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ACYCLIC IMPROPER COLORINGS OF GRAPHS[1]

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Abstract. In this paper, we introduce the new notion of acyclic improper colorings of graphs. An improper coloring of a graph is a vertex-coloring in which adjacent vertices are allowed to have the same color, but each color class V_i satisfies some condition depending on i. Such a coloring is acyclic if there are no alternating 2-colored cycles.

We prove that every outerplanar graph can be acyclically 2-colored in such a way that each monochromatic subgraph has degree at most five and that this result is best possible. For planar graphs, we prove some negative results and state some open problems.

Keywords. Acyclic coloring, Improper coloring, Planar graph, Outerplanar graph.

1 Introduction

For every graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. A *k-coloring* $c : V(G) \longrightarrow \{1, 2, \ldots, k\}$ of a graph G may be equivalently regarded as a partition of $V(G)$ into *color classes* $V_1, V_2, \ldots, V_k$ where for every i, $1 \le i \le k$, V_i is the set of vertices with color i. In the following, we shall use the term "improper" in order to stress the fact that adjacent vertices are not required to get distinct colors. Improper colorings are usually defined by requiring each color class V_i to satisfy some property depending on i [6, 10]. In particular, Borowiecki and Mihók considered in [4] improper colorings such that each color class V_i satisfies a so-called *hereditary* property $\mathcal{P}_i$ (a property $\mathcal{P}$ is *hereditary* if every subgraph of a graph which satisfies $\mathcal{P}$ also satisfies $\mathcal{P}$).

Following their notation, we say that a graph G satisfies the property $\mathcal{P}_1 \cdot \mathcal{P}_2 \cdot \ldots \cdot \mathcal{P}_k$ if there exists a partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$ such that for every i, $1 \le i \le k$, the subgraph $G[V_i]$ induced by V_i satisfies $\mathcal{P}_i$. Such a partition will be called a *$(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-partition* of G. If $\mathcal{P}_1 = \mathcal{P}_2 = \ldots = \mathcal{P}_k = \mathcal{P}$ we denote the property $\mathcal{P}_1 \cdot \mathcal{P}_2 \cdot \ldots \cdot \mathcal{P}_k$ by $\mathcal{P}^k$. By a *$(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-coloring* of G we mean a k-coloring such that the partition $(V_1, V_2, \ldots, V_k)$ given by the color classes is a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-partition of G.

For convenience, we shall also denote by $\mathcal{P}$ the family of graphs satisfying the property $\mathcal{P}$. We shall consider in particular the properties $\mathcal{D}_k$ of having degree at most k ($\mathcal{D}_0^k$ is thus the property of being properly k-colorable), $\mathcal{T}_k$ of having treewidth at most k ($\mathcal{T}_1$ is thus the property of being a forest) and $\mathcal{O}$ of being outerplanar.

A proper k-coloring of a graph G is *acyclic* if all the subgraphs induced by any two colors have no cycle [5]. The *acyclic chromatic number* of G is then defined as the minimum k such that G has an acyclic proper k-coloring. We extend this notion to the case of improper colorings as follows. Let $(V_1, V_2, \ldots, V_k)$ be a $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-partition of G. By an *(i,j)-edge* we mean an edge linking a vertex in V_i to a vertex in V_j. Such a partition is said to be *acyclic* if for every pair (i,j), $i \ne j$, the subgraph induced by all the (i,j)-edges has no cycle; $G[V_i \cup V_j]$ may contain bichromatic cycles but any such cycle must contain at least one monochromatic edge. A bichromatic cycle having no monochromatic edge will be called an *alternating bichromatic cycle*. We say that a graph G has property $\mathcal{P}_1 \odot \mathcal{P}_2 \odot \ldots \odot \mathcal{P}_k$ if it admits an acyclic $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-partition. If $\mathcal{P}_1 = \mathcal{P}_2 = \ldots = \mathcal{P}_k = \mathcal{P}$ we will denote the property $\mathcal{P}_1 \odot \mathcal{P}_2 \odot \ldots \odot \mathcal{P}_k$ by $\mathcal{P}^{(k)}$ ($\mathcal{D}_0^{(k)}$ is thus the property of being acyclically properly k-colorable).

