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ACYCLIC k-STRONG HYPERGRAPH COLORING OF MAPS ON SURFACES

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Abstract. A vertex coloring of a graph is acyclic if the ends of each edge have different colors and each bicolored subgraph is acyclic. Let each face of size at most k, where $k \geq$, in a map on a surface S^N be replaced by a clique on the same vertices. Then the resulting map can be acyclically colored with a number of colors that depends linearly both on N and on k. Such results were previously known only for $1 \leq N \leq 2$ and $3 \leq k \leq 4$.

Keywords. Acyclic coloring, Acyclic strong coloring.

1 Introduction

The problems of coloring graphs embedded on surfaces is an important part of graph theory. In particular, the well-known Four Color Problem belongs to this area.

We denote by $V(G)$ the set of vertices of a graph G and by $E(G)$ its set of edges. A (proper) *k-coloring* of G is a mapping $f : V(G) \longrightarrow \{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ whenever x and y are adjacent in G.

A vertex coloring of a graph is called *k-cyclic* or *k-strong hypergraph coloring* if every two distinct vertices lying in the boundary of the same face of size at most k have different colors. (We use below both terms.) This type of coloring may be looked at as an admissible coloring of a pseudograph obtained by replacing each face of size at most k by a clique on the same vertices. Let $\chi_k(G)$ be the minimum number of colors sufficient to color G k-cyclically, and let $\chi_k(S^N)$ be the minimum number of colors sufficient to color k-cyclically each map on surface S^N with Euler characteristic N. (Sometimes we drop the arguments.)

The case $k = 3$ corresponds to the ordinary vertex colouring, and the Four Color Theorem by Appel and Haken [1] and Heawood's theorem [2] give precise upper bounds for the plane and for the higher surfaces, respectively.

The case $k = 4$ may also be formulated in terms of the simultaneous vertex-face coloring and the vertex coloring of 1-embeddable graphs. For the plane, Borodin [3] proved, confirming Ringel's conjecture in [4], precise upper bound $\chi_4 \leq 6$; for the projective plane, Schumacher [5] proved precise upper bound $\chi_4 \leq 7$; and for the other surfaces, Ringel [6] proved that $\chi_4(\mathcal{S}^N) \leq \frac{2H(N)}{\sqrt{3}}$, where $H(N) = \lfloor \frac{7+\sqrt{49-24N}}{2} \rfloor$ is the Heawood number.

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For arbitrary $k \geq 3$, Ore and Plummer [7] proved that each plane graph G is k-cyclically $2k$-colorable, and Borodin, Sanders and Zhao [8] improved this bound to $\chi_k(G) \leq \frac{9k}{5}$.

A vertex coloring of a graph is *acyclic* if it is admissible, i.e., the ends of each edge are colored differently, and there is no bicolored cycle. Note that a loop gives a unicolored edge and two multiple edges give a bicolored cycle. We define a vertex coloring of a map to be acyclic provided that no two ends of an edge e are coloured the same unless e is a loop, and no bicolored cycle of length > 2 exists. Borodin [9] proved that each plane graph is acyclically 5-colorable, which bound is best possible. Albertson and Berman [10] proved that each graph embedded on the oriented surface S^N, where $N < 0$, is acyclically $(8-2N)$-colorable. Alon, Mohar and Sanders [11] proved, using the acyclic 5-colorability of plane graphs, that each graph on the projective plane is acyclically 7-colorable, and showed this bound to be best possible. Also, they prove that each graph embedded on an arbitrary surface S^N is acyclically $\mathcal{O}(N^{4/7})$-colorable, and this bound is tight up to the factor of $(\log N)^{1/7}$.

The acyclic coloring has a number of applications to other coloring problems, for the definitions see [12, 13, 14, 15, 16]. Suppose one has proved that $a(G) \leq a$. Then G has the star chromatic number at most $a2^{a-1}$ (Grünbaum [13]) and the oriented chromatic number at most $a2^{a-1}$ (Raspaud and Sopena [16]); every m-coloring of the edges of G can be homomorphically mapped on that of a graph with at most am^{a-1} vertices (Alon and Marshall [12]); every m-coloring of the edges of mixed graph G combined with its n-coloring can be homomorphically mapped on a graph with at most $a(2n+m)^{a-1}$ vertices (Nešetřil and Raspaud [15]).

