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ACYCLIC COLOURING OF 1-PLANAR GRAPHS

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Abstract. A graph is 1-planar if it can be drawn on the plane in such a way that every edge crosses at most one other edge. We prove that the acyclic chromatic number of every 1-planar graph is at most 20.

Keywords. Acyclic colouring, planar graphs, 1-planar graphs.

1 Introduction

We denote by $V(G)$ the set of vertices of a graph G and by $E(G)$ its set of edges. A (proper) k-*coloring* of G is a mapping $f : V(G) \longrightarrow \{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ whenever x and y are adjacent in G.

A *1-planar* graph is a graph which can be drawn on the plane so that every edge crosses at most one other edge. One of the reasons for introducing this notion by Ringel [14] is that the graph G_{vf} of adjacency/incidence of the vertices and faces of each plane graph G is 1-planar. (On a map, take a capital of each country and join adjacent capitals by a railroad via a common border; also join each capital with all the corners of the country.) Every 1-planar graph with the maximum number of edges can be obtained from a plane graph whose faces have size 3 or 4 by adding two crossing diagonals into each 4-face. Every G_{vf} arises similarly from a plane quadrangulation.

In 1965, Ringel [14, 15] conjectured that each 1-planar graph is 6-colorable; this was confirmed by Borodin [6] in 1984. (A new simplier proof was given in [7].) It follows, the vertices and faces of each plane graph can be colored with 6 colors so that every two adjacent or incident elements have different colors. The 3-prism needs 6 colors for such a coloring, because it has 11 elements to be colored, and no three of them can be colored the same. Archdeacon [4] proved that the vertices and faces of each bipartite plane graph are simultaneously 5-colorable.

A proper vertex coloring of a graph is *acyclic* if every cycle uses at least three colors (Grünbaum [9]). The *acyclic chromatic number* of G, denoted by $a(G)$, is the minimum k such that G admits an acyclic k-coloring.

Borodin [5] proved Grünbaum's conjecture that every planar graph is acyclically 5-colorable. This bound is best possible. Moreover, there are bipartite 2-degenerate planar graphs G with $a(G) = 5$ (Kostochka and Mel'nikov, [11]). However, if a plane graph G has no cycles of length less than 5, then $a(G) \leq 4$, and if it has no cycles of length less than 7, then $a(G) \leq 3$ (Borodin, Kostochka and Woodall [8]).

Acyclic colorings turned out to be useful for obtaining results about other types of colorings. Suppose one has proved that $a(G) \leq a$. Then G has the star chromatic number at most $a2^{a-1}$

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(Grünbaum [9]) and the oriented chromatic number at most $a2^{a-1}$ (Raspaud and Sopena [13]); every m-coloring of the edges of G can be homomorphically mapped on that of a graph with at most am^{a-1} vertices (Alon and Marshall [2]); every m-coloring of the edges of mixed graph G combined with its n-coloring can be homomorphically mapped on a graph with at most $a(2n+m)^{a-1}$ vertices (Nešetřil and Raspaud [12]).

Also, Hakimi, Mitchem and Schmeichel [10, p.38-39] proved that $E(G)$ can be partitioned into $a(G)$ star forests (whose every component is a star). Using [5], this confirms the conjecture of Algor and Alon [1] that the edges of every planar graph can be partitioned into five star forests.

In this paper we study the acyclic coloring of 1-planar graphs. Our main result is the following:

Theorem 1 *Every 1-planar graph is acyclically 20-colorable.*

The best lower bound known to us is 7: the 3-dimensional cube with all the diagonals added cannot be colored acyclically with fewer than 7 colors.

Theorem 1 has a number of applications to other coloring problems, listed below. For the precise definitions of the notions used see [2, 9, 10, 12, 13].

Corollary 2 *Every plane graph has an acyclic simultaneous coloring of vertices and faces with at most* 20 *colors.*

Corollary 3 *Every 1-planar graph has star chromatic number at most* $20 \cdot 2^{19}$.

Corollary 4 *Every 1-planar graph has oriented chromatic number at most* $20 \cdot 2^{19}$.

Corollary 5 *Every plane graph has oriented simultaneous coloring of vertices and faces with at most* $20 \cdot 2^{19}$ *colors.*

Corollary 6 *The edges of each 1-planar graph can be partitioned into 20 star forests.*

Note that a loop yields a unicolored edge and two multiple edges yield a bicolored cycle. Instead of Theorem 7, it is easier to prove a bit more:

Theorem 7 *Every 1-plane pseudograph can be 20-colored so that no ends of an edge e are colored the same unless e is a loop, and no bichromatic cycles of length* > 2 *exists.*

Note also that the proof below is valid for pseudographs 1-embedded into the projective plane. The only difference is that Euler's formula for it says $|V| - |E| + |F| \leq 1$. Accordingly, the extensions of Theorem 1 and Corrolaries 2-6 to the projective plane also take place.

