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# ON THE MAXIMUM AVERAGE DEGREE AND THE ORIENTED CHROMATIC NUMBER OF A GRAPH

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**Abstract.** The oriented chromatic number $o(H)$ of an oriented graph $H$ is defined as the minimum order of an oriented graph $H'$ such that $H$ has a homomorphism to $H'$. The oriented chromatic number $o(G)$ of an undirected graph $G$ is then defined as the maximum oriented chromatic number of its orientations. In this paper we study the links between $o(G)$ and $mad(G)$ defined as the maximum average degree of the subgraphs of $G$.

**Keywords.** Oriented graph colouring, Graph homomorphism, Maximum average degree.

## 1 Introduction and statement of results

For every graph $G$ we denote by $V(G)$, with $v_G = |V(G)|$, its set of vertices and by $E(G)$, with $e_G = |E(G)|$, its set of arcs or edges. A homomorphism from a graph $G$ to a graph $H$ is a mapping $\varphi$ from $V(G)$ to $V(H)$ which preserves the edges (or the arcs), that is $xy \in E(G) \Rightarrow \varphi(x)\varphi(y) \in E(H)$.

Homomorphisms of graphs have been studied in the literature [5, 7, 8, 9, 10, 12, 16, 17, 18, 19] as a generalization of graph coloring. It is not difficult to observe that an undirected graph $G$ has chromatic number $k$ if and only if $G$ has a homomorphism to $K_k$ but no homomorphism to $K_{k-1}$ (where $K_n$ denotes the complete graph on $n$ vertices). Therefore the chromatic number of an undirected graph $G$ can equivalently be defined as the minimum number of vertices in an undirected graph $H$ such that $G$ has a homomorphism to $H$. We will often say that a graph $G$ is $H$-*colorable* if $G$ has a homomorphism to $H$ and the vertices of $H$ will be called *colors*.

Oriented graphs are directed graphs without opposite arcs. In other words an oriented graph is an orientation of an undirected graph, obtained by assigning to every edge one of the two possible

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orientations. We can similarly define the *oriented chromatic number* $o(H)$ of an oriented graph $H$ as the minimum number of vertices in an oriented graph $H'$ such that $H$ has a (oriented) homomorphism to $H'$. The oriented chromatic number $o(G)$ of an undirected graph $G$ is then defined as the maximum oriented chromatic number of its orientations. In some sense, this notion measures how "bad" an orientation of an undirected graph can be. For instance, bipartite graphs may have arbitrarily large oriented chromatic number (see Observation 7 below).

Oriented chromatic number of (undirected) graphs with bounded degree or with bounded treewidth has been studied in [19]. In [18] it has been proved that every planar graph has oriented chromatic number at most 80. The proof depends on the acyclic 5-colorability of planar graphs proved in [2]. Despite many efforts, no better upper bound is known up to now. The study of planar graphs is thus particularly challenging in this context. This bound can be significantly decreased under some large girth assumption [16] (recall that the girth $g(G)$ of a graph $G$ is the length of a shortest cycle in $G$). The links between the oriented chromatic number and other parameters of a graph (arboricity, maximum degree, acyclic chromatic number) have been studied in [12].

In this paper, we study the relationship between the oriented chromatic number and the maximum average degree of a graph. The maximum average degree $mad(G)$ of a graph $G$ is defined as the maximum of the average degrees $ad(H) = 2e_H/v_H$ taken over all the subgraphs $H$ of $G$. Our first result is the following :

**Theorem 1**

1. *For every graph $G$ with $mad(G) < 7/3$, $o(G) \leq 5$.*
2. *For every graph $G$ with $mad(G) < 11/4$ and girth $g(G) \geq 5$, $o(G) \leq 7$.*
3. *For every graph $G$ with $mad(G) < 3$, $o(G) \leq 11$.*
4. *For every graph $G$ with $mad(G) < 10/3$, $o(G) \leq 19$.*

Concerning the maximum average degree of planar graphs, we have the following :

**Observation 2** *For every planar graph $G$ with girth at least $g$, $mad(G) < 2g/(g-2)$.*

To see that, observe that if $G$ has girth at least $g$ then the number of faces in $G$ is at most $2e(G)/g$. By Euler's formula we then get that $\frac{2e(G)}{v(G)} \leq \frac{2ge(G)}{2g+(g-2)e(G)}$ and thus $mad(G) < 2g/(g-2)$.

In case of planar graphs, Theorem 1 therefore leads to the following corollary, which improves some results given in [16] :

**Corollary 3** *Let $G$ be a planar graph, then*

1. *if $G$ has girth at least 14 then $o(G) \leq 5$,*
2. *if $G$ has girth at least 8 then $o(G) \leq 7$,*
3. *if $G$ has girth at least 6 then $o(G) \leq 11$,*
4. *if $G$ has girth at least 5 then $o(G) \leq 19$.*

It is easy to see that Observation 2, and hence Corollary 1, holds for projective plane as well.

In a companion paper [3] we continue the study of planar graphs and improve some of the above results.

