6,6)$ Archimedian solid. The following theorem gives a sufficient condition for a plane triangulation to have a face of weight at most 17; no parameter of this condition can be weakened.

Theorem 3 *If a plane triangulation has*

1. *neither a* ≤ 3*-vertex,*
2. *nor a 4-vertex adjacent to a* ≤ 6*-vertex,*
3. *nor a 7-vertex adjacent to a 4-vertex and to two other minor vertices,*

then it has a face of weight at most 17 not incident with 4-vertices.

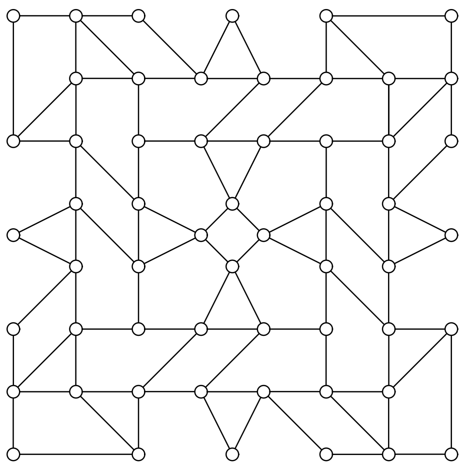
2 Proof of Theorem 3

First we show that no assumption 1-3 in Theorem 3 can be dropped or weakened.

(1) If 3-vertices are allowed, we can have a plane triangulation with only 3- and 10-vertices in which every face has weight equal to 23 by putting a vertex inside each face of the icosahedron and joining it with the boundary vertices of the face.

(2) Take a new vertex in each face of the 3-dimensional cube and join it to all the middle points of its edges, considering them as new vertices. The result is a quadrangulation with all faces of the type $(3,4,4,4)$. By putting a vertex inside each face and joining it with all vertices of the face, we obtain a triangulation with only 4-, 6-, and 8-vertices in which every face has weight 18.

(3) Take six copies of the graph shown in the figure below as the faces of the 3-cube. Now put a new vertex inside each nontriangular face to obtain a triangulation without ≤ 3-vertices in which the weight of every face is at least 18 and no 4-vertex is adjacent to a ≤ 6-vertex.



Suppose T is a counterexample to Theorem 3. Euler's formula $|V(T)| - |E(T)| + |F(T)| = 2$ may be rewritten as

$$\sum_{v\in V(T)} (d(v) - 6) = -12.$$

We set the *initial charge* of every vertex v of T to be $ch(v) = d(v) - 6$. Then we use the discharging procedure, leading to a *final charge* ch^*, defined by applying the following rules:

R1. Every ≥ 7-vertex v gives every face f containing v the following charge :

- $\frac{1}{6}$ if $d(v) = 7$ and f is incident with a 5-vertex;
- $\frac{1}{4}$ if $d(v) = 7$ and f is incident with a 4-vertex;
- $\frac{1}{4}$ if $d(v) \geq 8$ and f is incident with a minor vertex.

R2. Every face f either shares the charge obtained by R1 between the two 5-vertices incident with f, or gives all obtained by R1 to the only minor vertex incident with f.

Since the above procedure preserves the total charge, we have:

$$\sum_{v \in V(T)} ch(v) = \sum_{v \in V(M)} ch^*(v) = -12.$$

We shall get a contradiction by proving that for every vertex $v \in V(T)$ the inequality $ch^*(v) \geq 0$ holds.

If $d(v) \geq 8$ then $ch^*(v) \geq d(v) - 6 - \frac{d(v)}{4} = \frac{3(d(v)-8)}{4} \geq 0$.

Suppose $d(v) = 7$. If v is incident with a 4-vertex, then it can have at most two minor neighbors. This yields $ch^*(v) \geq 1 - 4 \times \frac{1}{4} = 0$. If a face containing v contains also two 5-vertices, then its weight is 17, and we are done. Otherwise, by parity, v can be incident with at most six faces incident with a 5-vertex, which yields $ch^*(v) \geq 1 - 6 \times \frac{1}{6} = 0$.

If $d(v) = 6$ then clearly $ch(v) = ch^*(v) = 0$.

Suppose $d(v) = 5$. If v is adjacent to two 5-vertices (which would be nonconsecutive), then $ch^*(v) \geq -1 + 4 \times \frac{1}{8} + 2 \times \frac{1}{4} = 0$.