Also, Hakimi, Mitchem and Schmeichel [14, p.38-39] proved that $E(G)$ can be partitioned into $a(G)$ star forests (whose every component is a star). Directly using [9], this confirms the conjecture of Algor and Alon [17] that the edges of every planar graph can be partitioned into five star forests.

In this paper we consider a coloring that is both acyclic and k-cyclic. Namely, we color acyclically the result of replacing all faces of size at most k in a map by cliques of the same sizes. So, each face of size at most k is assumed to have all 'invisible' diagonals. For $k = 3$ this type of coloring coincides with the acyclic coloring.

Our main result is

Theorem 1 *Every map on a surface S^N is acyclically k-strong hypergraph colorable with $c_N k + d_N$ colors whenever $k \geq 4$ and $N \leq 0$.*

In fact, we prove the result with $c_N = \max\{999, 117 - 471N\}$ and $d_N = 39 - 156N$. The argument is intentionally kept simple; more complicated argument may be used to decrease c_N and d_N.

Corollary 2 *Every map on the plane ($N = 2$) or on the projective plane ($N = 1$) is acyclically k-strong hypergraph colorable with $c_0 k + d_0$ colors whenever $k \geq 4$.*

Proof. Follows from the facts that each plane or projective plane map is also a map on the torus or Klein bottle, respectively. □

In [18], we prove that each projective plane graph (and hence each plane graph) is acyclically 4-strong hypergraph 20-colorable, i.e., each graph 1-embedded on the projective plane is acyclically 20-colorable. Thus, Theorem 1 and Corollary 2 augment the results of [9] ($N = 2$, $k = 3$), of [11] ($N \leq 1$, $k = 3$), and of [18] ($1 \leq N \leq 2$, $k = 4$).

2 Proof of Theorem 1

The *size*, $s(f)$, of a face f is the number of edges in its boundary, $\partial(f)$, counting multiplicities. For instance, a cut edge appears twice on the boundary of a certain face. To simplify the argument, we restrict ourselves to the case when $\partial(f)$ is connected whenever $f \in F$. The *degree* of a vertex v, i.e., the number of incident edges (loops are counted twice), is denoted by $d(v)$. By a $\geq k$-vertex we mean that of degree at least k, etc.

Given S^N and k, let P''' be a counterexample with the minimum number of vertices. For brevity, the acyclically k-strong hypergraph coloring with $c_N k + d_N$ colors we are searching for will be called *good.*

Delete from P''' the common edge of two ajacent 3-faces, if any. Repeat this operation until the resulting counterexample, P'', has no adjacent 3-faces.

Triangulate all $> k$-faces of P'' by adding diagonals to obtain another counterexample, P', with the minimum number of vertices. (A good coloring of P' would be good for P''.)

Remove from P' each loop e that forms a 1-face. This decreases the size of the other face incident with e by 1. Similarly, for a 2-face $f = e_1 e_2$ in P', remove one of the boundary edges, say e_1. This results in a face of the same size as the former face that was incident with e_1 and different from f.

If the so obtained graph P were well colorable, then recovering loops and repeated edges would not spoil the good coloring. Thus, P is also a counterexample with the fewest vertices. We have actually proved the following

Lemma 3 *If f is a face of P, then $3 \leq s(f) \leq k$; no two 3-faces of P are adjacent.*

Informally speaking, only adding new nontrivial adjacencies, represented by 'visible' edges or 'invisible diagonals', can spoil the good coloring. Our concern therefore is not to loose adjacencies while transforming P into a smaller pseudograph, which already admits a good coloring.

The following two observations should be kept in mind during the rest of the proof.

Observation 4 *Contracting an edge $e = vz$ into a vertex $v * z$ decreases the size of each face f in P by 0,1 or 2, depending on how many times e appears on $\partial(f)$.*

The obtained map R is well colorable by the minimality of P. Transfer its coloring to P with z colored as $v * z$ and v uncolored. Then in order that to get a good coloring for P, it remains to find a color for v without creating unicolored edges other than loops and bicolored cycles other than 2-cycles.