Relationships between oriented colorings and acyclic colorings have been discussed in [12, 18]. Recall that a $k$-coloring of an undirected graph $G$ is said to be *acyclic* if every cycle in $G$ uses at least three colors. The acyclic chromatic number $acn(G)$ of $G$ is then defined as the minimum number $k$ of colors such that $G$ has an acyclic $k$-coloring. It has been proved in [18] that $acn(G) \leq k$ implies $o(G) \leq k \cdot 2^{k-1}$. Considering the relationship between the $acn$ and $mad$ parameters we prove :

**Theorem 4** *For every $m > 15$ and every graph $G$ with $mad(G) < 4(1 - 1/(m-2))$, $acn(G) \leq m$.*

Using the above-mentioned theorem of [18], Theorem 4 leads to the following corollary :

**Corollary 5** *For every $n > 25$, if $mad(G) < 4(1-2/(n+1))$ then $o(G) \leq (n+5)\cdot 2^{\frac{n+1}{2}}$.*

Our next result is the following :

**Theorem 6** *For every $k$, there exists a graph $G$ of arbitrarily large girth with $mad(G) < 4(1-1/k)$ and $o(G) \geq k$.*

In case of bipartite graphs, we have the following :

**Observation 7** *For every $n > 0$ there exists a bipartite graph $B_n$ with girth 6 such that $o(B_n) \geq n$ and $mad(B_n) = 4(1-2/(n+1))$.*

To see that, let $B_n$ be the oriented graph obtained from the complete graph $K_n$ by replacing every edge by a directed 2-path. Clearly the graph $B_n$ thus obtained has oriented chromatic number at least $n$ (all the "original" vertices from $K_n$ must have distinct colors) and girth $g = 6$. Moreover, no proper subgraph of $B_n$ can have the average degree greater than $B_n$ itself and thus :

$$mad(B_n) = \frac{2e_{B_n}}{v_{B_n}} = \frac{2n(n-1)}{\frac{n(n-1)}{2}+n} = 4\left(1-\frac{2}{n+1}\right).$$

Corollary 2 thus indicates that the *mad* parameter is a robust parameter. Theorem 3 extends Observation 7 to graphs of arbitrarily large girth.

In all the following we will simply say that an undirected graph $G$ has a homomorphism to an oriented graph $T$, when G, with any orientation $L$, has a homomorhism to $T$.
The rest of this paper is devoted to the proofs of our theorems. For Theorem 1, we will prove that every graph satisfying the corresponding conditions has a homomorphism to some special (oriented) circulant graph. Recall here that the circulant graph $G = G(n; c_1, \ldots, c_d)$ is defined as $V(T) = \{0, 1, \ldots, n-1\}$ and $xy \in E(T)$ if and only if $y = x + c_i \pmod{n}$ for some $i$, $1 \leq i \leq d$.

If $n$ is a prime number of the form $4k+3$ and the $c_i's$ are the non-zero quadratic residues of $n$ then $d = \lfloor \frac{n-1}{2} \rfloor$ and $G = G(n; c_1, \ldots, c_d)$ is a tournament. We will extensively use the following property of this kind of tournament :

**Proposition 8** *Let $G = G(n; c_1, \ldots, c_d)$ be a circulant graph such that the $c_i's$ are the non-zero quadratic residues of $n$, where $n$ is a prime number of the form $4k+3$. Then for any two distinct vertices u,v in $G$ and any orientation of a 2-path, there exist at least $\frac{n-3}{4}$ 2-paths from $u$ to $v$ of given orientation.*

**Proof.** By Lemma 6.24 in [14] we have the following: if $n = 3 \pmod{4}$ is a prime number then for any b (except 0), the number of solutions of the equation $x^2 + y^2 = b \pmod{n}$ is $n+1$ and the number of solutions of the equation $x^2 - y^2 = b$ is $n-1$. This implies that for any two distinct vertices $u$,$v$ in the circulant and any orientation of a 2-path, there exist at least $\frac{n-3}{4}$ 2-paths from $u$ to $v$ of given orientation □

Let $H$ be any target graph with $n$ vertices called colors. Suppose that we want to construct a homomorphism $f$ of a given graph $G$ to $H$ and let $x, y$ be any two vertices to be coloured in $G$. We will say that $y$ *allows $k$ colors* for $x$ if for a given choice of the color of $y$ we still have in any case at least $k$ possible choices for coloring $x$. Similarly, we will say that $y$ *forbids $k$ colors* for $x$ if for a given choice of the color of $y$ we still have in any case $n-k$ possible choices for coloring $x$.