If v is adjacent to one 5-vertex then v is adjacent to two $\geq$ 8-vertices, an $\geq$ 7-vertex and a $\geq$ 6-vertex, so that $ch^*(v) \geq -1 + 2 \times \frac{1}{8} + 2 \times \frac{1}{6} + 2 \times \frac{1}{4} > 0$.

If v is not adjacent to 5-vertices then v is adjacent to three $\geq$ 7-vertices, which yields $ch^*(v) \geq -1 + 6 \times \frac{1}{6} = 0$.

Suppose $d(v) = 4$. Then v is surrounded by $\geq$ 7-vertices, so that $ch^*(v) \geq -2 + 8 \times \frac{1}{4} = 0$.

This completes the proof of Theorem 3.

3 Deducing Theorem 1 from Theorem 3

Let P with a P-list L be a counterexample to Theorem 1 on the fewest vertices. Then by adding diagonals, we can obtain a triangulation T without loops and multiple edges, on the same vertices. It follows that T is also a minimal counterexample, with the same list L, to Theorem 1.

(1) *There is no ≤ 3-vertex x in T.*

Otherwise we choose from L an acyclic coloring of $T - x$, and choose a color for x from $L(x)$ that does not appear on its neighbour vertices. Clearly, no unicolored K_2's or bicolored cycle can arise.

(2) *No 4-vertex x in T can be adjacent to a ≤ 6-vertex y.*

Suppose otherwise; let a, b, c, y be the neigbours of x in the clockwise order, and let $a, x, c, d_1, \ldots, d_{d(y)-3}$ be the neigbours of y in the clockwise order. Since T has no multiple edges, the endvertices of the edges ax and cx are different, i.e., $a \neq c$. Delete the vertex x. If a and c are not adjacent, then we add the edge ac. We can choose from L an acyclic coloring ϕ of the graph obtained, because it has fewer vertices than T. It follows from the construction that $\phi(a) \neq \phi(c)$. In $L(x)$ there are at least three colors that do not appear on vertices adjacent to x. We must choose one of them so that no bicolored cycle arises.

If $\phi(b) \neq \phi(y)$, then we can take any $\phi(x) \in L(x) \setminus \{\phi(a), \phi(b), \phi(c), \phi(y)\}$. Otherwise, we choose a color for x from the colors in $L(x)$ not appearing on the vertices adjacent to x or y (there are at most six restrictions, while $|L(x)| \geq 7$). As a result, we obtain an acyclic coloring of G chosen from L.

(3) *If a 7-vertex y is adjacent to a 4-vertex x, then y cannot be adjacent to another minor vertex.*

Suppose the contrary; let $a, x, c, d_1, \ldots, d_4$ be the neigbours of y in the clockwise order, and let b be the fourth neighbor of x. Replace x by the edge ac if $ac \notin E(G)$, or simply remove x otherwise. Let ϕ be an acyclic coloring of the obtained graph, chosen from L. Suppose $\phi(a, b, c) = (1, 2, 3)$. The argument given in the previous case does not work only if $\phi(y) = 2$, $\phi(d_1, \ldots, d_4) = (4, \ldots, 7)$, and any attempt to color x differently from 1,2, and 3 creates a bicolored cycle going through $bxyd_i$, where $1 \le i \le 4$. In particular, it follows that $L(x) = \{1, 2, \ldots, 7\}$.

However, one of d_i's is minor; w.l.o.g., suppose it is d_4. If y can be recolored with a color greater than 7, say 8, then there are no bicolored cycles, since all the neighbours of y now have pairwise different colors. Then it is possible to color x by a color not belonging to $\{1, 2, 3, 8\}$ and obtain an acyclic coloring of T.

Otherwise, $L(y) = \{1, 2, \ldots, 7\}$; then we recolor y with 7. It is now easy to color x, and we must only recolor d_4. If $d(d_4) = 4$ then it is easy, since the neighbours of d_4 are now colored by pairwise different colors.

Suppose $d(d_4) = 5$, and let the neighbors of d_4 in the clockwise order be a, y, d_3, u, v. Recall that due to the existence of a bicolored (2,7)-path between b and d_4, one of u, v has the color 2. W.l.o.g., suppose $\phi(u) = 2$. The present color 7 of y does not appear on the other neighbours of d_4; therefore, the only obstacle for coloring d_4 with a color $\alpha \in L(d_4) \setminus \{\phi(v), 1, 2, 6, 7\}$ could be a bicolored $(6, \alpha)$-cycle $d_3 d_4 v \ldots$. However, such a cycle is clearly prevented by the bicolored $(2, 7)$-path from b to u. Hence d_4 can be recolored.

