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ACYCLIC IMPROPER COLOURINGS OF GRAPHS WITH BOUNDED DEGREE

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Abstract. In this paper, we continue the study of acyclic improper colourings of graphs introduced in a previous work. An improper colouring of a graph G is a mapping c from the set of vertices of G to a set of colours such that for every colour i, the subgraph induced by the vertices with colour i satisfies some property depending on i. Such an improper colouring is acyclic if for every two distinct colours i and j, the subgraph induced by all the edges linking a i-coloured vertex and a j-coloured vertex is acyclic.

We consider in this paper the case of graphs with bounded degree. We prove some positive and negative results for graphs with maximum degree three and generalize some of the negative results to graphs with maximum degree k.

Keywords. Acyclic colouring, Improper colouring, Bounded degree.

1 Introduction

Let G be a graph. We denote by $V(G)$ the vertex set of G and by $E(G)$ the edge set of G. A (proper) *k-colouring* of G is a mapping c from $V(G)$ to a set of k colours such that $c(x) \neq c(y)$ whenever xy is an edge in G. We say that G is *k-colourable* if G admits a k-colouring. The *chromatic number* $\chi(G)$ of G is defined as the minimum k such that G is k-colourable. A k-colouring of a graph G may be equivalently regarded as a partition $V_1, V_2, \ldots, V_k$ of $V(G)$ where for every i, $1 \leq i \leq k$, V_i is the set of vertices with colour i. Elements of such a partition are usually called *colour classes*.

Many variations and extensions of graph colourings have been considered in the literature (see [8] for a general overview). In particular *improper* colourings (sometimes called *generalized*, *defective* or *relaxed* colourings) have extensively been studied [6, 15, 16]. A colouring of a graph G is said to be improper whenever two adjacent vertices may be assigned the same colour. In such colourings, the adjacency constraint is replaced by a constraint on the structure of *monochromatic subgraphs*, that is subgraphs induced by colour classes.

Following Borowiecki and Mihók [4], improper colourings may be defined as follows. Let $P_1, P_2, \ldots, P_k$ be graph properties. A $(P_1, P_2, \ldots, P_k)$*-partition* of a graph G is a partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$ such that for every i, $1 \leq i \leq k$, the subgraph of G induced by V_i, denoted by $G[V_i]$, has property P_i. If a graph G has such a partition we say that G has property $P_1 \circ P_2 \circ \ldots \circ P_k$. If $P_1 = P_2 = \ldots = P_k = P$ we will denote by P^k the property $P_1 \circ P_2 \circ \ldots \circ P_k$. If $\mathcal{D}_0$ denotes the property to be without edges, then $\mathcal{D}_0^k$ is the property to be k-colourable.

In the following we will call $(P_1, P_2, \ldots, P_k)$*-colouring* of a graph G a mapping c from $V(G)$ to the set of colours $C = \{1, 2, \ldots, k\}$ such that the partition $(V_1, V_2, \ldots, V_k)$ given by $V_i = \{x \in V(G) : c(x) = i\}$, $1 \leq i \leq k$, is a $(P_1, P_2, \ldots, P_k)$-partition of G. Moreover, for every graph property P we will denote by $F(P)$ the family of all graphs satisfying P.

A proper k-colouring of a graph G is *acyclic* if for every two colours i and j the subgraph induced by $V_i \cup V_j$ has no cycle [1, 3, 7]. The *acyclic chromatic number* of a graph G is then defined as the

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minimum k such that G has an acyclic proper k-colouring. This notion can be extended to the case of improper colourings as follows. Let $V_1, V_2, \ldots, V_k$ be a $(P_1, P_2, \ldots, P_k)$-partition of a graph G. Let us call a (i,j)*-edge* an edge linking a vertex in V_i to a vertex in V_j. Such a partition is said to be *acyclic* if for every i, j, $1 \leq i < j \leq k$, the subgraph induced by all the (i,j)-edges has no cycle. The reader should note here that the subgraph induced by any two colour classes may contain bichromatic cycles but that any such cycle has to contain at least one monochromatic edge. A bichromatic cycle with no monochromatic edge will be called an *alternating bichromatic cycle*. If a graph H has an acyclic $(P_1, P_2, \ldots, P_k)$-partition we say that G has property $P_1 \odot P_2 \odot \ldots \odot P_k$. If $P_1 = P_2 = \ldots = P_k = P$ we will denote by $P^{(k)}$ the property $P_1 \odot P_2 \odot \ldots \odot P_k$. Observe that $\mathcal{D}_0^{(k)}$ is the property of being acyclically (properly) k-colourable in the usual sense.

