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NILPOTENT FAMILIES OF ENDOMORPHISMS OF $(\mathcal{P}(V)^+, \cup)$

D. Fon-Der-Flaass[1]
Institute of Mathematics, Novosibirsk, Russia 630090

A.V. Kostochka[2]
Institute of Mathematics, Novosibirsk, Russia 630090 and University of Illinois at Urbana-Champaign, Urbana, IL, 61801

J. Nešetřil
Dept. of Appl. Math., Charles University, Prague, Czech Republic

A. Raspaud[3] **and E. Sopena**[4]
LaBRI, Universite Bordeaux I, 33405 Talence Cedex, France

Abstract. An oriented graph G is *nice* if there exists a positive integer k such that for every two vertices u, v (allowing $u = v$), and for every orientation of edges of the path of length k, there exists a walk of length k in G beginning at u and ending at v whose orientation of edges coincides with the given one. (Such a graph is also called k-nice.) We generalize this notion using the notion of a nilpotent semigroup of endomorphisms of $(\mathcal{P}(V)^+, \cup)$, and consider two basic problems:
(1) find bounds for the nilpotency class of such semigroups in terms of their generators (in graph–theoretical language: provided that a graph G on n vertices is nice, find the smallest k such that G is k-nice);
(2) find a way to demonstrate non-nilpotency of such semigroups (find as simple as possible characterization of non-nice graphs).

Keywords. Nice graphs, Nilpotent semigroup of endomorphisms.

1 Introduction

The notion of a nice graph first was implicitly used in the papers [1, 2, 5] as a useful tool for studying oriented chromatic number of graphs. Later, in [3], nice graphs were studied for their own sake, and some further generalizations were introduced.

An oriented graph G is called *k-nice* if for every two vertices u, v (allowing $u = v$), and for every orientation of edges of the path of length k, there exists a walk of length k in G beginning at u and ending at v whose orientation of edges coincides with the given one. An oriented graph G is *nice* if it is k-nice for some positive integer k (Note that every k-nice graph is $(k+1)$-nice). Similarly, a non-oriented (multi)graph G whose edges are coloured by c colours is called *k-nice* if for every two vertices u, v (allowing $u = v$), and for every edge colouring of the path of length k, there exists a walk of length k in G beginning at u and ending at v whose colouring coincides with the given one. Also, such a (multi)graph is *nice* if it is k-nice for some positive integer k.

In [3], characterizations of non-nice graphs in terms of so called 'black holes' (see the next section) were found. It was proved that in every non-nice graph there exist black holes of relatively simple structure (especially simple for non-coloured oriented graphs).

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In this paper we show that "niceness" of graphs is a special case of a more general and natural notion of nilpotency of semigroups of endomorphisms of the upper semilattice $(\mathcal{P}(V)^+, \cup)$ of all non-empty subsets of a finite set V of size n. In fact, this notion corresponds to nice edge-coloured oriented (multi)graphs.

In Section 3 we establish some bounds for the nilpotency class of such semigroups in terms of their generators, and the size n of the ground set. We show that the nilpotency class can be exponentially large if the number of generators is n, and give a polynomial in n upper bound on the nilpotency class when the generators have a special form related to the original graph-theoretical formulation: given a nice (oriented or simple multicoloured) graph on n vertices, what is the smallest k for which it is k-nice? As a corollary we get polynomial polynomial in n upper bounds on k for non-coloured oriented nice graphs and 2-edge-coloured non-oriented (multi)graphs.

In Section 4 we study the structure of shortest words in generators of a non-nilpotent semigroup, which demonstrate its non-nilpotency; that is, which have a black hole. We prove that there are no restrictions on these words; except that such a word cannot be a composite power.

Finally we pose some problems and conjectures.