(Here and in what follows, we use the obvious fact that two bicolored paths with disjoint color sets cannot cross each other. In particular, $\alpha \notin \{2, 7\}$ in the last case above.)

By above, G satisfies the conditions of Theorem 3 and thus contains a face of weight at most 17. Moreover, we now prove that this is impossible.

(4′) *A 5-vertex x in T cannot form a face with two ≤ 6-vertices y and z.*

We only give a proof for the most difficult case $d(y) = d(z) = 6$; the same argument works if one or both of y, z have degree 5, and it is left to the reader.

Suppose we have a vertex x with a clockwise neighborhood $N(x) = (a, b, y, z, h)$, and let $N(y) = (x, b, c, d, e, z)$, $N(z) = (x, y, e, f, g, h)$. Remove x and add an edge bh if such an edge does not already exist in T, and let ϕ be an acyclic coloring of the obtained graph T' according to L. W.l.o.g., suppose $\phi(h) = 1$, $\phi(a) = 2$, $\phi(b) = 3$.

If $\phi(y) = 4$ and $\phi(z) = 5$, then it easy to color x. Also observe that we cannot have $\phi(y) = 1$ and $\phi(z) = 3$ due to the impossibility of the nontrivial $(1, 3)$-cycle $byzh$ in T'.

If $2 \notin \{\phi(y), \phi(z)\}$, then, by symmetry, we may suppose in addition that $\phi(y) = 1$ and $\phi(z) = 4$. There are at least three colors in $L(x) \setminus \{1, 2, 3, 4\}$. These colors, say 5, 6, 7 should appear on the neighbors of y, for otherwise we are done. Moreover, there should exist all the three $(\alpha, 1)$-paths joining $\{c, d, e\}$ with h, where $\alpha \in \{5, 6, 7\}$.

If $2 \in L(y)$ then we recolor y with 2, and now not all the three $(\alpha, 2)$-paths from a to $\{c, d, e\}$ can exist, where $\alpha \in \{5, 6, 7\}$ (for example, bicolored paths from c to h and from d to a cannot co-exist), and we can color x with such an α. If $2 \notin L(y)$ then we simply recolor y with an $8 \in L(y)$.

The last case to consider is $2 \in \{\phi(y), \phi(z)\}$, or, w.l.o.g., $\phi(y) = 2$, $\phi(z) \in \{3, 4\}$. If $\phi(z) = 4$, then there are at least three colors > 4 in $L(x)$, say 5, 6, and 7. The only obstacle for coloring x with one of them is the existence of (2,5)-, (2,6)-, and (2,7)-paths from a to $\{c, d, e\}$, which implies that each color 5, 6, and 7 is the color of precisely one vertex in $\{c, d, e\}$. Then we recolor y and arrive at one of the already considered cases.

Finally, suppose $\phi(z) = 3$. Then $L(x)$ has at least four colors greater that 3, say 4, 5, 6, and 7. Suppose none of the vertices c, d, e is colored 4. We see that the only obstacle for coloring x with 4 is a bicolored (3,4)-path from b to $\{f, g\}$. This forbids all $(\alpha, 2)$-paths from a to $\{c, d, e\}$. It now suffices to color x with a color > 4 that does not appear on $\{e, f, g\}$.

(4″) *A 7-vertex x in T cannot form a face with two 5-vertices y and z.*

Suppose we have a vertex x with a clockwise neighborhood $N(x) = (a, b, c, y, z, g, h)$, and let $N(y) = (x, c, d, e, z)$, $N(z) = (x, y, e, f, g)$. Delete y and z and add those of the edges ce, eg and cg that do not exist in T. Let ϕ be an acyclic coloring of the obtained graph T' according to L. W.l.o.g., suppose $\phi(e) = 1$, $\phi(c) = 2$, $\phi(g) = 3$. Also suppose that all lists consist of positive integers.

Case 1. $\phi(x) = 4$.

Subcase 1.1. $\phi(f) \in \{2, 5\}$. W.l.o.g. assume that $\phi(d) \in \{3, 4, 5, 6\}$. If $\phi(d) \neq 4$, then it suffices to color y and z (in this order) with colors greater than 5 from their lists.