All the proofs will be based on the so-called method of *reducible configurations*, contributed to Heesh [6], and used in particular by Appel and Haken [1] in their proof of the four-color theorem. But we must note that Franklin [4] had already used it in the 20s. We first provide a (small) set of *forbidden configurations*, that is a set of graphs that a minimal counter-example $G$ to our theorem cannot contain as subgraphs. We will then assume that every vertex $v$ in $G$ is valued by its degree deg($v$) and define a *discharging procedure* which specifies some transfers of values among the vertices

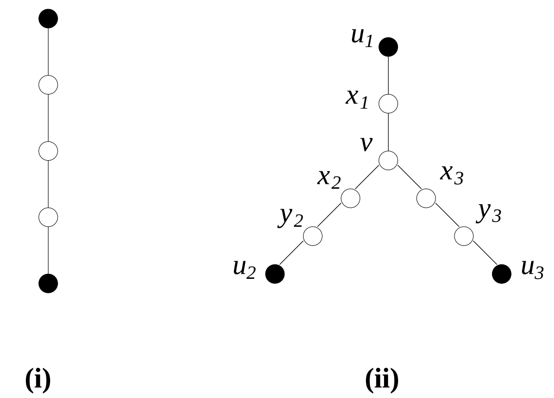


Figure 1: Forbidden configurations for Theorem 1(1).

in $G$ by keeping constant the sum of all the values. We will then get a contradiction by considering the *modernized degree* $\deg^*(v)$ of every vertex $v$, that is the value obtained by $v$ thanks to the discharging procedure.

**Drawing conventions :** In all the figures depicting forbidden configurations we will draw vertices with prescribed degrees as "white" vertices and vertices with unbounded degree as "black vertices". All the neighbours of white vertices are drawn. Unless otherwise specified, two or more black vertices may coincide in a single vertex, provided they do not share a common white neighbour.

## 2 Proof of Theorem 1(1)

We will use here the circulant graph $T_5 = G(5; 1, 2)$ as target graph. The following can easily be checked :

**Observation 9** *The graph $T_5$ is such that*

1. *For every pair of (not necessarily distinct) vertices $u, v \in V(T_5)$ there exists a directed 4-path connecting $u$ with $v$ for any of the 16 possible orientations of such a directed 4-path (this follows from Proposition 8 in [16]).*
2. *For every pair of distinct vertices $u, v \in V(T_5)$ there exists a directed 3-path connecting $u$ with $v$ for any of all 8 possible orientations of such a directed 3-path.*

We will now prove that every graph with maximum average degree less than 7/3 has a homomorphism to $T_5$. Assume $G$ with an orientation $L$ is a minimum counter-example to the Theorem. Then $G$ must avoid all the configurations depicted in Figure 1 :

**(i)** This follows directly from Observation 9.

**(ii)** Let $f$ be any $T_5$-coloring of $G \setminus \{v, x_1, x_2, y_2, x_3, y_3\}$. The vertex $u_1$ allows three colors for $v$, so that we can choose for $v$ a color distinct from the colors of $u_2, u_3$. By Observation 9 we can then extend $f$ to a $T_5$-coloring of $G$. This is a contradiction.

We now use the following discharging procedure : each vertex of degree at least 3 gives 1/3 to each of its 2-neighbours having itself a 2-neighbour and 1/6 to each of its other 2-neighbours.

Let us check that the modernized degree $\deg^*$ of each vertex is at least 7/3 which contradicts the assumption $mad(G) < 7/3$. We consider the possible cases for old degree $\deg(v)$:

1. $\deg(v) = 2$ : by (i), $v$ has at least one neighbour of degree at least 3. If $v$ has a 2-neighbour then it receives 1/3 so that $\deg^*(v) = 7/3$. If $v$ has no 2-neighbour then it receives 1/6 from each of its neighbours and thus $\deg^*(v) = 7/3$.

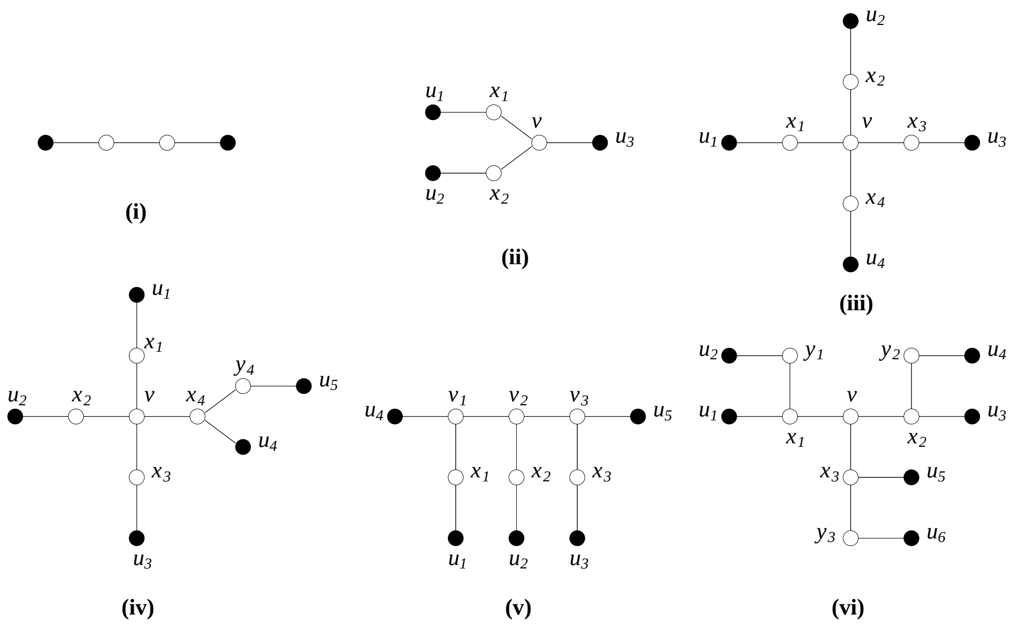


Figure 2: Forbidden configurations for Theorem 1(2).