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The study of acyclic improper colorings was motivated by the problem of improving the upper bound for the oriented chromatic number of planar graphs [8]. Oriented colorings and their links with acyclic improper colorings are introduced in the next Section. In Section 3 we consider the case of outerplanar graphs and prove that every outerplanar graph admits an acyclic $(\mathcal{D}_5, \mathcal{D}_5)$-partition and that this result is in some sense optimal. We give some negative results concerning planar graphs in Section 4 and state some open questions.

In a companion paper [1], acyclic improper colorings of graphs with bounded degree are considered.

2 Acyclic improper colorings and oriented colorings

Let $\vec{G}$ be an oriented graph (a digraph without 2-cycles). An *oriented k-coloring* of $\vec{G}$ is a mapping $c : V(\vec{G}) \longrightarrow \{1, 2, \ldots, k\}$ such that (i) for every arc $\vec{xy}$ in $E(\vec{G})$, $c(x) \neq c(y)$ (an oriented coloring is thus a proper coloring) and (ii) for every two arcs $\vec{xy}$ and $\vec{zt}$ in $E(\vec{G})$, $c(x) = c(t) \Longrightarrow c(y) \neq c(z)$. Intuitively speaking, condition (ii) means that the colors assigned to any two adjacent vertices x and y uniquely determine the orientation of the edge xy. The *oriented chromatic number* $\vec{\chi}(\vec{G})$ of $\vec{G}$ is defined as the minimum k such that $\vec{G}$ has an oriented k-coloring. The oriented chromatic numbers of an undirected graph G and a family $\mathcal{F}$ of graphs are defined by $\vec{\chi}(G) = \max\{\vec{\chi}(\vec{G}) : \vec{G}$ is an orientation of $G\}$ and $\vec{\chi}(\mathcal{F}) = \max\{\vec{\chi}(G) : G \in \mathcal{F}\}$.

Oriented chromatic numbers of graphs have been studied in [7, 8, 9]. Raspaud and Sopena proved in [8] that every graph with acyclic chromatic number k has oriented chromatic number at most $k \times 2^{k-1}$. A result of Borodin [2] states that every planar graph has acyclic chromatic number at most 5. Thus every planar graph has oriented chromatic number at most 80. Improving this upper bound seems to be particularly challenging. Our initial interest in acyclic improper colorings was motivated by this problem since we can prove the following extension of Raspaud and Sopena's result:

Theorem 1 *Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ be graph properties such that for every i, $1 \leq i \leq k$, the family $\mathcal{P}_i$ has oriented chromatic number at most α_i. Then we have:*

$$\vec{\chi}(\mathcal{P}_1 \odot \mathcal{P}_2 \odot \ldots \odot \mathcal{P}_k) \ \leq \ 2^{k-1} \times \sum_{i=1}^{k} \alpha_i.$$

Proof. Let $\mathcal{P} = \mathcal{P}_1 \odot \mathcal{P}_2 \odot \ldots \odot \mathcal{P}_k$, G be a graph in $\mathcal{P}$, $\vec{G}$ be any fixed orientation of G and $(V_1, V_2, \ldots, V_k)$ be an acyclic $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-partition of $V(G)$. For every i, $1 \leq i \leq k$, let $c_{i,i}$ be any oriented α_i-coloring of $\vec{G}[V_i]$. We assume in the following that for every i, j, $1 \leq i < j \leq k$, the oriented colorings $c_{i,i}$ and $c_{j,j}$ use disjoint sets of colors, none of them containing colors 1 or 2.