Observation 5 *Suppose a vertex v is incident with edges $e_i = vz_i$, where $0 \leq i \leq d(v) \leq k-1$, in a cyclic order. (Of course, z_i's are not all necessarily distinct.) Split each nontriangular face $f_i = \ldots z_i v z_{i+1}$, where subscripts are taken modulo $d(v)$, into a triangle $z_i v z_{i+1}$ and a face f_i'. Then $s(f_i') < s(f_i)$. Remove v and all e_i's, then a new face arises of size $d(v) \leq k$, so that the obtained pseudograph can be well colored. Hence, all distinct z_i's are colored differently. If we choose a color α for v not appearing on any vertex in $\cup_{0 \leq i \leq k-1} \partial(f_i)$, then the only possibility for the obtained coloring to be bad is a bicolored α, β-cycle (i.e., consisting of vertices alternatively colored with α and β) through v and some vertex u such that u is colored with β and $u \in \partial(f_i) \setminus \{z_i, z_{i+1}\}$ for some f_i.*

Lemma 6 *If $v \in V(P)$ then $d(v) \geq 2$.*

Proof. If $d(v) = 1$, we contract the edge vz, transfer the good coloring of the obtained pseudograph to P, and color v differently from those at most k vertices that belong to the same faces as v. □

Lemma 7 *Each face in P is incident with at least three vertices (not necessarily distinct) of degree at least three.*

Proof. If a face f in P has only two vertices u, w (not necessarily distinct) of degree greater than 2 in $\partial(f)$, then we remove the longer of the two paths that form $\partial(f)$. This creates a new face f' of size $\leq k$. From a good coloring of the resulting graph, it is easy to obtain a good coloring of P, since all vertices on $\partial(f')$ are colored pairwise differently.

The case that there is only one $>$ 2-vertex in $\partial(f)$ easily reduces to the previous one. If all vertices in $\partial(f)$ have degree 2, then P is a cycle; a contradiction. □

If every face of P is an open 2-cell, then Euler's formula says that $|V|-|E|+|F|=N$; otherwise

$$|V|-|E|+|F|\geq N.$$

Using obvious equalities $2|E|=\sum_{v\in V}d(v)=\sum_{f\in F}s(f)$, this may be rewritten as

$$\sum_{v\in V}(d(v)-4)+\sum_{f\in F}(s(f)-4)\leq -4N,$$

or

$$-2n_2+\sum_{v\in V_{3+}}(d(v)-4)+\sum_{f\in F}(s(f)-4)<-4N+1, \tag{1}$$

where n_i is the number of i-vertices in P, and V_{i+} is the set of $\geq i$-vertices in P.

Denote by $n_2(f)$ the number of 2-vertices on the boundary of face f, counting multiplicities. The *reduced size*, $s^*(f)$, of a face f is defined to be $s(f)-n_2(f)$. In fact, $s^*(f)$ is the number of >2-vertices in $\partial(f)$, counting multiplicities. Then (1) may be put as

$$\sum_{v\in V_3^+}(d(v)-4)+\sum_{f\in F}(s^*(f)-4)<-4N+1. \tag{2}$$

By Lemma 7, $s^*(f)\geq 3$ whenever $f\in F$. Let $f_i^*(v)$ be the number of i^*-faces at v, i.e., those having $s^*(f)=i$. Then

$$\sum_{v\in V_{3+}}(d(v)-4-\frac{f_3^*(v)}{3})+\sum_{f\in F^*_{4+}}(s^*(f)-4)<-4N+1, \tag{3}$$

where F^*_{i+} is the set of $\geq i^*$-faces in P, or

$$\sum_{v\in V_{3+}}(d(v)-4-\frac{f_3{}^*(v)}{3}+\frac{4N-1}{n^*})+\sum_{f\in F^*_{4+}}(s^*(f)-4)<0, \tag{4}$$

where $n^*=|V_{3+}|=|V|-n_2$.