Acyclic improper colourings originally arose in the study of oriented colourings. An *orientation* of a graph G is a digraph obtained from G by giving to every edge of G one of the two possible orientations. A digraph is called an *oriented graph* if it is an orientation of some undirected graph. An *oriented k-colouring* of an oriented graph $\vec{G}$ is a mapping c from $V(\vec{G})$ to a set of k colours such that (i) for every arc $\vec{xy}$ in $E(\vec{G})$, $c(x) \neq c(y)$ (an oriented colouring is thus a proper colouring) and (ii) for every two arcs $\vec{xy}$ and $\vec{zt}$ in $E(\vec{G})$, $c(x) = c(t) \Longrightarrow c(y) \neq c(z)$. The *oriented chromatic number* $\vec{\chi}(\vec{G})$ of an oriented graph $\vec{G}$ is then defined as the minimum k such that $\vec{G}$ has an oriented k-colouring.

Oriented chromatic numbers have been studied in [9, 11, 12, 13]. In [2] we have proved the following result which gives a link between acyclic improper colourings and oriented colourings:

Theorem 1 *Let $P_1, P_2, \ldots, P_k$ be graph properties such that for every i, $1 \leq i \leq k$, every orientation of every graph in the family $F(P_i)$ has oriented chromatic number at most α_i. Then every orientation of every graph in $F(P_1 \odot P_2 \odot \ldots \odot P_k)$ has oriented chromatic number at most $2^{k-1} \times \sum_{i=1}^{k} \alpha_i$.*

Acyclic improper colourings of outerplanar and planar graphs have been considered in [2]. We continue here the study of acyclic improper colourings by considering the case of graphs with bounded degree. We will denote by $\mathcal{D}_k$ the property of having degree at most k and by $\mathcal{T}_1$ the property of being a forest. The main result concerning improper colourings of graphs with bounded degree is due to Lovàsz [10]:

Theorem 2 [Lovàsz, 1966] *For every $p, q \geq 0$, $F(\mathcal{D}_{p+q+1}) \subseteq F(\mathcal{D}_p \circ \mathcal{D}_q)$.*

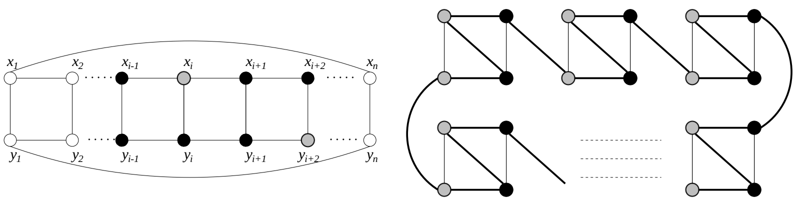
The problem of determining the acyclic chromatic number of graphs with bounded degree appears to be more difficult and it is still an open question [8]. Alon *et al.* [1] proved the following result:

Theorem 3 [Alon *et al.*, 1991] *For every k, there exist positive constants a_k and b_k such that:*
1. Every graph with maximum degree k has acyclic chromatic number at most $a_k \times k^{4/3}$.
2. There exist graphs with maximum degree k and acyclic chromatic number at least $b_k \times k^{4/3} \times (\log k)^{-1/3}$.

For small values of k, Grünbaum proved in [7] that every graph with maximum degree three has acyclic chromatic number at most four (this result is optimal). In [5] Burstein proved that every graph with maximum degree four has acyclic chromatic number at most five (he also mentionned that this result was independently obtained by Kostochka). Using our notation we thus get that $F(\mathcal{D}_3) \subseteq F(\mathcal{D}_0^{(4)})$ and $F(\mathcal{D}_4) \subseteq F(\mathcal{D}_0^{(5)})$.