For every i, j, $1 \leq i < j \leq k$, denote by $G_{i,j}$ the subgraph of G induced by all the (i, j)-edges. For every connected component C of $G_{i,j}$ let x_C be any arbitrarily chosen vertex and denote by A_0 the set of those x_C's. Let now A_m, $m \geq 1$ be the set of vertices at distance m from A_0 in $G_{i,j}$ (that is at distance m from one vertex in A_0). Observe that since the partition is acyclic, the graph $G_{i,j}$ is a forest and thus every vertex in A_m has exactly one neighbor in A_{m-1}. We now inductively define a 2-coloring $c_{i,j}$ of $G_{i,j}$ as follows:

1. $c_{i,j}(x_0) = 1$ for every $x_0 \in A_0$,
2. for every $x_m \in A_m$, $m \geq 1$, with neighbor x_{m-1} in A_{m-1}, let $c_{i,j}(x_m) = c_{i,j}(x_{m-1})$ if $x_m \in V_i$ and $\vec{x_m x_{m-1}}$ is an arc in $\vec{G}$ or $x_m \in V_j$ and $\vec{x_{m-1} x_m}$ is an arc in $\vec{G}$ and let $c_{i,j}(x_m) = 3 - c_{i,j}(x_{m-1})$ otherwise.

In other words, two adjacent vertices are assigned the same color if and only if the corresponding arc in $\vec{G}$ is directed from the vertex belonging to the subset of lower index towards the vertex belonging to the subset of higher index in the partition.

We claim that the mapping c defined by

$$c(x) = (c_{1,i}(x), \ldots, c_{i-1,i}(x), c_{i,i}(x), c_{i,i+1}(x), \ldots, c_{i,k}(x))$$

for every i and $x \in V_i$ is an oriented coloring of $\vec{G}$. From above, we get that this mapping uses exactly $2^{k-1} \times \sum_{i=1}^{k} \alpha_i$ colors. Now let $x \in V_i$ and $y \in V_j$ be two adjacent vertices in $\vec{G}$. If $i = j$ then the definition of $c_{i,i}$ implies $c(x) \neq c(y)$, while if $i \neq j$ then $c(x)$ and $c(y)$ clearly differ on their ith and jth components. Finally, let us suppose that there exist two arcs $\vec{xy}$ and $\vec{zt}$ in $\vec{G}$ such that $c(x) = c(t)$ and $c(y) = c(z)$. Since all the $c_{i,i}$'s are oriented colorings, these four vertices cannot belong to the same V_i. Then w.l.o.g. we have $x, t \in V_i$ and $y, z \in V_j$ for some $i < j$. Thus $c_{i,j}(x) = c_{i,j}(t)$ and $c_{i,j}(y) = c_{i,j}(z)$, which is a contradiction since the definition of $c_{i,j}$ implies that $c_{i,j}(x) = c_{i,j}(y)$ and $c_{i,j}(z) \neq c_{i,j}(t)$. □

Theorem 1 provides a new way to attack the problem of determining an upper bound for the oriented chromatic number of some families of graphs. Our main interest concerns partitions whose color classes induce subgraphs with bounded degree or bounded treewidth since some upper bounds for the oriented chromatic numbers of these families are known [7, 9]. Recall that a *k-tree* is inductively defined as follows: the complete graph K_k is a k-tree; if G is a k-tree and $x_1, x_2, \ldots, x_k$ are k vertices inducing a complete subgraph in G then the graph obtained from G by adding a new vertex y adjacent to $x_1, x_2, \ldots, x_k$ is also a k-tree. A graph is then said to have treewidth at most k if it is a subgraph of some k-tree.

3 Outerplanar graphs

An outerplanar graph is a planar graph that can be drawn in such a way that every vertex lies on the outer face. An outerplanar graph is *maximal* if adding to it one edge makes it no longer outerplanar. Every outerplanar graph has at least two vertices with degree two and has treewidth at most two. By using an easy inductive argument it can be proved that the acyclic chromatic number of an outerplanar graph is at most 3. We shall thus consider acyclic partitions with two color classes, each inducing either a forest or a graph with bounded degree. We shall prove that an outerplanar graph admits an acyclic partition with two color classes, each inducing a subgraph with maximum degree at most 5, and that this result is best possible if we require the color classes to induce subgraphs having bounded degree or bounded treewidth.

We first prove the following negative results:

Proposition 2
1. $\mathcal{O} \not\subseteq \mathcal{T}_1 \odot \mathcal{T}_1$,
2. for every $k \geq 0$, $\mathcal{O} \not\subseteq \mathcal{T}_1 \odot \mathcal{D}_k$,
3. for every $k \geq 0$, $\mathcal{O} \not\subseteq \mathcal{D}_4 \odot \mathcal{D}_k$.