Alon, Mohar and Sanders [3] showed that the acyclic 5-colorability of the plane graphs easily implies the acyclic 7-colorability of the projective plane graphs.

2 Proof of Theorem 7

Let P_0 be a counterexample with the fewest vertices. Clearly, $|V(P_0)| \geq 21$. Observe that P_0 has no separating cliques, hence is 2-connected. In particular, P_0 has no vertices of degree less than 3. Recall that the degree of a vertex and the size of a face in a plane pseudograph (a map) is the number of incident edges.

We fix a 1-plane representation of P_0 with the minimum number of crossings. Then for each pair of edges ab, cd that cross each other at point s, their end vertices are paiwise distinct: $|\{a,b,c,d\}| = 4$. For each such a pair, we add edges ac, cb, bd and da 'close to s' i.e. so that they form triangles asc, csb, bsd and dsa with empty interior, respectively. The so obtained 1-plane graph P_1 is also a counterexample to Theorem 7, because any acyclic 20-coloring of P_1 is valid for P_0.

Denote by M_1 the plane map obtained from P_1 by removing all the crossed edges. Apply to $M_)$ as many as possible of the following operations, in any order:

- Delete a loop that forms a face of size 1.
- Delete one of two edges that form a face of size 2.
- Delete a common edge of two adjacent faces of siz 3.
- Triangulate each face of size at least 5 by adding diagonals.

The resulting map is denoted by M.

Observation 8 *M is connected, has no faces of size other than 3 or 4, and has no triangles with an edge in common.* □

Let P be obtained from M by inserting two crossed diagonals inside each 4-face. Then P is a counterexample to Theorem 7 with the fewest vertices. This is because adding an edge does not decrease the acyclic chromatic number, while adding loops or repeated edges cannot change it. (In other words: if the underlying graphs of pseudographs G and H coincide, then $a(G) = a(H)$.) The proof of Theorem 7 consists in establishing a set of structural properties of P that will be shown to contradict each other.

We shall mainly work with M. The degree of a vertex v in M, i.e., the number of incident edges (loops are counted twice), is denoted by $d(v)$. The size of a face f, i.e., the number of incident edges, counting multiplicities, is denoted by $s(v)$. By a $\geq k$-vertex we mean that of degree at least k, etc.

Denote by $D(v)$ the degree of v in P. Clearly, $D(v) = 2q(v)+t(v)$, where $q(v)$ and $t(v)$, respectively, stand for the number of quadrangles and triangles incident with v. It is easy to see that $d(v) = q(v) + t(v)$. $D(v)$ will be referred to as the *cyclic degree* of v in M. A vertex v will be called a *minor* vertex if $D(v) \leq 7$.

Lemma 9 *M has no separating cycle of length at most four.*

Proof. Otherwise, we split M along such a cycle S into two smaller maps, transform them into 1-planar pseudographs P' and P'' by adding all the diagonals, and combine their acyclic 20-colorings to obtain an acyclic 20-coloring for P. (The vertices of S are colored pairwise differently in the colorings both of P' and of P'' .) □

Corollary 10 *If $v \in V(P)$ then $D(v) \geq 5$ and $d(v) \geq 3$.*

Euler's formula $|V(M)| - |E(M)| + |F(M)| = 2$ may be rewritten as

$$\sum_{v \in V(M)} (d(v) - 4) + \sum_{f \in F(M)} (s(f) - 4) = -8, \tag{1}$$

where $F(M)$ is the set of faces in M.

We set the *initial charge* of every vertex v of M as $ch(v) = d(v) - 4$ and of every face f of M as $ch(f) = s(f) - 4$. We then use the discharging procedure, leading to a *final charge* ch^*, defined by applying the following (ordered) rules:

R0. Every vertex gives to every triangle containing it the charge $\frac{1}{3}$.

R1. Every vertex v with $D(v) = D \geq 8$ sends across every incident 4-face the charge $1 - 8/D$ and along every triangle edge the charge $\frac{1}{12} + \frac{1}{4}(1 - 8/D)$. In particular, every 8-vertex contained in a triangle sends along every triangle edge the charge $\frac{1}{12}$.

R2. If a vertex v with $D(v) = D \geq 8$ gets something across some incident 4-face f, then v gives it back to f and this charge is evenly distributed among the minor vertices belonging to this face (if such vertices exist).

If a vertex v with $D(v) = D \geq 8$ gets something along a triangle edge, then v sends it along the other edge of the same triangle.

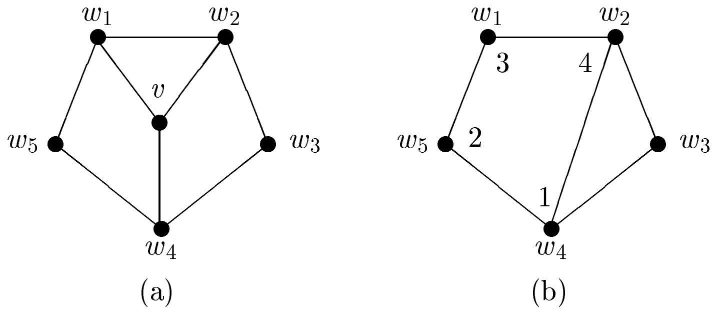

Figure 1: Configurations for Lemma 3

R3. If after fulfilling R1 and R2 some vertex v with $6 \le D(v) = D \le 7$ have a positive charge, then this charge is distributed evenly among the 5-vertices connected with v by triangle edges (if such vertices exist).