We will consider acyclic improper colourings of graphs with bounded degree whose colour classes induce either forests or graphs with bounded degree. In the case of graphs with maximum degree three we know from Grünbaum's result that we only have to consider acyclic improper colourings using at most three colours. Moreover, since there exist infinitely many graphs with maximum degree three and acyclic chromatic number four we clearly get that $F(\mathcal{D}_3) \not\subseteq F(\mathcal{T}_1 \odot \mathcal{D}_0)$ (and thus $F(\mathcal{D}_3) \not\subseteq F(\mathcal{D}_1 \odot \mathcal{D}_0)$). In addition, we will prove the following negative results:

Theorem 4
1. $F(\mathcal{D}_3) \not\subseteq F(\mathcal{D}_0 \odot \mathcal{D}_2)$,
2. $F(\mathcal{D}_3) \not\subseteq F(\mathcal{D}_1 \odot \mathcal{D}_1)$.



(a) The circular grid T(2.n) (b) Cyclic arrangement of copies of K4-e

Figure 1: Negative results for graphs with maximum degree three

Our second theorem gives some positive results:

Theorem 5
1. $F(\mathcal{D}_3) \setminus \{K_4\} \subseteq F(\mathcal{T}_1 \odot \mathcal{T}_1)$,
2. $F(\mathcal{D}_3) \subseteq F(\mathcal{T}_1 \odot \mathcal{D}_2)$,
3. $F(\mathcal{D}_3) \subseteq F(\mathcal{D}_1 \odot \mathcal{D}_1 \odot \mathcal{D}_1)$.

The general case of graph with maximum degree k seems to be quite difficult to handle. Nethertheless, we can prove the following negative results:

Theorem 6
1. For every $k \geq 2$, $F(\mathcal{D}_k) \not\subseteq F(\mathcal{D}_{k-2} \odot \mathcal{D}_{k-2})$,
2. For every p, q, $pq \neq 0$, $p + q \geq 4$, $F(\mathcal{D}_{p+q}) \not\subseteq F(\mathcal{D}_p \odot \mathcal{D}_q)$.

In the following sections we will prove our three theorems and we will end this paper by stating some open problems.
Drawing conventions. In all the forthcomming figures, grey (resp. black) vertices will represent vertices with colour 1 (resp. 2). Uncoloured vertices will be drawn as white vertices.

2 Negative results for graphs with maximum degree three

We prove in this section the negative statements of Theorem 4. In every case we will exhibit infinitely many graphs having no corresponding acyclic colouring.

Proof of first statement. Let $T(2 \times n)$, $n \geq 3$, be the circular grid of dimension two with 2 rows of n vertices each (see Figure 1(a)). Let $\{x_1, x_2, \ldots, x_n\}$ (resp. $\{y_1, y_2, \ldots, y_n\}$) denote the vertices of row 1 (resp. of row 2). Since every vertex has degree 3, at least one of them, say x_i, must get colour 1. Vertices x_{i-1}, x_{i+1}, y_i then necessarily get colour 2. In order to avoid alternating bichromatic cycles, vertices y_{i-1} and y_{i+1} must also get colour 2. Since they both have two neighbours already coloured with colour 2, vertices y_{i-2} and y_{i+2} must get colour 1 and so on. It is not difficult to observe that the pattern $\left(\begin{array}{c} 2122 \\ 2221 \end{array}\right)$ must be repeated all along the grid. Therefore, if $n \equiv 0 \pmod 4$, the colouring thus obtained is an acyclic $(\mathcal{D}_0, \mathcal{D}_2)$-colouring, if $n \equiv 1 \pmod 4$, the colour 1 does not induce a graph with no edges, if $n \equiv 2 \pmod 4$, the colour 2 does not induce a graph with maximum degree two and if $n \equiv 3 \pmod 4$, we get an alternating bichromatic cycle. □