Proof.

1. Let G_1 be the outerplanar graph shown in Figure 1(a). Suppose that c is an acyclic $(\mathcal{T}_1, \mathcal{T}_1)$-coloring of G. The triangle abc must then use 2 colors. Suppose $c(a) = 2$, $c(b) = c(c) = 1$. If $c(d) = 1$ we have a monochromatic triangle while if $c(d) = 2$ we have an alternating bichromatic cycle, a contradiction.

2. Let $G_{2,k}$ be the outerplanar graph shown in Figure 1(b). We first claim that if c is an acyclic $(\mathcal{T}_1, \mathcal{D}_k)$-coloring of $G_{2,k}$ then $c(x) = 1$. To see that, suppose that $c(x) = 2$. Since at most k neighbors of x may be assigned color 2, there exists some i, $1 \leq i \leq 2k - 1$, such that $c(y_i) = c(y_{i+1}) = 1$. But in that case, if $c(z_i) = 1$ we have a monochromatic triangle and if $c(z_i) = 2$ we have an alternating bichromatic cycle, a contradiction. Consider now the outerplanar graph $H_{2,k}$ obtained by taking three disjoint copies of $G_{2,k}$ and joining their corresponding x-vertices in a triangle (see Figure 1(c)). At least one of the x-vertices must be assigned color 2 and thus $H_{2,k}$ cannot be acyclically $(\mathcal{T}_1, \mathcal{D}_k)$-colored.

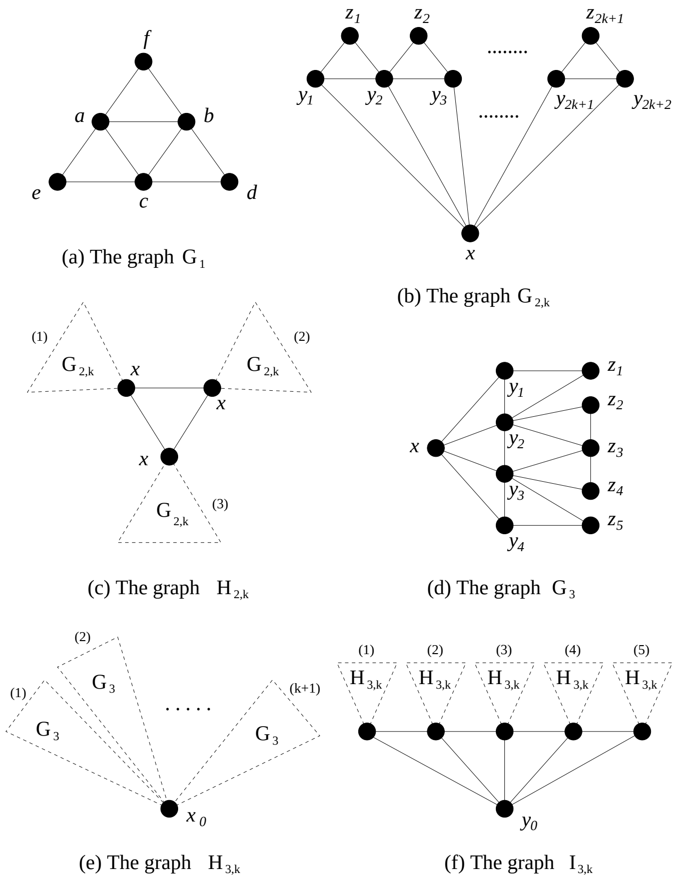


Figure 1: Impossibility results for outerplanar graphs.