Since the above procedure preserves the total charge, we have:

$$\sum_{v \in V(M)} ch(v) + \sum_{f \in F(M)} ch(f) = \sum_{v \in V(M)} ch^*(v) + \sum_{f \in F(M)} ch^*(f) = -8.$$

Observe that $ch^*(q) = 0$ for every quadrangle q, and by rule R0 $ch^*(t) = 1$ for every triangle t.

We shall get a contradiction by proving that for every vertex v the inequality $ch^*(v) \ge 0$ holds. The proof will be delivered in a series of Lemmas.

Lemma 11 *If $D(v) \ge 8$ then $ch^*(v) \ge 0$.*

Proof. It is enough to prove that v has a nonnegative charge after fulfilling R0 and R1, since at steps R2 and R3 vertices with a positive charge preserve the positivity of their charge. Let v belong to t triangles and to q quadrangles. Then $D = t + 2q$ and $ch(v) = t + q - 4$. By R0, v gives to triangles exactly $t/3$. By R1, it sends $q(1 - 8/D)$ across quadrangles and at most $\frac{2t}{12} + \frac{2t}{4}(1 - 8/D)$ along the triangle edges. Even if it gets nothing from other vertices, it is left with at least

$$t + q - 4 - \frac{t}{3} - q(1 - \frac{8}{D}) - \frac{t}{6} - \frac{t}{2}(1 - \frac{8}{D}) = (\frac{t}{2} + q)(1 - (1 - \frac{8}{D})) - 4 = \frac{D}{2}\frac{8}{D} - 4 = 0.$$

□

Lemma 12 *Let a 5-vertex v have the cyclic neighbours $w_1, \dots, w_5$ (in this cyclic order), where w_1, w_2 and v form a triangle, and w_4v is an edge (see Figure 1(a)). Then $D(w_3) \ge 19$ and $D(w_5) \ge 19$. Moreover, if $D(w_3) = 19$ then $D(w_1) \ge 20$ and if $D(w_5) = 19$, then $D(w_2) \ge 20$.*

Proof. First we work with w_3. Delete v and add the edge w_2w_4. By the minimality of M, there exists an acyclic 20-coloring ϕ of the resulting multigraph. Let $\phi(\{w_4, w_5, w_1, w_2\}) = \{1, 2, 3, 4\}$ (in this order), as depicted in Figure 1(b). If $\phi(w_3) \notin \{2, 3\}$, then we can color v, a contradiction.

CASE 1. $\phi(w_3) = 3$. For every $\alpha \in \{5, \dots, 20\}$, the only obstacle for coloring v with α can be a $(3, \alpha)$-path between w_3 and w_1. Thus in this case $D(w_3) \ge 19$ and $D(w_1) \ge 20$.

CASE 2. $\phi(w_3) = 2$. For every $\alpha \in \{5, \dots, 20\}$, the only obstacle for coloring v with α can be a $(2, \alpha)$-path between w_3 and w_5. Moreover, if $D(w_3) = 19$ then we can recolor w_3 with 3 and get Case 1.

Similarly, we work with w_5. □

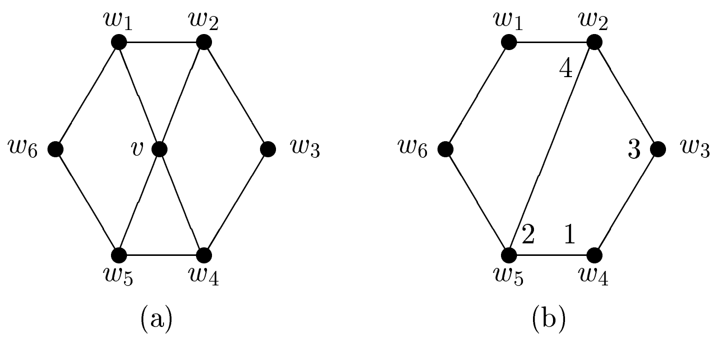

Figure 2: Configurations for Lemma 4

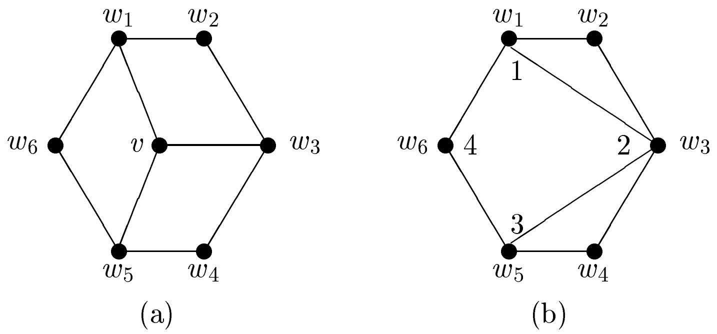

Figure 3: Configurations for Lemma 5

Lemma 13 *Let a 6-vertex v have the cyclic neighbours $w_1, \ldots, w_6$ (in this cyclic order), where w_1, w_2, v and w_4, w_5, v form triangles (see Figure 2(a)). Then $ch^*(v) \geq 0$.*