2. $\deg(v) = 3$ : by (ii) $v$ gives at most $max\{2{\cdot}1/3, 1/3{+}2{\cdot}1/6, 3{\cdot}1/6\} = 2/3$ and thus $\deg^*(v) \geq 7/3$.
3. $\deg(v) = k \geq 4$ : $v$ gives at most $k \cdot 1/3$ and thus $\deg^*(v) \geq k - k/3 > 7/3$.

Therefore, every vertex in $G$ gets a modernized degree at least 7/3 and the theorem is proved.

By using more sophisticated techniques we prove in our companion paper [3] that every graph $G$ with $mad(G) < 12/5$ is $T_5$-colorable. By Observation 2 we thus get that every planar graph with girth at least 12 has oriented chromatic number at most 5.

# 3 Proof of Theorem 1(2)

We will use here the circulant graph $T_7 = G(7; 1, 2, 4)$ as target graph. The following observation is a direct consequence of Proposition 8 :

**Observation 10** *The graph $T_7$ is such that for each $u, v \in V(T)$ with $u \neq v$, there exist a directed 2-paths connecting $u$ with $v$ of all four possible orientations. It follows that there is a directed 3-path of any possible orientation linking any two vertices (not necessarily distinct).*

Observe that if the arcs of a directed 2-path have opposite directions, then in $T_7$ there is a directed 2-path from any vertex $v$ to itself. All that essentially means that if we have a path $vxu$ in $G$ with $x$ of degree 2 then $v$ "forbids" for $u$ at most one color (actually its own color, if the two arcs have the same direction). More precisely, we have the following :

**Observation 11** *Let $G$ be an oriented graph and $u, v$ be two neighbour vertices in $G$. Let $f$ be some $T_7$-coloring of $G \setminus \{v\}$. If for a fixed coloring of the neighbours of $v$ distinct from $u$ we can use $i$ distinct colors for $v$, then $v$ forbids for $u$ at most two colors for $i = 2$, one color for $i = 3, 4$ and no colors for $i > 4$.*

We will now prove that every graph with maximum average degree less than 11/4 and girth at least 5 has a homomorphism to $T_7$. Assume $G$ with an orientation $L$ is a minimum counter-example to the Theorem. Then $G$ must avoid all the configurations depicted in Figure 2 :

**(i)** This follows directly from Observation 10.

**(ii)** Let $f$ be any $T_7$-coloring of $G \setminus \{v, x_1, x_2\}$. Then $u_3$ allows exactly three colors for $v$ while $u_1, u_2$ forbid at most two of them. Thus $f$ can be extended to a $T_7$-coloring of $G$, a contradiction.

**(iii)** Let $f$ be any $T_7$-coloring of $G \setminus \{v, x_1, x_2, x_3, x_4\}$. All the $u_i$'s forbid at most four colors for $v$ and thus $f$ can be extended to a $T_7$-coloring of $G$, a contradiction.

**(iv)** Observe first that since $G$ has girth at least 5, $u_1, u_2, u_3, u_4$ are distinct but one of $u_1, u_2, u_3$ may coincide with $u_5$. Let $f$ be any $T_7$-coloring of $G \setminus \{v, x_1, x_2, x_3, x_4, y_4\}$. Then $u_4$ and $u_5$ allow at least two colors for $x_4$ and thus at least five colors for $v$ while $u_1, u_2, u_3$ forbid at most three of them. Thus $f$ can be extended to a $T_7$-coloring of $G$, a contradiction.

**(v)** Observe first that since $G$ has girth at least 5, $u_3$ and $u_4$ (resp. $u_1$ and $u_5$) may be identified. Let $f$ be any $T_7$-coloring of $G \setminus \{v_1, v_2, v_3, x_1, x_2, x_3\}$. The vertices $u_1$ and $u_4$ (resp. $u_3$ and $u_5$) allow at least two colors for $v_1$ (resp. for $v_3$). Then $v_1$ and $v_3$ allow at least three colors for $v_2$ while $u_2$ forbids at most one of them. Thus $f$ can be extended to a $T_7$-coloring of $G$, a contradiction.

**(vi)** Observe first that since $G$ has girth at least 5 no "black" vertex can coincide with a "white" one. Let $f$ be any $T_7$-coloring of $G \setminus \{v, x_1, x_2, x_3, y_1, y_2, y_3\}$. All the $u_i$'s allow at least two colors for each $x_i$ and thus at least one color for $v$. Thus $f$ can be extended to a $T_7$-coloring of $G$, a contradiction.

We now use the following discharging procedure : each vertex of degree at least 3 gives $3/8$ to each of its 2-neighbours and $1/8$ to each of its 3-neighbours having a 2-neighbour.