3. Let c be an acyclic $(\mathcal{D}_4, \mathcal{D}_k)$-coloring of the graph G_3 depicted in Figure 1(d). We first claim that we cannot have $c(x) = 2$ and $c(y_i) = 1$ for every i, $1 \leq i \leq 4$. If this is the case then $c(z_1) = c(z_3) = c(z_5) = 1$, otherwise there would be an alternating bichromatic cycle. Thus vertices y_2 and y_3 both have four neighbors with color 1 and we must have $c(z_2) = c(z_4) = 2$ which creates an alternating bichromatic 6-cycle. Now let $H_{3,k}$ be the outerplanar graph obtained by taking $k+1$ disjoint copies of G_3 and identifying their corresponding x-vertices into one unique vertex x_0 (see Figure 1(e)). Let c be an acyclic $(\mathcal{D}_4, \mathcal{D}_k)$-coloring of $H_{3,k}$. We claim that $c(x_0) = 1$. If not, then at most k y-vertices have color 2, so that at least one of the $k+1$ copies of G_3 has all its y-vertices of color 1, a contradiction as discussed above. Taking five distinct copies of $H_{3,k}$, we now construct the outerplanar graph $I_{3,k}$ as shown in Figure 1(f). If c is an acyclic $(\mathcal{D}_4, \mathcal{D}_k)$-coloring of I_3 then $c(y_0) = 2$, since otherwise at least one copy of $H_{3,k}$ would have a x-vertex with color 2, a contradiction. Finally, by taking one copy of $H_{3,k}$, one copy of $I_{3,k}$ and identifying their corresponding x_0- and y_0-vertices, we obtain an outerplanar graph that cannot be acyclically $(\mathcal{D}_4, \mathcal{D}_k)$-colored. $\square$

The main result of this section is the following:

Theorem 3 *$\mathcal{O} \subseteq \mathcal{D}_5 \odot \mathcal{D}_5$.*

An inductive proof of this result can be obtained from a theorem of Borodin and Woodall [3] which states that every maximal outerplanar graph contains a triangle of a special type. We give here a more constructive proof from which we can easily derive a linear coloring algorithm.

Proof. Let G be an outerplanar graph. W.l.o.g. we assume that G is maximal outerplanar and consider a fixed outerplane embedding of G. Let x_0 be any vertex in G. For every $i \geq 0$ let W_i be the

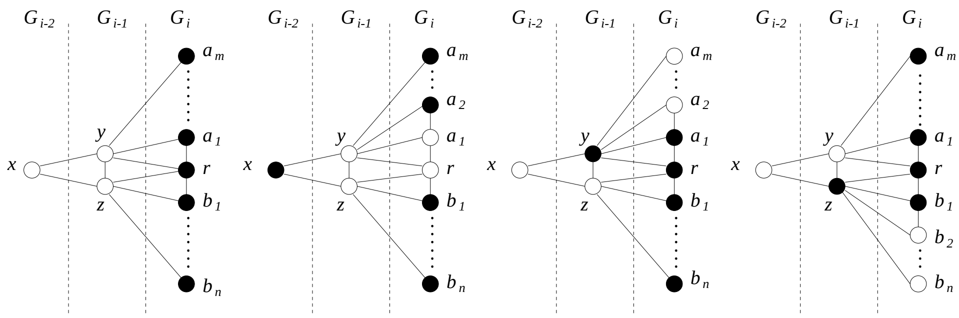


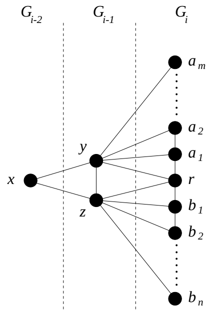
Figure 2: Configurations of Rule R1.

set of vertices which are at distance i from x_0 in G. If d denotes the maximum distance between x_0 and any other vertex in G then $(W_0, W_1, \dots, W_d)$ defines a partition of $V(G)$. For every i, $0 \leq i \leq d$, we denote $G[W_i]$ by G_i. The following simple facts are easy to prove:

Fact 1. *For every i, $0 \leq i \leq d$, G_i is a union of disjoint paths.*

Fact 2. *Each vertex in G_i is adjacent to vertices in at most two paths in G_{i+1}.*

Fact 3. *For every i, $2 \leq i \leq d$, every maximal path $P = a_m a_{m-1} \dots a_1 r b_1 b_2 \dots b_n$ in G_i (as seen from x_0 in clockwise order) is linked to G_{i-1}, G_{i-2} as follows:*



The unique vertex r in P having two neighbors in G_{i-1} will be called the *root* of P, the triangle xyz will be called the *basis* of P, and the vertex x will be called the *head* of the basis. For any color α in $\{1, 2\}$, we denote by $\overline{\alpha}$ the color $3 - \alpha$. We now inductively define a coloring c of G using the following rules:

Rule R0. We set $c(x_0) = 1$ and $c(x) = 2$ for every $x \in G_1$.