Proof. We have $ch(v) = 4 - 4 = 0$, and by R0 v gives $2/3$ to incident 3-faces. Therefore, it suffices to prove that v receives either at least $1/6$ from w_1 or at least $1/3$ from w_6, and then use the double symmetry.

Delete v and add the edge w_2w_5. By the minimality of M, there exists an acyclic 20-coloring ϕ of the resulting multigraph. Let $\phi(\{w_5, w_4, w_3, w_2\}) = \{1, 2, 3, 4\}$ (in this order), as depicted in Figure 2(b). If $\{\phi(w_1), \phi(w_6)\}) \cap \{2, 3\} = \emptyset$, then we can color v, a contradiction.

CASE 1. $\phi(w_1) = 5$. Then every color $\alpha \in \{6, \ldots, 20\}$ must be adjacent to w_6. Thus $D(w_6) \geq 18$, and by rule R1, v gets at least $(1 - 8/18) = 10/18 > 1/3$ from w_6.

CASE 2. $\phi(w_1) \in \{2, 3\}$ and $\phi(w_6) = 5$. Now every color $\alpha \in \{6, \ldots, 20\}$ must be adjacent to w_1. Thus $D(w_1) \geq 19$, and by rule R1 v gets at least $1/12 + 1/4(1 - 8/19) = 13/57 > 1/6$ from w_1.

CASE 3. $\phi(w_1) \in \{2, 3\}$ and $\phi(w_6) \in \{2, 3\}$. Now every color $\alpha \in \{5, \ldots, 20\}$ must be adjacent either to w_1 or to w_6. Hence $(D(w_1) - 4) + (D(w_6) - 3) \geq 16$, so that either $D(w_1) \geq 12$ or $D(w_6) \geq 12$. Then by rule R1, v gets either at least $1/6$ from w_1 or at least $1/3$ from w_6, respectively.

Now, due to the horizontal symmetry v gets at least $1/3$ from w_1, w_5 and w_6, and due to the vertical symmetry, at least $2/3$ from all its neighbours. □

Lemma 14 *Let a 6-vertex v have the cyclic neighbours $w_1, \ldots, w_6$ (in this cyclic order), where w_1v, w_3v and w_5v are edges in M. Then $ch^*(v) \geq 0$.*

Proof. We have $ch(v) = 3 - 4 = -1$ and v has no incident 3-face. Therefore, we have to prove that

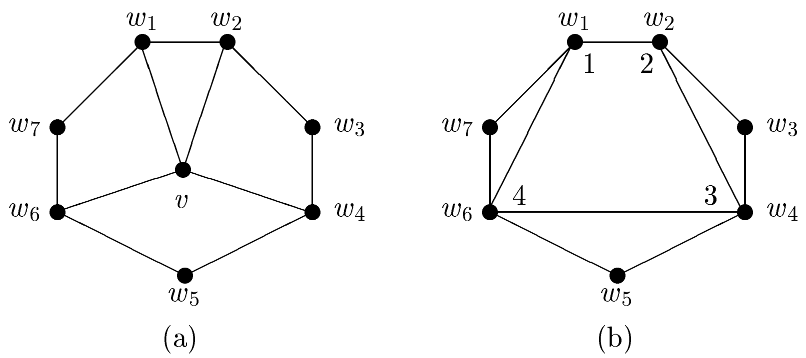

Figure 4: Configurations for Lemma 6

v gets at least 1 from its cyclic neighbours.

We may assume that $D(w_2) \leq D(w_4) \leq D(w_6)$. Delete v and add the edges w_1w_3 and w_3w_5. By the minimality of M, there exists a 20-coloring ϕ of the resulting multigraph. Let $\phi(\{w_1, w_3, w_5, w_6\}) = \{1, 2, 3, 4\}$ (in this order).

CASE 1. The color of w_2 or w_4 is not in $\{1, 2, 3, 4\}$ (say, $\phi(w_2) = 5$). Then every color $\alpha \in \{6, \ldots, 20\}$ must be adjacent to w_4. Thus $D(w_6) \geq D(w_4) \geq 18$ and v gets at least $2(1 - 8/18) > 1$.