Proof of second statement. Observe first that the graph $K_4 - e$ (that is K_4 minus one edge) has a unique acyclic $(\mathcal{D}_1, \mathcal{D}_1)$-colouring. By arranging several copies of $K_4 - e$ into a cycle (see Figure 1(b)) we get infinetely many graphs having no acyclic $(\mathcal{D}_1, \mathcal{D}_1)$-colouring: if every colour induces a graph with maximum degree one, we necessarily have an alternating bichromatic cycle (drawn in thick edges). □

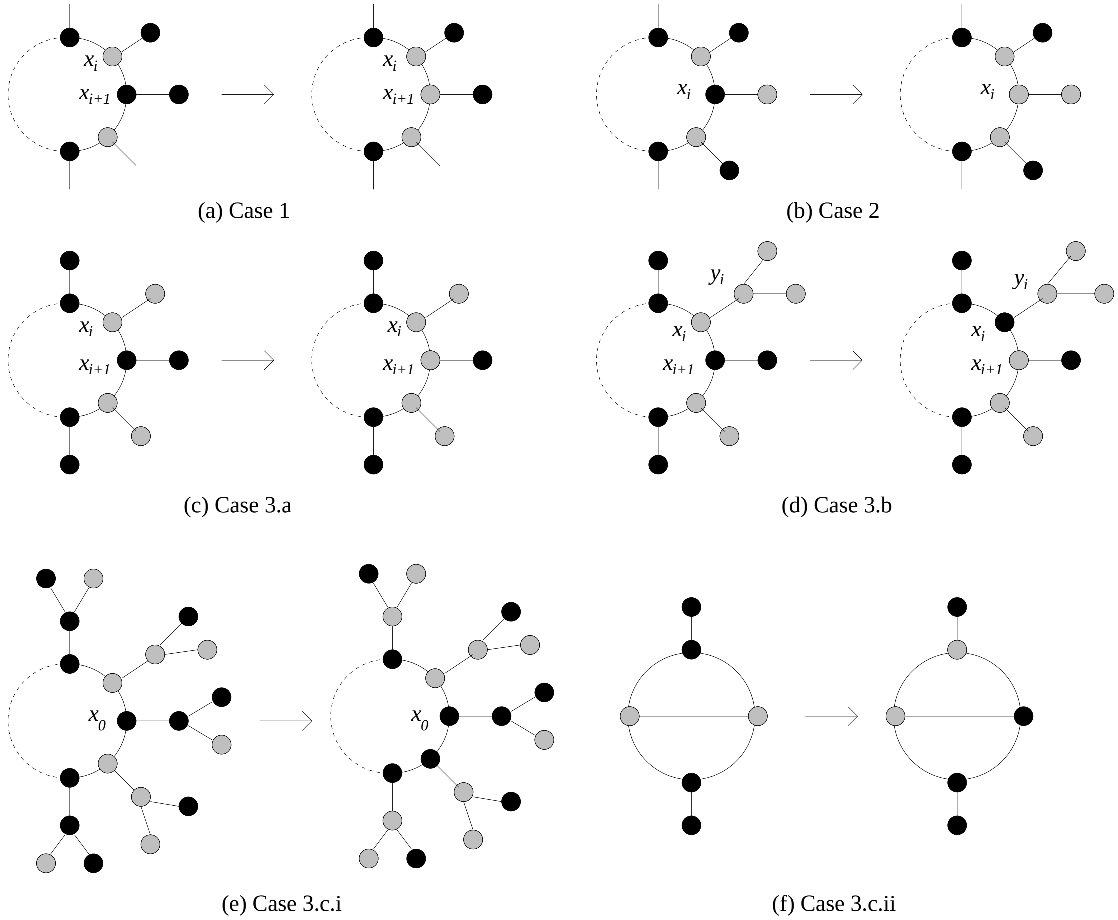


Figure 2: Deleting an alternating bichromatic cycle in a $(\mathcal{T}_1, \mathcal{T}_1)$-colouring

3 Positive results for graphs with maximum degree three

We prove in this section the positive statements of Theorem 5. Observe first that it is sufficient to prove these statements for 3-regular graphs. From Lovàsz' Theorem, we know that $F(\mathcal{D}_3) \subseteq F(\mathcal{T}_1 \circ \mathcal{T}_1)$ and $F(\mathcal{D}_3) \subseteq F(\mathcal{T}_1 \circ \mathcal{D}_2)$. In the following we will prove that we can modify any (not acyclic) such improper colouring in order to strictly decrease the number of alternating bichromatic cycles. By repeating this step we will get in each case an acyclic improper colouring.