Put $ch(v)=d(v)-4-\frac{f_3{}^*(v)}{3}+\frac{4N-1}{n^*}$ whenever $v\in V_{3+}$, $ch(f)=0$ if f is a 3^*-face, and $ch(f)=s^*(f)-4$ whenever $f\in F^*_{4+}$. Then (4) becomes

$$\sum_{v\in V_{3+}}ch(v)+\sum_{f\in F}ch(f)<0. \tag{5}$$

The rest of our proof consists in redistributing *the charge* $ch(x)$ on $x\in V_{3+}\cup F$, preserving the sum of charges, so that *the new charge*, $ch^*(x)$, is nonnegative whenever $x\in V_{3+}\cup F$. The contradiction with (5) will complete the proof.

First observe the following fact.

Lemma 8 $n^*>39(-4N+1)$.

Proof. Euler's formula (1) for the map P^* obtained from P by contracting each path $uv_1\dots v_sw$ such that $d(u)\geq 3$, $d(v_1)=\dots=d(v_s)=2$, $d(w)\geq 3$ into an edge uw, may be written as

$$\sum_{v\in V(P^*)}(d_{P^*}(v)-6)+\sum_{f\in F(P^*)}(2s_{P^*}(f)-6)\leq -6N,$$

which implies

$$\sum_{v\in V(P^*)}(d_{P^*}(v)-6)\leq -6N,$$

or else
$$|E(P^*)| \leq 3n^* - 3N.$$

By Lemma 1,
$$n \leq n^* + k|E(P^*)|.$$

But $n > c_N k + d_N$, for otherwise P has a trivial good coloring (all vertices have different colors), and so

$$n^*(1+3k) - 3kN > c_N k + d_N \geq (117 - 471N)k + 39 - 156N = 39(-4N+1)(1+3k) - 3kN,$$

as desired (compare the leftmost and the rightmost expressions). □

Lemma 9 *If $d(v) \geq 7$ then $ch(v) \geq 0$.*

Proof. Indeed,
$$ch(v) \geq d(v) - 4 - \frac{f_3{}^*(v)}{3} + \frac{4N-1}{n^*} \geq d(v) - 4 - \frac{d(v)}{3} + \frac{4N-1}{n^*} \geq \frac{2}{3} + \frac{4N-1}{n^*},$$
and we are done by Lemma 8. □

If $v \in V$ and $ch(v) < 0$, then v is called *poor*. Put
$$\xi = \frac{-4N+1}{n^*}$$
and
$$\varepsilon = \frac{1}{39}.$$
By Lemmas 4 and 5, $\xi \leq \varepsilon$, and each poor vertex v satisfies $3 \leq d(v) \leq 6$.

Our rules of redistribution of charge are:

R1. Each ≥ 14-vertex gives $\frac{2}{3} + \varepsilon$ to each incident $\geq 4^*$-face and $\frac{1}{3} + \varepsilon$ to each incident 3^*-face.

R2. Each poor vertex v receives $\frac{2}{3} + \varepsilon$

- from each incident $\geq 4^*$-face f such that there is a ≥ 14-vertex z in $\partial(f)$ not joined to v along $\partial(f)$ by a path of 2-vertices, and
- from each incident $\geq 13^*$-face f.

R3. If a 3^*-face f is incident with a ≥ 14-vertex, then f gives $\frac{1}{6} + \frac{\varepsilon}{2}$ to every incident poor vertex.

Lemma 10 *If v is poor, then v receives $\frac{1}{6} + \frac{\varepsilon}{2}$ by R3 from each of every two consecutive 3^*-faces at v.*

Proof. We first prove that if a 3^*-face f is incident with a 2-vertex u, then the ≥ 3-vertex which is opposite to u in $\partial(f)$ is in fact a ≥ 14-vertex.

Let $\partial(f) = xx_1x_2\dots x_{k(x)}yy_1y_2\dots y_{k(y)}zz_1z_2\dots z_{k(z)}$, where all x_i's, y_i's, and z_i's are all 2-vertices, while x, y, and z are ≥ 3-vertices. Suppose that $k(x) \geq 1$, i.e., x_1 exists. Denote by f_{xy} the face lying on the other side of path $xx_1x_2\dots x_{k(x)}xy$ than f (perhaps $f_{xy} = f$). The faces f_{yz} and f_{zx} are defined similarly.