CASE 2. The colors of w_2 and w_4 are in $\{1, 2, 3, 4\}$. Now every color $\alpha \in \{5, \ldots, 20\}$ must be adjacent either to w_2 or to w_4. If $D(w_4) \geq 16$ then also $D(w_6) \geq 16$ and v gets at least $2(1-8/16) = 1$. Otherwise, since we cannot recolor w_2 or w_4 with a color $\alpha \in \{5, \ldots, 20\}$ so to reduce to Case 1 (despite $D(w_4) \leq 15$), not all the neighbours of each of them have different colors. We conclude that $(D(w_2) - 4) + (D(w_4) - 4) \geq 16$. Therefore, $D(w_2) + D(w_4) \geq 24$ and, since $D(w_4) \leq 15$, v gets from w_2, w_4 and w_6 at least

$$2(1 - \frac{8}{D(w_4)}) + (1 - \frac{8}{D(w_2)}) \geq \min_{9 \leq x \leq 12} \{2(1 - \frac{8}{24 - x}) + (1 - \frac{8}{x})\} = 47/45 > 1.$$

□

Lemma 15 *Let a 7-vertex v have the cyclic neighbours $w_1, \ldots, w_7$ (in this cyclic order), where w_1, w_2 and v form a triangle, and w_4v and w_6v are edges in M (see Figure 4(a)). Then w_3, w_5 and w_7 together give v by R1 at least $1/3$. Therefore, $ch^*(v) \geq 0$.*

Proof. We have $ch(v) = 4 - 4 = 0$ and, by R0, v gives $1/3$ to its incident 3-face. Therefore, we have to prove that v gets at least $1/3$ from its cyclic neighbours.

Observe first that if $w_i \in \{w_3, w_5, w_7\}$ is such that $D(w_i) \geq 12$ then w_i gives v by R1 at least $1 - 8/12 = 1/3$ and we are done. Suppose now that $D(w_i) < 12$, $i = 3, 5, 7$. Delete v and add the edges w_2w_4, w_4w_6 and w_6w_1. Let $\phi(\{w_1, w_2, w_4, w_6\}) = \{1, 2, 3, 4\}$ (in this order), as depicted in Figure 4(b). If all the colors of $w_1, \ldots, w_7$ are distinct, then we can color v. Thus, it is not the case.

CASE 1. There are $u, z \in \{w_3, w_5, w_7\}$ covering all possible bicolored cycles through v (when we color v by some or other color). Then $(D(u) - 3) + (D(z) - 3) \geq 14$ and u and z give to v at least

$$\min_{9 \leq x \leq 10} \{1 - \frac{8}{x} + 1 - \frac{8}{20 - x}\} = 1 - \frac{8}{9} + 1 - \frac{8}{11} = \frac{38}{99} > \frac{1}{3}.$$

CASE 2. No two $u, z \in \{w_3, w_5, w_7\}$ cover all possible bicolored cycles through v. Now we have 16 free colors. Since we cannot recolor w_3, w_5, w_7, we have $(D(w_3) - 4) + (D(w_5) - 4) + (D(w_7) - 4) \geq 16$. Hence, the minimum amount w_3, w_5 and w_7 give v is

$$(1 - \frac{8}{8}) + (1 - \frac{8}{9}) + (1 - \frac{8}{11}) = \frac{38}{99} > \frac{1}{3}.$$

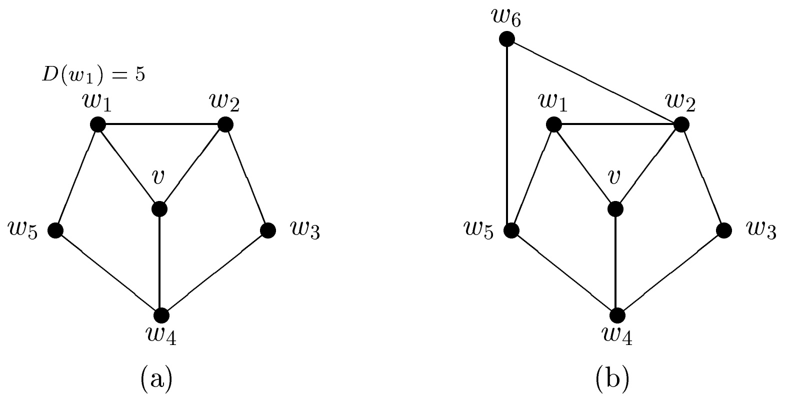

Figure 5: Configurations for Lemma 7

□

Lemma 16 *Let a 5-vertex v have the cyclic neighbours $w_1, \dots, w_5$ (in this cyclic order), where w_1, w_2 and v form a triangle, and w_4v is an edge. If w_1 is a 5-vertex, then $ch^*(v) \geq 0$.*

Proof. We have $ch(v) = 3 - 4 = -1$ and, by R0, v gives $1/3$ to its incident 3-face. Therefore, we have to prove that v gets at least $4/3$ from its cyclic neighbours.