Rule R1. Let $P = a_m \dots a_1 r b_1 \dots b_n$ be a maximal path in G_i, $i > 1$, with root r and basis xyz. We then set:

(1) $c(a_2) = \dots = c(a_m) = \overline{c(y)}$

(2) $c(b_2) = \dots = c(b_n) = \overline{c(z)}$

(3) $c(a_1) = c(r) = \overline{c(x)}$

(4) $c(b_1) = \alpha$ where α is the color that appears at most once on the basis.

The different configurations corresponding to Rule R1 are depicted in Figure 2.

We claim that the coloring c thus obtained is such that each color class induces a graph with maximum degree at most 5 and that c produces no alternating bichromatic cycle. Let a *j-neighbor* be a neighbor with color j and a *j-basis* be a basis whose head has color j.

Observe first that by Rule R0, vertex x_0 has no 1-neighbor. Now let u be any vertex in G_i, $1 \leq i \leq d$ and assume w.l.o.g. that $c(u) = 1$. Let $\lambda(u)$ denote the number of 1-neighbors of u. Observe first that all the neighbors of u belong to G_{i-1}, G_i or G_{i+1}. The vertex u has at most two 1-neighbors in G_i (by Fact 1), at most two in G_{i-1} (by Fact 3) and at most $2 \times 2 = 4$ in G_{i+1} (by Fact 2 and Rule R1). Thus $\lambda(u) \leq 8$. We now prove that $\lambda(u) \leq 5$.

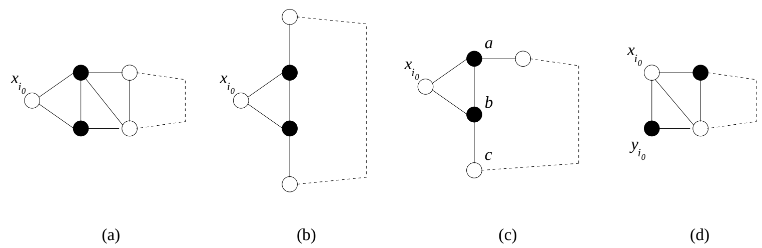
1. If u does not belong to a 2-basis then u has no 1-neighbor in G_{i+1} (by Rule R1) and thus $\lambda(u) \leq 4$.

2. If u belongs to exactly one 2-basis then it has at most two 1-neighbors on the corresponding path in G_{i+1} and none on the other path. Moreover, it has at most one 1-neighbor in G_{i-1} and thus $\lambda(u) \leq 5$.

3. Finally, suppose that u belongs to two 2-bases (which means that u has no 1-neighbor in G_{i-1}). If u has a 2-neighbor in G_i, then it has at most two 1-neighbors on each path in G_{i+1} and thus $\lambda(u) \leq 5$. If both neighbors of u in G_i have color 1 then, by Rule R1, u has at most one 1-neighbor on one path in G_{i+1} and at most two on the other path. Thus $\lambda(u) \leq 5$.

Therefore, in each case, $\lambda(u) \leq 5$.

Now suppose that G contains an alternating bichromatic cycle C. From the coloring rules, we first observe that no three consecutive vertices of C can belong to the same G_i. Thus if i_0 is minimal such that G_{i_0} and C intersect, then G contains one of the following four configurations:



But by Rule R1, configurations (a), (b) and (d) cannot occur; and (c) also cannot occur since, if it did, then by considering the shortest paths from x_0 to x_{i_0} and from x_0 to c we would see that b did not lie on the outer face and so G was not outerplanar. Thus G contains no alternating bichromatic cycle and c is therefore an acyclic $(\mathcal{D}_5, \mathcal{D}_5)$-coloring of G. □

4 Planar graphs

The result of Borodin [2] states that every planar graph can be acyclically 5-colored. We are thus interested in acyclic improper colorings of planar graphs that use at most 4 colors.