Let us check that the modernized degree $\deg^*$ of each vertex is at least $11/4$ which contradicts the assumption $mad(G) < 11/4$. We consider the possible cases for old degree $\deg(v)$:

1. $\deg(v) = 2$: by (i), $v$ receives exactly $3/4$, and thus $\deg^*(v) = 11/4$.
2. $\deg(v) = 3$: if $v$ has a 2-neighbour then by (ii) and (v), $v$ gives at most $3/8 + 1/8 = 1/2$ but then it also receives $1/8 + 1/8$ so that $\deg^*(v) \geq 11/4$; if $v$ has no 2-neighbour then by (vi), it gives at most $2/8$ to its 3-neighbours having a 2-neighbour and thus $\deg^*(v) \geq 11/4$.
3. $\deg(v) = 4$: by (iii) and (iv), $v$ gives at most $9/8$ and thus $\deg^*(v) \geq 23/8 > 11/4$.
4. $\deg(v) = k > 4$: $v$ gives at most $3k/8$ and thus $\deg^*(v) \geq 5k/8 \geq 25/8 > 11/4$.

Therefore, every vertex in $G$ gets a modernized degree at least $11/4$ and the theorem is proved.

## 4 Proof of Theorem 1(3)

Here we use as target graph the circulant $T_{11} = G(11; 1, 3, 4, 5, 9)$. From Proposition 8 we have :

**Observation 12** *The graph $T_{11}$ is such that for every two distinct vertices $u, v$ in $T_{11}$, there are are least two distinct 2-paths linking $u$ and $v$ for every possible orientation of this 2-path and there is a 3-path of any possible orientation linking any two (not necessarily distinct) vertices in $T_{11}$.*

We will now prove that every graph with maximum average degree less than 3 has a homomorphism to $T_{11}$. Assume $G$ with an orientation $L$ is a minimum counter-example to the Theorem. Then $G$ must avoid all the configurations depicted in Figure 3 :

**(i)** This directly follows from Observation 12.

**(ii)** Let $f$ be some $T_{11}$-coloring of $G \setminus \{v\}$. By Observation 12, we can choose $f$ such that $f(x) \neq f(u_3)$ and thus $f$ can be extended to a $T_{11}$-coloring of $G$.

**(iii)** Let $f$ be some $T_{11}$-coloring of $G \setminus \{v, x_1, x_2, x_3\}$. Then $u_4$ allows five colors for $v$ and $u_1, u_2, u_3$ forbid at most three colors. Thus $f$ can be extended to a $T_{11}$-coloring of $G$.

**(iv)** As before, let $f$ be some $T_{11}$-coloring of $G \setminus \{v, x_1, x_2, x_3, x_4\}$. Then $u_5$ allows five colors for $v$ and $u_1, u_2, u_3, u_4$ forbid at most four colors. Thus $f$ can be extended to a $T_{11}$-coloring of $G$.

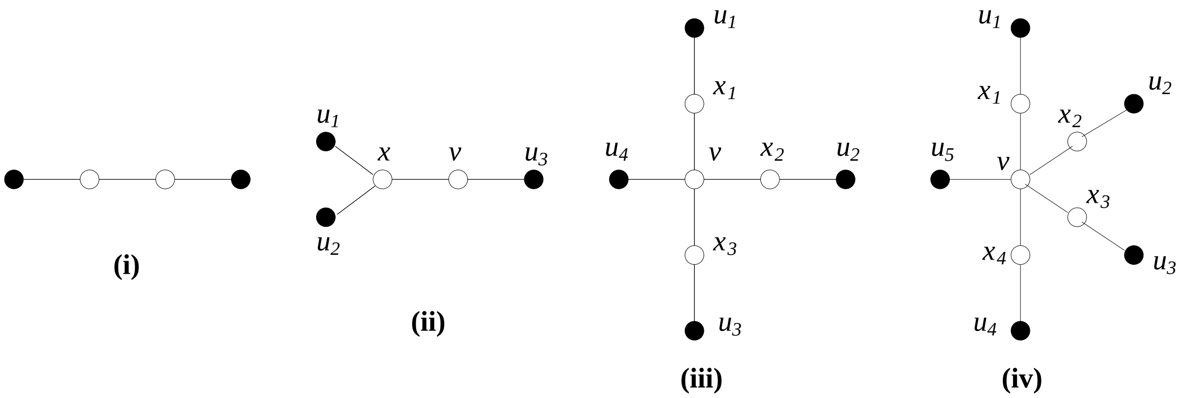


Figure 3: Forbidden configurations for Theorem 1(3).

We now use the following discharging procedure : each vertex of degree at least 4 gives 1/2 to each of its 2-neighbours.

Let us check that the modernized degree $\deg^*$ of each vertex is at least 3 which contradicts the assumption $mad(G) < 3$. We consider all possible cases for old degree $\deg(v)$:

1. $\deg(v) = 2$: by (i) and (ii), $v$ receives exactly 1 and thus $\deg^*(v) = 3$.
2. $\deg(v) = 3$: no changes.
3. $\deg(v) = 4$: by (iii), $v$ gives at most 1 and thus $\deg^*(v) \geq 3$.
4. $\deg(v) = 5$: by (iv), $v$ gives at most 3/2 and thus $\deg^*(v) \geq 7/2 > 3$.
5. $\deg(v) = k \geq 6$: $v$ gives at most $k/2$ and thus $\deg^*(v) \geq k/2 \geq 3$.