Let w_6 be the neighbour of w_2 and w_5 which sees w_1. By Observation 8, w_6 does not coincide with w_3 (otherwise $w_3w_4w_5$ would be a separating cycle) and is not adjacent to it (otherwise $w_3w_4w_5w_6$ would be a separating cycle). In particular, $D(w_2) \geq 7$. By Lemma 12 (applied to vertices v and w_1), the degree of each of w_3, w_4, w_5 and w_6 is at least 19. Thus, if $D(w_2) \geq 8$, then by R1 and R2, each of v and w_1 receives at least $3(1 - 8/19) = 33/19 > 4/3$.

Let $D(w_2) = 7$. Then by Lemma 12, the degree of each of $= w_3$, w_4, w_5 and w_6 is at least 20 and w_2 receives at least $2(1 - 8/20)$ from w_4 and w_5. Hence it sends by R3 at least $6/5 - 1/3$ to v and w_1, so that each of them gets in total at least

$$2(1 - \frac{8}{20}) + \frac{1}{2}(\frac{6}{5} - \frac{1}{3}) = \frac{6}{5} + \frac{1}{2} \times \frac{13}{15} = \frac{49}{30} > 4/3.$$

□

Lemma 17 *Let a 5-vertex v have the cyclic neighbours $w_1, \dots, w_5$ (in this cyclic order), where w_1, w_2 and v form a triangle, and w_4v is an edge (see Figure 6(a)). If w_1 is a 6-vertex, then $ch^*(v) \geq 0$.*

Proof. As before, we have to prove that v gets at least $4/3$ from its cyclic neighbours.

Suppose that the lemma is false. Let w_1 have the cyclic neighbours $w_2, v, w_4, w_5, w_6, w_7$ (in this cyclic order), where w_5, w_6 and w_1 form a triangle (see Figure 6(b)). If $D(w_2) = 5$ we have the configuration of Lemma 16 and we are done. We thus suppose that $D(w_2) > 5$. By Lemma 12, we also get $D(w_5) \geq 19$ and $D(w_3) \geq 20$.

CASE 1. w_6 is a 5-vertex. Then by Lemma 12, $D(w_2) \geq 19$, $D(w_3) \geq 20$, and by R1 v gets from w_2, w_3 and w_5 at least

$$\frac{1}{12} + \frac{1}{4}(1 - \frac{8}{19}) + (1 - \frac{8}{20}) + (1 - \frac{8}{1 = 9}) = \frac{1}{12} + \frac{1}{4} \times \frac{11}{19} + \frac{3}{5} + \frac{11}{19} = \frac{7}{3} - \frac{88}{95} > \frac{4}{3}.$$

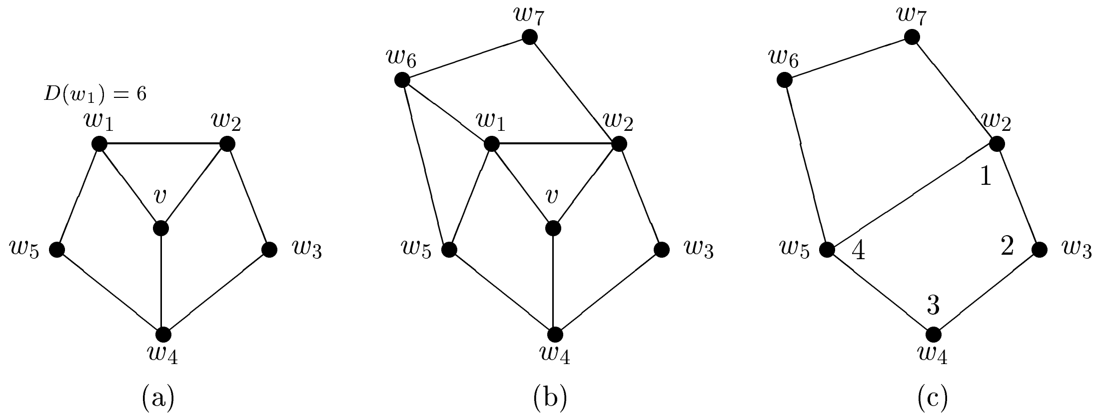

Figure 6: Configurations for Lemma 8

CASE 2. w_6 is not a 5-vertex. By Lemma 12, v gets from w_3 at least $1-8/20$ and from w_2 and w_5 together at least $1-8/20$. Thus, if it gets from w_1 by R3 at least $2/15$, it is happy. Delete v and w_1 and add the edge w_2w_5. By the minimality of M, there exists an acyclic 20-coloring ϕ of the resulting multigraph. Let $\phi(\{w_2, w_3, w_4, w_5\}) = \{1, 2, 3, 4\}$ (in this order), as depicted in Figure 6(c). If neither of w_6 and w_7 is colored by 3, then it is easy to complete the coloring of M by setting $\phi(v) = 3$ and $\phi(w_1) = x \notin \{\phi(w_6), \phi(w_7)\}$.