Suppose $\phi(d) = 4$. If y can be colored with a color greater than 4 without creating a bicolored cycle $dyx\ldots$, then we are done by coloring z with any color greater than 5. Suppose the contrary, i.e., that there are $(4, \alpha)$-paths between d and $\{b, a, h\}$ for three distinct $\alpha > 4$, $\alpha \in L(y)$. Then we recolor x by any color not belonging to $\{2, 3, \phi(a), \phi(b), \phi(h)\}$, and color z and y, in this order, with colors greater than 5; the only additional requirement is that if $\phi'(x) \in \{1, 5\}$, then we, moreover, choose $\phi(z) \neq \phi(h)$. (Now the bicolored $(4, \phi(h))$-path from d to h forbids all bicolored cycles of the type $\ldots zx \ldots$.)

Subcase 1.2. $\phi(d) = \phi(f) = 4$. First observe that we can choose at least one color $\alpha > 4$ for y so that no bicolored cycle arises, z being still uncolored.

Indeed, otherwise there should exist a bicolored $(4, \alpha)$-path joining d with $\{b, a, h\}$ for each of at least three colors $\alpha > 4$ in $L(y)$. It follows that each color $\alpha > 4$ in $L(y)$ must occur on $\{b, a, h\}$. But then we obtain a coloring of T as follows: first, recolor x, using the fact that the colors of the five already colored neighbors of x are pairwise distinct. Now if $\phi'(x) \neq 1$, we just choose any color greater than 4 for y and z. If $\phi'(x) = 1$, we color y and z with colors greater than 4 different from $\phi(h)$. We see that bicolored $(1, \beta)$-paths from e to $\{b, a\}$, where $\beta > 4$, are now forbidden by the bicolored $(4, \phi(h))$-path from d to h.

We have thus proved that at least one color $\alpha > 4$ can be chosen for y to get an acyclic coloring of $T - z$. By symmetry, a color $\beta > 4$ can be chosen for z to obtain a coloring of $T - y$. If $\alpha \neq \beta$, this readily gives an acyclic coloring of T. We are already done, unless there exists only one such admissible α for y, only one admissible β for z, and $\alpha = \beta$. Assume this to be the case, with $\alpha = \beta = 5$.

It follows that we have (at least) two bicolored paths from d to $\{b, a, h\}$ and two from f to $\{b, a, h\}$. These four paths must have a vertex in common in $\{b, a, h\}$. Let this vertex w be colored 6.

Observe that now x can be recolored. Indeed, the only obstacle for doing so is a bicolored cycle going through x and two vertices u, v of the same color in $S = \{g, h, a, b, c\}$. Clearly, u and v cannot be consecutive on the path $P = ghabc$. We know that at least two of the five vertices of P are colored with 2 or 3 and at least two have colors greater than 4. It is convenient to consider two possibilities: $\phi(u) < 4$ and $\phi(u) > 4$. In each of them, we see that the vertex w defined above separates u and v along P. Now recall that w is joined to d and f by a $(4, > 4)$-path, which clearly forbids a bicolored cycle $\ldots uxv \ldots$ when a color for x is different from 4 and does not appear on P.

If the new $\phi'(x) \neq 1$, we are done. Suppose $\phi'(x) = 1$. Since there are (4,6)-paths from d and from f to $\{b, a, h\}$, only $(1, 6)$-paths from e to $\{b, a, h\}$ are possible. Thus, we can color y and z with any admissible colors greater than 4 and different from 6 (each of them has a choice of at least two colors).

Case 2. $\phi(x) = 1$. W.l.o.g., suppose $\phi(f) \in \{2, 4\}$, $\phi(d) \in \{3, 4, 5\}$. If for at most one $\alpha > 4$ there is a $(1, \alpha)$-path joining e to $\{b, a, h\}$, we can color with colors > 4 first y and then z to obtain an acyclic coloring of T. So, suppose there are at least two such pathes, with $\alpha_1 > 4$ and $\alpha_2 > 4$.

If x cannot be recolored, then there is a color $\gamma \in \{2, 3, \alpha_1, \alpha_2\}$ that appears on the path $P = ghabc$ at least twice. Furthermore, the vertices u and v in P colored γ must be joined by a (β, γ)-path, where $\beta \notin \{1, 2, 3, \alpha_1, \alpha_2\}$. It is easy to see (as in Subcase 1.2) that u and v should be separated along P by vertex w colored α_1 or α_2; w.l.o.g, assume $\phi(w) = \alpha_1$. Moreover, w is the only vertex in P colored with that color. It follows that w is joined to e by a $(1, \alpha_1)$-path. But this makes bicolored paths of the type $\ldots uxv \ldots$ impossible.

Since x can be recolored, we have Case 1.

Thus our counterexample T to Theorem 1 contradicts Theorem 3. This completes the proof of Theorem 1.

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