Contract the edge xx_1 transfer a good coloring of the obtained map to $P - x_1$. If we color x_1 with a color that does not appear on the boundaries of f and the faces incident with z, then the obtained coloring is good. This is impossible only if $d(z) \geq 14$.

To complete the proof of our Lemma, observe that by the second statement of Lemma 3, at least one of every two consecutive 3^*-faces, f_1 and f_2, at v is incident with a 2-vertex, z. If z is incident both with f_1 and with f_2, then we are done by R3 and the statement just proved. Otherwise, z is opposite to the ≥ 3-vertex, w, which is incident both with f_1 and with f_2, because by the above statement, z cannot be opposite to the poor vertex v which has degree at most six due to Lemma 5. It follows that $d(w) \geq 14$, and we again use R3. $\square$

Lemma 11 *If $d(v) = 3$ then v is incident with at least two faces each of which gives $\frac{2}{3} + \varepsilon$ to v by R2.*

Proof. Let v be incident with paths $vx_1x_2 \ldots x_{k(x)}x$, $vy_1y_2 \ldots y_{k(y)}y$, and $vz_1z_2 \ldots z_{k(z)}z$, where all x_i's, y_i, and z_i's are 2-vertices while x, y, and z are all $>$ 2-vertices. Let $\partial(f_1) = \ldots xx_{k(x)} \ldots x_2x_1vy_1y_2 \ldots y_{k(y)}y$, $\partial(f_2) = \ldots yy_{k(y)} \ldots y_2y_1vz_1z_2 \ldots z_{k(z)}z$, and $\partial(f_3) = \ldots zz_{k(z)} \ldots z_2z_1vx_1x_2 \ldots x_{k(x)}x$.

Suppose the contrary, namely, by symmetry, that neither f_1 nor f_2 gives $\frac{2}{3} + \varepsilon$ to v. Then by R2 each of f_1 and f_2 is a $\leq 12^*$-face, and each vertex in $\partial(f_1) \setminus \{x, y\}$ and in $\partial(f_2) \setminus \{y, z\}$ has degree at most 13.

Contract the path $vy_1y_2 \ldots y_{k(y)}y$ to a vertex $v * y$. Transfer a good coloring of the obtained map to P. Color $y_1, y_2, \ldots, y_{k(y)}$ and v pairwise differently using colors that do not appear on the boundaries of the faces f_1, f_2, f_3, and those at most $2 \times 9 \times (13 - 1)$ faces that are incident with $>$ 2-vertices in $\partial(f_1) \setminus \{x, y\} \cup \partial(f_2) \setminus \{y, z\}$. The number of restrictions is less than $3k + 2 \times 9 \times (13 - 1)k < c_N k + d_N$, and no nontrivial unicolored edges or bicolored cycles appear. $\square$

Lemma 12 *If $d(v) = 4$ then v is incident with at least one face giving $\frac{2}{3} + \varepsilon$ to v by R2.*

Proof. Let v be incident with paths $vx_1^ix_2^i \ldots x_{k(x^i)}^ix^i$ in a cyclic order, where $0 \leq i \leq 3$, all x_j^i's are 2-vertices, and all x^i's are $>$ 2-vertices.

Let the face f^i have

$$\partial(f^i) = x^ix_{k(x^i)}^i \ldots x_2^ix_1^ivx_1^{i+1}x_2^{i+1} \ldots x_{k(x^{i+1})}^{i+1}x^{i+1} \ldots,$$

where upper indices are taken modulo 4.

Suppose that none of f^i gives $\frac{2}{3} + \varepsilon$ to v. Then by R2 each f^i is a $\leq 12^*$-face, and each vertex in $\partial(f^i) \setminus \{x^i, x^{i+1}\}$ has degree at most 13.

Add the edge x^ix^{i+1} into f^i whenever $0 \leq i \leq 3$, unless such an edge already exists in $\partial(f^i)$. Delete v and transfer a good coloring of the obtained map to P. Color v using a color that does not appear on the boundaries of the faces incident with vertices in $\partial(f^i) \setminus \{x^i, x^{i+1}\}$ whenever $0 \leq i \leq 3$. The number of restrictions is less than $4 \times 9 \times 13k \leq c_N k + d_N$, and it is not hard to see that no nontrivial unicolored edges or bicolored cycles appear. $\square$

Lemma 13 *If $v \in V_{3+}$, then $ch^*(v) \geq 0$.*

Proof. First suppose v is poor, i.e., has $ch(v) < 0$. Then, by Lemma 9, $d(v) < 7$. If $d(v) = 3$ then $ch(v) = -1 - \xi$, and we are done due to Lemma 11, because $2 \times (\frac{2}{3} + \varepsilon) > 1 + \xi$.