2 Nilpotent families

Let V be a finite set, $|V| = n$. By $\mathcal{P}(V)^+$ we denote the set of all non-empty subsets of V. We consider endomorphisms of the semi-lattice $(\mathcal{P}(V)^+, \cup)$, that is, mappings $\varphi : \mathcal{P}(V)^+ \to \mathcal{P}(V)^+$ satisfying the identity

$$\varphi(X \cup Y) = \varphi(X) \cup \varphi(Y).$$

Every such mapping φ is uniquely determined by the values it takes on one-element subsets:

$$\varphi(X) = \cup_{x \in X} \varphi(\{x\}).$$

It is often convenient to view such mappings as oriented graphs on the vertex set V in which (x, y) is an arc if and only if $y \in \varphi(\{x\})$. Occasionally we shall abuse the notation by writing $\varphi(x)$ instead of $\varphi(\{x\})$.

The set $M(V)$ of all such endomorphisms is a semigroup. It has a right zero element Ω, $\Omega(X) = V$ for all $X \in \mathcal{P}(V)^+$. Let $M_\Omega(V) \subset M(V)$ be the sub-semigroup of those elements φ for which $\varphi(V) = V$. Ω is the unique two-sided zero of $M_\Omega(V)$.

Let $\Lambda = \{\varphi_1, \ldots, \varphi_r\} \subseteq M_\Omega(V)$ be any collection of endomorphisms. By Λ^k we denote the set of all products $x_1 \ldots x_k$ of k elements from Λ. The collection Λ is called *nilpotent of class k*, or k-nilpotent, if $\Lambda^k = \{\Omega\}$ for some $k > 0$. This terminology is in accordance with traditional usage of the word "nilpotent". Obviously, if Λ is k-nilpotent then it is m-nilpotent for every $m > k$.

We say that $\emptyset \neq X \subseteq V$ is a *black hole* for φ if $\varphi(X) = X$; if $X \neq V$ then the black hole is called *non-trivial*. Note that if for some X we have $\varphi(X) \subseteq X$ then some non-empty subset of $\varphi(X)$ is a black hole. In particular, if $\varphi(V) \neq V$ then φ has a non-trivial black hole.

It is easy to see that Λ is nilpotent if and only if no composition of mappings from Λ has a black hole. If Λ is nilpotent then, starting from any non-empty subset $X \subseteq V$ and applying mappings from Λ in any sequence, we eventually shall reach the whole set V, and in not more than $2^n - 2$ steps (because each proper subset of V can appear at most once; otherwise it would be a black hole for some composition of mappings from Λ). Thus, every nilpotent family is $2^n - 2$-nilpotent.

For some nilpotent collections $2^n - 2$ steps may be necessary. Here is an example. Let $X_1, \ldots, X_{2^n-1}$ be any linear extension of $\mathcal{P}(V)^+$ ordered by inclusion. For $i = 1, \ldots, 2^n - 2$ the mapping φ_i is defined by $\varphi_i(x) = X_{i+1}$ if $x \in X_i$, $\varphi_i(x) = V$ if $x \notin X_i$. Then the system $\Lambda = \{\varphi_1, \ldots, \varphi_{2^n-2}\}$ is nilpotent, and it takes exactly $2^n - 2$ steps to transform X_1 into $X_{2^n-1} = V$ if we take the mappings in their order. Later, in Example 3, we shall see that by being more careful we can obtain the same number of steps for a nilpotent family of only n different mappings.

3 Bounds for nilpotency class

In this section we shall try to obtain better upper bounds for the nilpotency class when the collection Λ is small and/or satisfies some additional properties.

We say that a mapping ψ is *opposite to* φ if $y \in \varphi(\{x\})$ exactly when $x \in \psi(\{y\})$; and that φ is *symmetric* if it is opposite to itself. These notions naturally arise in the graph-theoretical setting: taking the opposite of a mapping corresponds to inverting all arcs of an oriented graph; and symmetric mappings correspond to simple (non-oriented) graphs.

A mapping φ is called *increasing* if $X \subseteq \varphi(X)$ for every $X \subseteq V$. Note that if φ is increasing and nilpotent then it is strictly increasing; i.e. X is a proper subset of $\varphi(X)$. Also, if φ^+, φ^- are opposite then $\varphi^+\varphi^-$ is increasing.