The following negative result says that in every acyclic improper coloring of a planar graph with less than five colors in which each color class induces a subgraph with bounded treewidth or bounded degree, at least one color class has to induce a subgraph with treewidth at least 3. More precisely, we have:

Theorem 4 *There exist planar graphs having no acyclic $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4)$-partition where for every i, $1 \leq i \leq 4$, $\mathcal{P}_i = \mathcal{T}_2$ or $\mathcal{P}_i = \mathcal{D}_{k_i}$, $k_i \geq 0$.*

Proof. Consider the planar graph G depicted in Figure 3(a) whose (usual) acyclic chromatic number is 5. We define the planar graph G_p, $p > 0$, obtained from G by connecting to every edge xy a path $z_1 z_2 \ldots z_p$ as depicted in Figure 3(b), and let c be an acyclic $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4)$-coloring of G_p. In the following, the original vertices of G in G_p will be called *main* vertices.

Observe first that c induces a monochromatic edge linking two main vertices, say xy, with $c(x) = c(y) = i$. Now consider the path $z_1 z_2 \ldots z_p$ connected to xy. Since c is acyclic, all the colors but i can be used at most once on this path. Thus, $p-3$ vertices of this path have to be of color i. If $p \geq 8$ then two consecutive vertices z_j and z_{j+1} have color i and the subgraph induced by color i, which contains a K_4, has treewidth at least 3. If $p \geq k_i + 3$ then the subgraph induced by color i has degree at least $k_i + 1$. This concludes the proof. □

There is a striking dichotomy between Theorem 4 and the result of Borodin [2]: to acyclically color a planar graph in such a way that each color class induces a graph with bounded degree we have to use at least five colors, but in that case an acyclic proper coloring can be provided.

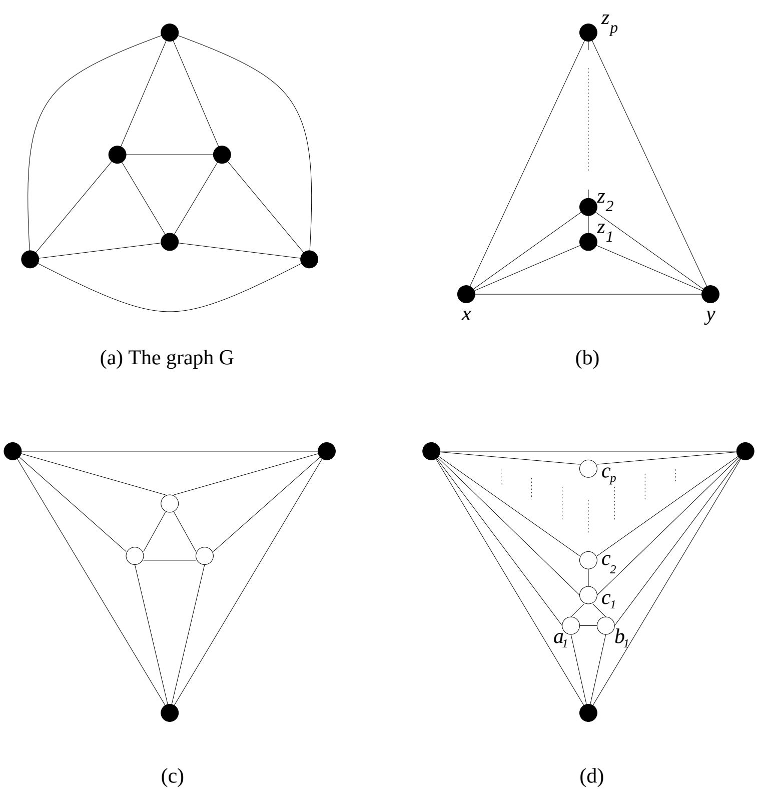


Figure 3: Impossibility results for planar graphs.