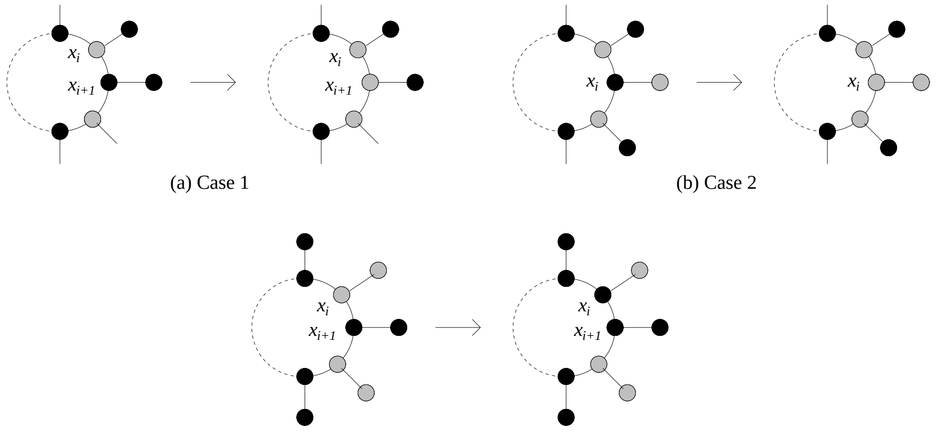
Let $C = (x_0, x_1, \ldots, x_{2n-1})$ be an alternating bichromatic cycle. For every i, $0 \leq i \leq 2n-1$, let y_i be the neighbour of x_i not belonging to C. We say that C is of type I if there exists some i such that y_i and y_{i+1} have the same colour (the subscripts are always taken modulo $2n$). We say that C is of type II if it is not of type I and for every i, x_i and y_i have distinct colours. Otherwise, we say that C is of type III (in that case, for every i, x_i and y_i have the same colour).

Unless otherwise specified, in the following pictures vertices not belonging to the alternating bichromatic cycle may be identified with vertices of the cycle provided they have the same color.

Proof of first statement. Let G be a 3-regular graph, c a (not acyclic) $(\mathcal{T}_1, \mathcal{T}_1)$-colouring of G and C an alternating bichromatic cycle of smallest type in G. We have three cases to consider:

1. *C is of type I.* Let i be such that y_i and y_{i+1} have the same colour. If x_i and y_i have the same colour it suffices to change the colour of x_i, otherwise it suffices to change the colour of x_{i+1} (see Figure 2(a)).

2. *C is of type II.* It suffices to change the colour of any of the $x_i's$ (see Figure 2(b)).



(a) Case 1 (b) Case 2 (c) Case 3

Figure 3: Deleting an alternating bichromatic cycle in a $(\mathcal{T}_1, \mathcal{D}_2)$-colouring

3. *C is of type III.* We have three subcases to consider:
   (a) if there exits an index i such that x_i and x_{i+2} does not belong to the same tree (induced by their colour), it suffices to change the colour of x_{i+1} (see Figure 2(c)).
   (b) Otherwise, if there exists a vertex y_i with the same colour as its three neighbours, it suffices to change the colour of x_i and x_{i+1} (see Figure 2(d)).
   (c) if none of the previous subcases happens, we still have two cases to consider:
      i. if none of the y_i's belongs to C it suffices to change the colour of vertices x_{2i}, $i \geq 0$, and of vertices y_{2j+1}, $j \geq 1$ (see Figure 2(e)).
      ii. if there exists y_i such that y_i belongs to C then vertices x_i and y_i belong to the same tree (induced by their colour) and this tree is of order two. If the length of C is at least 6, we are in the subcase 3(a). If C has length 4, since G is not K_4, we necessarily have the configuration depicted in Figure 2(f) and it suffices to change the colour of two consecutive vertices of C.