SUBCASE 2.1. $\phi(w_7) = 3$. If there is a color $\alpha \in \{5, \ldots, 20\} - \phi(w_6)$ not adjacent to w_4 or not adjacent to w_7, then we can color w_1 with α and then color v with a color not in $\{1, 2, 3, 4, \alpha\}$, a contradiction. Thus, there is no such α. It follows that $D(w_4) \geq 20$, $D(w_7) \geq 18$, and w_1 gets from them at least $(1-8/18) + (1-8/20) = 3/5 + 5/9$. Hence v gets from w_1 by R3 at least

$$\frac{3}{5} + \frac{5}{9} - \frac{2}{3} = \frac{22}{45} > \frac{2}{15}.$$

SUBCASE 2.2. $\phi(w_6) = 3$. Similarly to Subcase 2.2, we get $D(w_4) \geq 20$ and $D(w_6) \geq 19$. Hence w_1 sends by R3 to v at least

$$1 - \frac{8}{20} + \frac{1}{12} + \frac{1}{4}(1 - \frac{8}{19}) - \frac{2}{3} = \frac{3}{5} + \frac{1}{12} + \frac{11}{76} - \frac{2}{3} > \frac{2}{15}.$$

□

Lemma 18 *Let a 5-vertex v have the cyclic neighbours $w_1, \ldots, w_5$ (in this cyclic order), where w_1, w_2 and v form a triangle, and w_4v is an edge. If $ch^*(v) < 0$ then both w_1 and w_2 are 7-vertices.*

Proof. Suppose that the lemma is false for a 5-vertex v. Then by Lemmas 16 and 17, neither of w_1 and w_2 has the cyclic degree 5 or 6. If $8 \leq D(w_1), D(w_2) \leq 19$ then, by Lemma 13, we have $D(w_3) \geq 20$, $D(w_5) \geq 20$, and by R1 v gets from w_1, w_2, w_3 and w_5 at least

$$\frac{1}{12} + \frac{1}{12} + 2(1 - \frac{8}{20}) = \frac{1}{6} + \frac{6}{5} = \frac{41}{30} > \frac{4}{3}.$$

If $8 \leq D(w_1), D(w_2)$ and $D(w_1) \geq 20$ then v gets at least

$$\frac{1}{12} + \frac{1}{12} + \frac{1}{4}(1 - \frac{8}{20}) + (1 - \frac{8}{19}) + (1 - \frac{8}{20}) = \frac{1}{6} + \frac{5}{4} \times \frac{11}{19} + \frac{3}{5} = \frac{287}{195} > \frac{4}{3}.$$

In both cases, it contradicts the assumption $ch^*(v) < 0$.

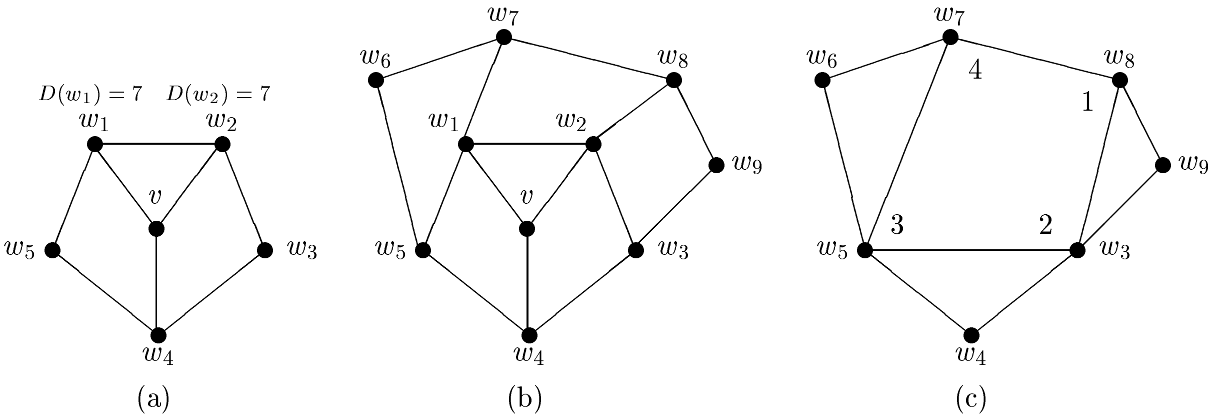

Figure 7: Configurations for Lemma 10

Suppose now that $D(w_1) = 7$ and $D(w_2) > 7$. Then w_2 sends $1/12$ to v and $1/12$ to w_1. But by Lemma 15, w_1 receives $1/3$ even without this $1/12$. Hence, it sends by R3 to v at least this $1/12$. Together with what v gets from w_3 and w_5, it receives at least

$$\frac{1}{12} + \frac{1}{12} + (1 - \frac{8}{19})(1 - \frac{8}{20}) = \frac{2}{12} + \frac{3}{5} + \frac{11}{19} = \frac{767}{570} > \frac{4}{3}.$$

This proves the lemma. □

Lemma 19 *Let a 5-vertex v have the cyclic neighbours $w_1, \ldots, w_5$ (in this cyclic order), where w_1, w_2 and v form a triangle, and w_4v is an edge (see Figure 7(a)). If both w_1 and w_2 are 7-vertices, then $ch^*(v) \geq 0$.*

Proof. As in Lemmas 16 and 17, we have to prove that v gets at least $4/3$ from its cyclic neighbours.