Therefore, every vertex in $G$ gets a modernized degree at least 3 and the theorem is proved.

# 5 Proof of Theorem 1(4)

Here we use as target graph the circulant $T_{19} = G(19; 1, 4, 5, 6, 7, 9, 11, 16, 17)$ on 19 vertices. From Proposition 8 we have :

**Observation 13** *For each distinct $u, v \in V(T_{19})$, and for each of all four possible orientations of a 2-path, there exist at least four distinct 2-paths connecting $u$ with $v$ with given orientation.*

By a tedious case study we also get the following :

**Observation 14** *For each distinct $u, v, w \in V(T_{19})$, and for each of all eight possible orientations of edges connecting $u, v$ and $w$ with a vertex, there exist a vertex $x \in V(T_{19})$ such that the edges connecting $x$ with $u, v$ and $w$ have given orientation.*

We will now prove that every graph with maximum average degree less than 10/3 has a homomorphism to $T_{19}$. Assume $G$ with an orientation $L$ is a minimum counter-example to the Theorem. Then $G$ must avoid all the configurations depicted in Figure 4 :

**(i)** Let $f$ be some $T_{19}$-coloring of $G \setminus \{vx\}$ (that is we delete the edge $vx$). By Observation 13, we have at least four choices for coloring $x$, so that we can choose for $x$ a color distinct from the colors of $u_1$ and $u_2$. By Observation 14, we can then find a color for $v$, which is a contradiction.

**(ii)** Let $f$ be some $T_{19}$-coloring of $G \setminus \{x_1, \ldots, x_{m-2}\}$. By Observation 13, we have at least four choices for coloring $v$ while $u_1, \ldots, u_{m-2}$ forbid at most three of them. Thus we can extend $f$ to a $T_{19}$-coloring of $G$, a contradiction.

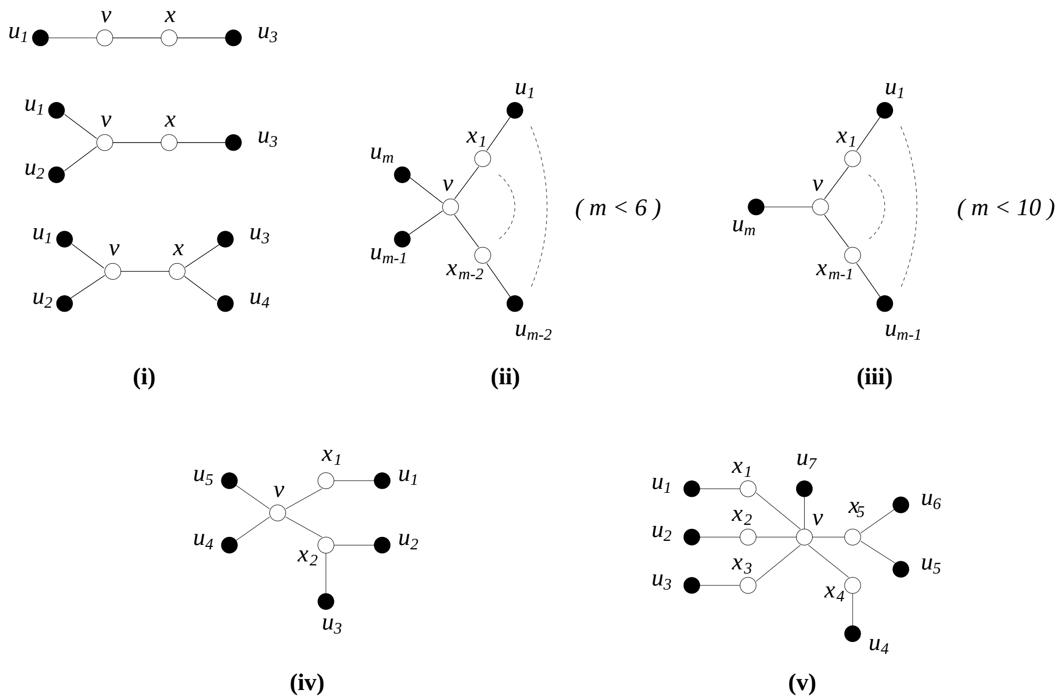


Figure 4: Forbidden configurations for Theorem 1(4).

**(iii)** Let $f$ be some $T_{19}$-coloring of $G \setminus \{x_1, \dots, x_{m-1}\}$. By Observation 13, we have at least nine choices for coloring $v$ while $u_1, \dots, u_{m-1}$ forbid at most eight of them. Thus we can extend $f$ to a $T_{19}$-coloring of $G$, a contradiction.

**(iv)** Let $f$ be some $T_{19}$-coloring of $G \setminus \{x_1\}$. By Observation 13, $u_4$ and $u_5$ allow at least four colors for $v$ while $u_2, u_3$ forbid at most two of them. Hence we can choose for $v$ a color distinct from the color of $u_1$ and extend $f$ to a $T_{19}$-coloring of $G$, a contradiction.