□

The reader should observe that K_4 is thus the only 3-regular graph having no acyclic $(\mathcal{T}_1, \mathcal{T}_1)$-colouring.

Proof of second statement. Let G be a 3-regular graph, c a (not acyclic) $(\mathcal{T}_1, \mathcal{D}_2)$-colouring of G and C an alternating bichromatic cycle of smallest type in G. We have three cases to consider:

1. *C is of type I.* Let i be such that y_i and y_{i+1} have the same colour. If x_i and y_i have the same colour it suffices to change the colour of x_i, otherwise it suffices to change the colour of x_{i+1} (see Figure 3(a)).
2. *C is of type II.* It suffices to change the colour of any of the $x_i's$ having colour 1 (see Figure 3(b)).
3. *C is of type III.* It suffices to change the colour of any of the $x_i's$ having colour 2 (see Figure 3(c)).

□

Proof of third statement. This statement can be proved by induction on the number of vertices of the graph. Due to the lack of space, we will not report the proof here. The complete proof can be found in [14]. □

4 Negative results for graphs with maximum degree k

We prove in this section the negative statements of Theorem 6. In every case we will exhibit infinitely many graphs having no corresponding acyclic colouring.

Proof of first statement. Let us show that the graph $G_k = K_{k+1} - e$ does not have any acyclic $(\mathcal{D}_{k-2}, \mathcal{D}_{k-2})$-colouring. Denote by u and v the two non-adjacent vertices of G_k and let c be any $(\mathcal{D}_{k-2}, \mathcal{D}_{k-2})$-colouring of G_k (we know by Lovàsz' Theorem that such a colouring exists). Since colours 1 and 2 both induce a graph of degree at most $k-2$ both of them must be used at least two times on $V(G_k) \setminus \{u\}$ and therefore c contains an alternating bichromatic cycle. Since G_k is not k-regular there are infinitely many graphs with maximum degree k containing G_k as a subgraph and thus having no acyclic $(\mathcal{D}_{k-2}, \mathcal{D}_{k-2})$-colouring. □

Proof of second statement. Let H_k^2, $k \geq 4$, be the graph obtained by cyclically connecting two copies of $K_{k-1,k-1}$, as depicted by Figure 4(a). Let us denote by $x_0^1, \ldots, x_k^1$ and $x_0^2, \ldots, x_k^2$ (resp. by $x_0^3, \ldots, x_k^3$ and $x_0^4, \ldots, x_k^4$) the vertices of the first (resp. the second) copy of $K_{k-1,k-1}$. A vertex x_j^i will be said to be of type i. Let c be any $(\mathcal{D}_p, \mathcal{D}_q)$-colouring of H_k^2 (we know by Lovàsz' Theorem that such a colouring exists).

We first claim that each copy of $K_{k-1,k-1}$ contains at least two vertices of each colour. If all vertices of a given copy have the same colour, say 1, then $p = k-1$. All their neighbours in the other copy are thus 2-coloured and we get $p + q = 2k - 2 > k$, a contradiction since $k \geq 4$. If all vertices but one, say x_0^1, in the first copy, have the same colour, say 1, then $p = k-1$ (all the x_j^1's, $j \neq 0$, have $k-1$ neighbours of type 2 with the same colour) and thus $q = 1$. But in that case, vertices $x_1^4, \ldots, x_k^4$ must have colour 2 and thus all the x_j^3's must have colour 1. It is not difficult to check that we then necessarily have an alternating bichromatic cycle.

Our second claim is that for every type of vertex, one colour is used at most once. This is obvious if $k = 4$. If $k > 4$, suppose on the contrary that two vertices of type 1 are 1-coloured and two vertices of type 1 are 2-coloured. In that case, no two vertices of type 2 may have the same colour in order to avoid alternating bichromatic cycles. We thus get $k = 3$, a contradiction.