If $d(v) = 4$ then $ch(v) = -\xi$. By Lemma 12, v gets at least $\frac{2}{3} + \varepsilon$ from an incident $\geq 4^*$-face. If v is incident with at most two 3^*-faces, then $ch^*(v) \geq \varepsilon - \xi \geq 0$. Otherwise, by Lemma 10, v gets at least $2(\frac{1}{6} + \frac{\varepsilon}{2})$ from the three incident 3^*-faces, and $ch^*(v) \geq 0$.

If $d(v) = 5$ then $ch(v) = 1 - \xi$, and we are done if v is incident with at most two 3^*-faces. If there are r such faces at v, where $3 \leq r \leq 5$, then Lemma 10 provides v with at least $2(\frac{1}{6} + \frac{\varepsilon}{2})$ if $r = 3$ and at least $4(\frac{1}{6} + \frac{\varepsilon}{2})$ if $r \geq 4$, whence $ch^*(v) \geq 0$.

If $d(v) = 6$ then $ch(v) = 2 - \xi$, so that we are done if v is incident with at most five 3^*-faces. Otherwise, by Lemma 10, v gets at least $3 \times (\frac{1}{6} + \frac{\varepsilon}{2})$ from the six incident 3^*-faces, and $ch^*(v) \geq 0$.

Now let v not be poor. If it does not give anything to neighbor vertices by R1 or R2, then $ch^*(v) = ch(v) \geq 0$. Otherwise, $d(v) \geq 14$. Then v makes at most $d(v)$ transfers by R1. It follows that

$$ch^*(v) = d(v) - 4 - \xi - d(v)(\frac{2}{3} + \varepsilon),$$

whence $ch^*(v) \geq 0$, because $\xi \leq \varepsilon$. $\square$

Lemma 14 *If $f \in F$, then $ch^*(f) \geq 0$.*

Proof. A 3^*-face does not participate in discharging if all its incident vertices have degree at most 13. Otherwise, it receives at least $\frac{2}{3} + \varepsilon$ by R1 and gives at most $2(\frac{1}{3} + \frac{\varepsilon}{2})$ to the incident poor vertices. Hence $ch^*(f) \geq 0$ in both cases.

Now suppose $f \in F^*_{4+}$. If there are at least two vertices of degree ≥ 14 incident with f, then

$$ch^*(f) \geq s^*(f) - 4 + 2(\frac{2}{3} + \varepsilon) - (s^*(f) - 2)(\frac{2}{3} + \varepsilon) = (s^*(f) - 4)(\frac{1}{3} - \varepsilon) \geq 0.$$

If there is only one ≥ 14-vertex, z, at f, then

$$ch^*(f) \geq s^*(f) - 4 + \frac{2}{3} + \varepsilon - (s^*(f) - 3)(\frac{2}{3} + \varepsilon) \geq 0,$$

because f gives $\frac{2}{3} + \varepsilon$ to at most $s^*(f) - 3$ poor vertices by R2: nothing is given to z itself and to its left and right neighbours of degree ≥ 3 along $\partial(f)$.

Suppose there is no vertex of degree ≥ 14 at f. If $s^*(f) \leq 12$, then $ch^*(f) = ch(f) = s^*(f) - 4 \geq 0$, because f does not participate in discharging. Finally, if $s^*(f) \geq 13$ then by R2,

$$ch^*(f) \geq s^*(f) - 4 - s^*(f)(\frac{2}{3} + \varepsilon) = s^*(f)(\frac{1}{3} - \varepsilon) - 4 \geq 0,$$

because $\varepsilon = \frac{1}{39}$. $\square$

The lemmas above imply together that $ch^*(x) \geq 0$ for every $x \in V_{3+} \cup F$. This contradiction with (5) completes the proof.

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