**(v)** Let $f$ be some $T_{19}$-coloring of $G \setminus \{v, x_1, x_2, x_3, x_4\}$. Then $u_7$ allows nine colors for $v$ while $u_1, \dots, u_4$ forbid at most four of them and $u_5, u_6$ at most two of them. Hence we can color $v$ with a color distinct from the colors of $u_1, \dots, u_7$. By Observation 14, we can color $x_5$. We have then extended $f$ to a $T_{19}$-coloring of $G$, a contradiction.

We now use the following discharging procedure : each vertex of degree at least 4 gives 2/3 to each its 2-neighbour and 1/9 to each its 3-neighbour.

Let us check that the modernized degree $\deg^*$ of each vertex is at least 10/3 which contradicts the assumption $mad(G) < 10/3$. We consider the possible cases for old degree $\deg(v)$:

1. $\deg(v) = 2$: by (i), $v$ receives exactly 4/3 and thus $\deg^*(v) \geq 10/3$.
2. $\deg(v) = 3$: by (i), $v$ receives exactly $3 \cdot 1/9 = 1/3$ and thus $\deg^*(v) = 10/3$.
3. $\deg(v) = 4$: if $v$ has no 2-neighbor then it gives at most $4 \cdot 1/9$ so that $\deg^*(v) \geq 32/9 > 10/3$. Otherwise, by (ii) and (iv) it gives exactly 2/3 and thus $\deg^*(v) = 10/3$.
4. $\deg(v) = 5$: by (ii), $v$ gives at most $2{\cdot}2/3{+}3{\cdot}1/9$ to its 2- and 3-neighbours so that $\deg^*(v) \geq 10/3$.
5. $\deg(v) = 6$: by (iii) and (v), $\deg^*(v) \geq 6 - \max\{3 \cdot 2/3 + 3 \cdot 1/9, 4 \cdot 2/3\} = 10/3$.
6. $7 \leq \deg(v) = n \leq 9$: by (iii), $\deg^*(v) \geq n - (n-2) \cdot 2/3 - 2 \cdot 1/9 = n/3 + 10/9 > 10/3$.
7. $\deg(v) = k \geq 10$: in that case $\deg^*(v) \geq n/3 \geq 10/3$.

Therefore, every vertex in $G$ gets a modernized degree at least 10/3 and the theorem is proved.

## 6 Proof of Theorem 2

We will prove in this section that every graph $G$ with $mad(G) \leq 4(1-1/(m-2))$, $m > 15$, has acyclic chromatic number at most $m$. Suppose $G$ is a minimum counter-example to the Theorem. Then $G$ satisfies the following properties :

**(i)** *No 2-vertex in $G$ is adjacent to some $j$-vertex for $j < m$.* Assume that $\deg(v) = 2$ and let $x_1, x_2$ be its neighbours, where $x_1$ has degree at most $m-1$. Let $f$ be some acyclic $m$-coloring of $G \setminus \{v\}$. If $f(x_1) \neq f(x_2)$ we can color $v$ with any color distinct from $f(x_1)$ and $f(x_2)$. Otherwise, we can color $v$ with a color distinct from $f(N(x_1))$ and $f(x_2)$. In both cases, $f$ can be extended to an acyclic $m$-coloring of $G$, a contradiction.

**(ii)** *For every 3-vertex $v$ in $G$ with $N(v) = \{x_1, x_2, x_3\}$, we have $\deg(x_i) + \deg(x_j) \geq m+1$ for every $1 \leq i < j \leq 3$.* Assume this is not the case and let $f$ be any acyclic $m$-coloring of $G \setminus \{v\}$. We have three cases to consider :

1. $f(x_1) \neq f(x_2) \neq f(x_3) \neq f(x_1)$. We can obviously color $v$ with any color distinct from the colors of its neighbours.
2. $1 = f(x_1) = f(x_2) \neq f(x_3) = 2$. If some color $c \in \{3, \ldots, m\}$ is not used in $N(x_1)$ or $N(x_2)$ then we can color $v$ with $c$. Otherwise, $\deg(x_1) \geq m-1$ and $\deg(x_2) \geq m-1$. But in that case, we get by (i) that $\deg(x_3) \geq 3$ and the result follows.
3. $f(x_1) = f(x_2) = f(x_3) = 1$. Suppose $\deg(x_1) + \deg(x_2) \leq m$. We can then color $v$ with a color $c \neq 1$ which is neither used in $N(x_1)$ nor in $N(x_2)$.

Hence, in all cases $f$ can be extended to an acyclic $m$-coloring of $G$, a contradiction.