Proposition 1 *Let $\Lambda = \{\varphi\}$. If Λ is nilpotent then it is $n(n-1)$-nilpotent. If, moreover, φ is symmetric then Λ is $(2n-2)$-nilpotent.*

Proof. Take a longest sequence $X_0, X_1, \ldots, X_m$ such that $X_{i+1} = \varphi(X_i)$ and $X_m \neq V$, and suppose, by way of contradiction, that $m \geq n(n-1)$. We can assume that $X_0 = \{x\}$ is a one-element set. In the oriented graph corresponding to φ the vertex x lies in some oriented cycle (otherwise the set $\cup_{i=1}^{\infty}\varphi^i(x)$ would be a non-trivial black hole). So, for some $k \leq n$ we have $x \in X_k$, or $X_0 \subseteq X_k$. The mapping φ preserves the relation of one set being a subset of another; therefore, applying it several times, we obtain that $X_j \subseteq X_{j+k}$ for every j. In particular,

$$X_0 \subseteq X_k \subseteq X_{2k} \subseteq \ldots \subseteq X_{(n-1)k}.$$

All inclusions in this chain are strict (otherwise we would have a black hole); and so $|X_{(n-1)k}| \geq n$. Thus, $X_{(n-1)k} = V$, contrary to our assumption about m. The first part is proved.

If φ is symmetric then φ^2 is an increasing mapping; so the above argument holds with the value $k = 2$, which proves the second claim. □

After submitting this present paper, the authors have learned that both bounds of this proposition have been previously obtained by other people in a different language. The general case was first studied by Wielandt in the paper [6]. There, the exact upper bound of $n(n-2)$ on the nilpotency class was obtained. The (exact) bound $2n-2$ for the symmetric case was obtained in the paper [4].

Proposition 2 *Let $\Lambda = \{a, b, c_1, \ldots, c_k\}$ be a nilpotent collection such that the mappings ab, ba, and all c_i are increasing. Then Λ is n^3-nilpotent.*

Proof. Take an arbitrary sequence $(f_1, f_2, \ldots, f_m)$ of mappings from Λ, an arbitrary non-empty $X_0 \subseteq V$, and let $X_i = f_i(X_{i-1})$ for $i = 1, 2, \ldots, m$. To each set X_i we assign a level, an integer value l_i, defined as follows:

$l_0 = 0$;
if $f_i = a$ then $l_i = l_{i-1} + 1$;
if $f_i = b$ then $l_i = l_{i-1} - 1$;
if $f_i = c_j$ then $l_i = l_{i-1}$.

Now we shall prove two claims, from which the proposition will immediately follow. Let $0 \leq p < q \leq m$.

Claim 1. If $l_q = l_p$ and $X_p \neq V$ then X_p is a proper subset of X_q.

We shall prove this claim by induction on $q - p$. If $q - p = 1$ then $f_q = c_j$ is increasing, as required. If $l_{p+1} \neq l_{q-1}$ then there is an r such that $p < r < q$ and $l_r = l_p$, and we apply the induction hypothesis to (p, r) and (r, q). Finally, let $l_{p+1} = l_{q-1} \neq l_p$. We can assume that $f_{p+1} = a$, $f_q = b$ (the other case is similar). By induction, we have $X_{p+1} \subseteq X_{q-1}$. So,

$$X_p \subset b(a(X_p)) = b(X_{p+1}) \subseteq b(X_{q-1}) = X_q,$$

and the claim is proved. (Here the first inclusion is proper since X_p is not a black hole.)

Claim 2. If $|l_q - l_p| \geq n(n-1)$ then $X_q = V$.

Let $l_q = l_p + n(n-1)$; the other case is treated similarly. For $i = 0, \ldots, n(n-1)$ let p_i be the smallest index such that $p \leq p_i \leq q$ and $l_{p_i} = l_p + i$. In particular, $p_0 = p$. For every $i = 1, \ldots, n(n-1)$ we have $l_{p_i - 1} = l_p + i - 1 = l_{p_{i-1}}$. So, by Claim 1, we have $X_{p_{i-1}} \subseteq X_{p_i - 1}$; and $a(X_{p_{i-1}}) \subseteq a(X_{p_i - 1}) = X_{p_i}$. All these inclusions together imply that $a^{n(n-1)}(X_p) \subseteq X_{p_{n(n-1)}}$, and by Proposition 1, $X_{p_{n(n-1)}} = V$. The claim is proved.