Concerning acyclic improper colorings of planar graphs with two colors we can prove the two following results:

Theorem 5 *There exist planar graphs having no acyclic $(\mathcal{T}_3, \mathcal{T}_3)$-partition.*

Proof. Consider again the graph G depicted in Figure 3(a). Since G is 4-regular, G is not a partial 3-tree. Thus every $(\mathcal{T}_3, \mathcal{T}_3)$-coloring of G must use two colors. Moreover, in every *acyclic* $(\mathcal{T}_3, \mathcal{T}_3)$-coloring of G one of the two colors has to be used exactly once, leading to 4 monochromatic triangles (or faces).

Consider now the graph G' obtained from G by inserting into each of the 8 faces of G a triangle connected to the vertices of the face as depicted by Figure 3(c) (every original face in G becomes in G' isomorphic to the graph G itself). Suppose that c is an acyclic $(\mathcal{T}_3, \mathcal{T}_3)$-coloring of G'. We still have 4 monochromatic triangles made of vertices originating from G. The second color has thus to be used on the four triangles included into these faces. It is then easy to check that we have an alternating bichromatic cycle, a contradiction. □

Theorem 6 *For every $k \geq 0$, there exist planar graphs having no acyclic $(\mathcal{T}_3, \mathcal{D}_k)$-partition.*

Proof. We consider again the graph G depicted in Figure 3(a). Let G'_p, $p > 0$, be the planar graph obtained from G by inserting a triangle $a_1b_1c_1$ and vertices $c_2, \ldots, c_p$ into each face of G as depicted by Figure 3(d). Using the same argument as before, we know that in every acyclic $(\mathcal{T}_3, \mathcal{D}_k)$-partition of G'_p we have 4 monochromatic triangles made of vertices originating from G. We have two cases to consider:

Case (i): These 4 triangles have color 1. For every $p \geq 1$, every corresponding inner triangle $a_1b_1c_1$ must use color 2, leading to an alternating bichromatic cycle as in the proof of the previous theorem.

Case (ii): These 4 triangles have color 2. For every $p > k-1$, at least two vertices c_i, c_j in each of these triangles must have color 1, leading again to alternating bichromatic cyles.

We thus get in each case a contradiction, which concludes the proof. □

Some questions concerning planar graphs remain open. Considering our original motivation, that is the study of oriented chromatic numbers of planar graphs, some questions are particularly challenging: is it true or not that every planar graph has an acyclic $(\mathcal{T}_3, \mathcal{D}_0, \mathcal{D}_0)$-partition ? an acyclic $(\mathcal{T}_3, \mathcal{D}_0, \mathcal{D}_1)$-partition ? A positive answer would imply that the oriented chromatic number of every planar graph is at most 72 or 76 respectively.

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References

[1] P. Boiron, E. Sopena and L. Vignal, Acyclic improper colorings of graphs with maximum degree three, *Dimacs Series in Discrete Math. and Theoret. Computer Sci.*, to appear.

[2] O.V. Borodin, On acyclic colorings of planar graphs, *Discrete Math.* **25** (1979), 211–236.

[3] O.V. Borodin and D.R. Woodall, Thirteen colouring numbers for outerplane graphs, *Bull. Inst. Comb. Appl.* **14** (1995), 87–100.

[4] M. Borowiecki and P. Mihók, Hereditary properties of graphs, In: V. R. Kulli, editor, Advances in Graph Theory. Vishwa International Publishers (1991), 41–68.

[5] B. Grünbaum, Acyclic coloring of planar graphs, *Israel J. Math.* **14** (1973), 390–412.

[6] T.R. Jensen and B. Toft, *Graph coloring problems*, Wiley Interscience (1995).

[7] A.V. Kostochka, E. Sopena and X. Zhu, Acyclic and oriented chromatic numbers of graphs, *J. Graph Theory* **24(4)** (1997), 331–340.

[8] A. Raspaud and E. Sopena, Good and semi-strong colorings of oriented planar graphs, *Inf. Processing Letters* **51** (1994), 171–174.

[9] E. Sopena, The chromatic number of oriented graphs, *J. Graph Theory* **25** (1997), 191–205.

[10] D.R. Woodall, Improper colourings of graphs, in: *Graph Colourings* (eds. R. Nelson and R.J. Wilson), vol. 218 of Pitman Research Notes in Mathematics Series (1990), 45–63.