Suppose that the lemma is false for a 5-vertex v. We may assume that the cyclic neighbours of w_1 are $w_2, v, w_4, w_5, w_6, w_7, w_8$ (in this cyclic order), and the cyclic neighbours of w_2 are $w_3, w_4, v, w_1, w_7, w_8, w_9$ (in this cyclic order), as depicted in Figure 7(b). Recall that by Lemma 12, v gets from w_3 and w_5 at least $6/5$. Therefore it still needs $2/15$.

Delete v, w_2 and w_1 and add the edges w_8w_3, w_3w_5 and w_5w_7. By the minimality of M, there exists an acyclic 20-coloring ϕ of the resulting multigraph. Let $\phi(\{w_8, w_3, w_5, w_7\}) = \{1, 2, 3, 4\}$ (in this order), as depicted in Figure 7(c). Now we will try to color w_1 and w_2 so that to get an acyclic coloring of $G - v$ such that the colors of $w_1, \ldots, w_5$ are all distinct. That would be enough to finish the proof.

CASE 1. $\phi(w_4) = 5$.

SUBCASE 1.1. $\phi(w_6), \phi(w_9) \notin \{1, 2, 3, 4\}$. If there are two colors in $\{6, \ldots, 20\} \setminus \{\phi(w_6), \phi(w_9)\}$ not adjacent to w_4 then we can color w_1 and w_2 with them. Otherwise, $D(w_4) \geq 5 + 12$ and it sends at least $9/17$ to each of w_1 and $= w_2$. Hence, according to Lemma 15, each of w_1 and w_2 gives by R3 to v at least $9/17 - 1/3 = 10/51 > 1/15$.

SUBCASE 1.2. $\phi(w_6) \in \{1, 2\}, \phi(w_9) \notin \{1, 2, 3, 4\}$. If $D(w_4) \geq 12$ and $D(w_9) \geq 10$ then they give to w_2 at least $4/12 + 2/10$ and, according to Lemma 15, w_2 gives to v by R3 at least $2/10 > 2/15$, a contradiction. Otherwise, there exists a color $\alpha \in \{6, \ldots, 20\} - \phi(w_9)$ such that coloring w_2 with α does not create bicolored cycles. Now, coloring w_1 with a color $\beta \in \{6, \ldots, 20\} - \alpha$ can be bad only if $\phi(w_6) = 1$ and a cycle of colors 1 and β arises. It follows that $D(w_6) \geq 14 + 3$ and $D(w_8) \geq 14 + 5$, and they give to w_1 at least $(1 - 8/17) + (1 - 8/19) = 9/17 + 11/19$. Hence, according to Lemma 15, w_1 gives by R3 to v at least $9/17 + 11/19 - 2/3 = 428/969 > 2/15$.

SUBCASE 1.3. $\phi(w_6) \in \{1,2\}, \phi(w_9) \in \{3,4\}$. An argument similar to that of Subcase 1.2 works.

CASE 2. $\phi(w_4) = 1$. The worst subcase here is clearly when $\phi(w_6) = 1$ and $\phi(w_9) = 4$.

SUBCASE 2.1. There exists a color $\alpha \in \{5, \ldots, 20\}$ such that coloring w_1 with α does not create bicolored cycles. If we cannot color w_2, then every $\beta \in \{5, \ldots, 20\} - \alpha$ must be adjacent either to w_7 and w_9 or to w_4 and w_8. If at least 8 colors in $\{5, \ldots, 20\} - \alpha$ are adjacent to w_7 and w_9 then $D(w_7) \geq 13$, $D(w_9) \geq 11$, and they give to w_2 at least $5/13 + 3/11$, so that w_2 gives by R3 to v at least $5/13 + 3/11 - 1/3 > 2/15$. Otherwise, at least 8 colors in $\{5, \ldots, 20\} - \alpha$ are adjacent to w_4 and w_8. In this case, $D(w_4) \geq 13$, $D(w_8) \geq 13$, and they together give to w_1 at least $2(1 - 8/13) = 10/13$, so that w_1 gives by R3 to v at least $10/13 - 1/3 = 17/39 > 2/15$.

SUBCASE 2.2. There exists no color $\alpha \in \{5, \ldots, 20\}$ such that coloring w_1 with α does not create bicolored cycles. This means that every color in $\{5, \ldots, 20\}$ is adjacent to at least two of w_4, w_6 and w_8. It follows that $(D(w_4) - 5) + (D(w_6) - 3) + (D(w_8) - 4) \geq 32$, and hence together they give to w_1 at least $1 - 8/28 > 2/3$. Thus, v gets from w_1 by R3 at least $1/3$. □

The lemmas above imply together that $ch^*(v) \geq 0$ for every $v \in V(M)$. This contradiction with Euler formula finishes the proof.

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