**(iii)** *Let $v$ be a vertex in $G$ with $N(v) = \{u, y_1, y_2, y_3, x_1, \ldots, x_t\}$ such that $x_1, \ldots, x_t$ have degree two and $y_1, y_2, y_3$ have degree at most three. Then $t \geq (m-10)(m-5)+1$.* Assume on the contrary that $t \leq (m-10)(m-5)$. For every $i$, $1 \leq i \leq t$, denote by $z_i$ the neighbour of $x_i$ distinct from $v$ and let $f$ be some acyclic $m$-coloring of $G \setminus \{v, x_1, \ldots, x_t\}$. Denote $A = f(\{u, y_1, y_2, y_3\} \cup N(y_1) \cup N(y_2) \cup N(y_3))$. Clearly, $|A| \leq 10$. There exists a color $c \in \{1, \ldots, m\} \setminus A$ such that at most $\frac{t}{m-10}$ $z_j$'s (counted with multiplicity) are colored with $c$. We color $v$ with $c$ and then color every $x_j$ such that $f(z_j) \neq c$ with any color distinct from $c$ and $f(z_j)$. Finally, we color $x_j$'s such that $f(z_j) = c$ with distinct colors in $\{1, \ldots, m\} \setminus (\{c\} \cup f(\{u, y_1, y_2, y_3\}))$. Since $t/(m-10) \leq m-5$, we can do that and thus extend $f$ to an acyclic $m$-coloring of $G$, a contradiction.

We now use the following two-step discharging procedure :
*Step 1 :* each vertex of degree at least 8 gives $1/2$ to each of its 3-neighbours,
*Step 2 :* each vertex of degree at least $m$ (before discharging) leaves 4 for itself and uniformly distributes the rest between its 2-neighbours.

Let us check that the modernized degree $\deg^*$ of each vertex is at least $4(1-1/(m-2))$ which contradicts the assumption of Theorem 3. We consider the possible cases for old degree $\deg(v)$ :

1. $\deg(v) \geq 8$ : after first step of discharging, $v$ has at least $0.5\deg(v)$, so it can afford to save 4 for itself and thus $\deg^*(v) \geq 4$.

2. $4 \leq \deg(v) \leq 7$ : by (i), $\deg^*(v) = \deg(v)$.

3. $\deg(v) = 3$ : by (ii), $v$ cannot have two neighbours with degree at most 7. Thus $v$ receives at least 1 and $\deg^*(v) \geq 4$.

4. $\deg(v) = 2$ : let us check that $v$ receives at least $\frac{m-4}{m-2}$ from each of its neighbors. Indeed, let $(v, u)$ be an edge in $E(G)$. By (i), the degree $d$ of $u$ is at least $m$. If $u$ has $a$ neighbors of degree

at least 4 and $b$ neighbors of degree 3 then $v$ receives from $u$ :

$$h(d,a,b) = \frac{d-4-b/2}{d-a-b} = 1 - \frac{4-b/2-a}{d-a-b}.$$

Hence if $a+b/2 \geq 4$, we are done. Suppose on the contrary that $a+b/2 \leq 3.5$. Under these conditions, $h(d,a,b)$ increases if any of $d,a,b$ increases and others are fixed or if $a+b$ is fixed and $a$ increases. We have three cases to consider :

(a) $a \geq 2$. By above, $h(d,a,b) \geq h(m,2,0) = \frac{m-4}{m-2}$.

(b) $a \leq 1, a+b \geq 5$. By our previous observation concerning the behaviour of the function $h$ and the fact that if $b=4$ then $a=1$, we have : $h(d,a,b) \geq h(m,0,5) = \frac{m-6.5}{m-5} > \frac{m-4}{m-2}$.

(c) $a \leq 1, a+b \leq 4$. Then by (iii), $\deg(u) \geq (m-10)(m-5)+5$ and

$$h(d,a,b) \geq h(m^2-15m+55,0,0) = \frac{m^2-15m+51}{m^2-15m+55} > \frac{m-4}{m-2}.$$

Thus we get that $\deg^*(v) \geq 2 + 2 \cdot \frac{m-4}{m-2} = 4(1-1/(m-2))$.

Therefore, every vertex in $G$ gets a modernized degree at least $4(1-1/(m-2))$ and the theorem is proved.

## 7 Proof of Theorem 3

In [13] Kriz constructed for every $k,g$ a graph $H_{k,g}$ having girth $g$ and chromatic number $k$. Moreover, this graph $H_{k,g}$ is the union of $(k-1)$ forests, whose every component is a star.

By Nash-Williams Theorem [15] it means that for every subgraph $H'$ of $H_{k,g}$ we have $e_{H'} \leq (k-1)(v_{H'}-1)$. We now define the oriented graph $H^*_{k,g}$ obtained from $H_{k,g}$ by replacing every edge by a directed 2-path. Clearly the graph $H^*_{k,g}$ has girth $2g$ and oriented chromatic number at least $k$.

We claim that $m = mad(H^*_{k,g})$ is at most $4(1-1/k)$. To see that let $G$ be any subgraph of $H^*_{k,g}$ whose average degree is exactly $m$. It is not difficult to see that $G = H'^*$ for some subgraph $H'$ of $H_{g,k}$ and thus $e_G = 2e_{H'^*}$ and $v_G = v_{H'^*} + e_{H'^*}$. Hence

$$m = \frac{2e_G}{v_G} = \frac{4e_{H'^*}}{v_{H'^*}+e_{H'^*}} = 4 \cdot \left(1 - \frac{v_{H'^*}}{v_{H'^*}+e_{H'^*}}\right)$$

Since $e_{H'^*} \leq (k-1)(v_{H'^*}-1)$ we get $v_{H'^*}+e_{H'^*} \leq kv_{H'^*} - (k-1)$ and thus $m \leq 4(1-1/k)$. The theorem is proved.

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