From these two claims we get that each copy of $K_{k-1,k-1}$ contains exactly two vertices of one colour, say 1. Colour 2 thus induces a graph of degree at least $k-2$. We have two cases to consider:

1. $p = 2$ and $q = k-2$. The subgraph induced by the 2-coloured vertices in the first copy of $K_{k-1,k-1}$ has degree $k-2$. The neighbours of these vertices in the second copy are thus 1-coloured and we get $p = q = 2$ and thus $k = 4$. In that case, we necessarily have an alternating bichromatic cycle (see Figure 4(b)).

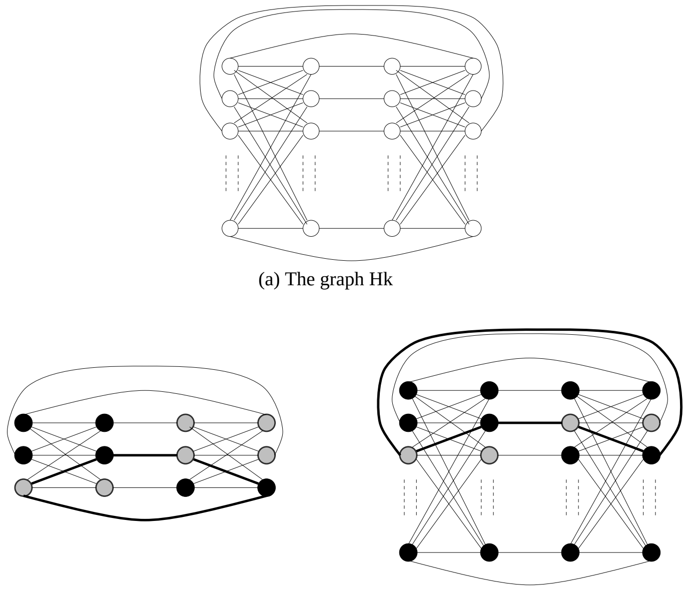
2. $p = 1$ and $q = k-1$. Every 1-coloured vertex in a copy of $K_{k-1,k-1}$ is adjacent to a 2-coloured vertex in the second copy and we also get an alternating bichromatic cycle (see Figure 4(c)).

Thus H_k^2 has no acyclic $(\mathcal{D}_p, \mathcal{D}_q)$-colouring. It is not so difficult to check that this is also the case for the graphs H_k^p obtained by cyclically connecting p copies of $K_{k-1,k-1}$. □

5 Discussion and open problems

None of the positive results given in Theorem 5 improves the best known upper bound for the oriented chromatic number of oriented graphs with maximum degree three (this bound has been settled to 11 in [14] and it has been conjectured in [13] that every *connected* oriented graph with maximum degree three has oriented chromatic number at most 7).

We do not know whether every (or almost every) graph with maximum degree three admits an acyclic $(\mathcal{D}_0, \mathcal{D}_0, \mathcal{D}_1)$-, $(\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_1)$-, $(\mathcal{D}_1, \mathcal{D}_2)$- or $(\mathcal{T}_1, \mathcal{D}_1)$-colouring. In the answer of this last question



(a) The graph Hk

(b) The case p = q = 2

(c) The case p = 1, q = k-1

Figure 4: Construction for statement 2 of Theorem 6

is YES, this would decrease to 10 the upper bound for the oriented chromatic number of oriented graphs with maximum degree three. We know that at least three graphs do not admit an acyclic $(\mathcal{T}_1, \mathcal{D}_1)$-colouring, namely the Petersen graph and the cyclic grids $T(2\times 3)$ and $T(2\times 4)$. We know by Lovàsz' Theorem that every graph in $\mathcal{D}_3$ admits a $(\mathcal{D}_2, \mathcal{D}_2)$-colouring. Concerning acyclic colourings, we have the following conjecture:

Conjecture 7 $F(\mathcal{D}_3) \subseteq F(\mathcal{D}_2 \odot \mathcal{D}_2)$.

The problem of acyclic improper colourings of graphs with degree at most k seems really difficult to handle. We have no idea of whether there exist or not such colourings whose colour classes induce graphs with bounded degree or forests.

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