Now, if some value of the level is assigned to n or more sets then Claim 1 implies that the last of these sets is equal to V. On the other hand, if we have more than $n(n-1)$ different values of the level, Claim 2 implies that we have reached V. Therefore we shall reach V after at most $n(n-1)(n-1)$ steps. □

Proposition 3 *Let* $\Lambda = \{\varphi_1, \varphi_2\}$*. If* Λ *is nilpotent and either both* φ_i *are symmetric, or they are opposite to each other, then* Λ *is* $(c \cdot n^3)$*-nilpotent.*

Proof. When φ_1 and φ_2 are opposite, Proposition 2 applies immediately, and Λ is n^3-nilpotent.

When both mappings are symmetric, define four new mappings: $a = \varphi_1\varphi_2$, $b = \varphi_2\varphi_1$, $c_1 = \varphi_1\varphi_1$, and $c_2 = \varphi_2\varphi_2$. It is easy to check that these mappings satisfy the conditions of Proposition 2. Now, every sequence of $2N$ mappings φ_i can be considered as a sequence of N mappings a, b, c_i. Therefore Λ is $2n^3$-nilpotent. □

The case of φ_1 and φ_2 being opposite corresponds precisely to the original notion of nice graphs [3]. Thus, Proposition 3 in this case asserts that "every nice graph on n vertices is n^3-nice".

We now give some examples. We do not have examples which would give lower bounds comparable with upper bounds from Propositions 2 and 3. The next two examples provide such bounds for Proposition 1. In all examples, we let $V = \{0, 1, \ldots, n-1\}$.

Example 1. Let the mapping a correspond to the path $0 - 1 - \ldots - (n-1)$ with a loop at the vertex $n-1$; this mapping is symmetric. The single-element collection $\{a\}$ is nilpotent, and if we start with the set $X = \{0\}$, we reach V only after $2n-2$ steps.

Example 2. Now, let a correspond to the oriented graph formed from the oriented cycle $0 \to 1 \to \ldots \to n-1 \to 0$ and an extra arc $(n-1) \to 1$. Again, $\{a\}$ is nilpotent, and if we start with the set $X = \{0\}$, we reach V only after $n(n-2)$ steps.

It was proved in [6] that this is the only mapping which attains this bound.

Example 3. For each $v \in V$ define the mapping a_v as follows: $a_v(\{v\}) = V$, $a_v(\{x\}) = \{v\}$ if $x < v$, $a_v(\{x\}) = \{x, v\}$ if $x > v$. All these mappings are symmetric. Let $X_1, \ldots, X_{2^n - 1}$ be the lexicographic ordering of $\mathcal{P}(V)^+$; each X_i is the set of positions at which the binary expansion of i has ones.

To make sure that the collection $\{a_0, \ldots, a_{n-1}\}$ is nilpotent, it is enough to check that in this ordering $a_i(X) > X$ for every i and every $X \subset V$. This is straightforward.

On the other hand, for $1 \leq i < 2^n - 1$, let $z(i)$ be the position of the first zero in the binary expansion of i; or, equivalently, $z(i) = \min\{j \mid j \in V \setminus X_i\}$. Now it is straightforward to check that $a_{z(i)}(X_i) = X_{i+1}$.

Thus, the collection $\{a_0, \ldots, a_{n-1}\}$ is nilpotent of class $2^n - 2$.

4 Black holes

Here we consider families of endomorphisms which are not nilpotent. This means that some element of the semigroup generated by the family has a black hole. So, we say that the family Λ has a black hole X with a pattern w if w is a word over the alphabet Λ, and $X \subset V$ is a black hole of the endomorphism corresponding to this word.

A non-nilpotent family can have many different patterns of black holes; in particular, every power of a pattern is again a pattern. So, the question arises: what can be said about the shortest pattern of a black hole of a non-nilpotent family?

Proposition 4 *Let Λ be a finite alphabet, and w an arbitrary word over it. The following statements are equivalent:*

(1) *w cannot be represented in the form v^n for a composite number n.*

(2) *There exists a set V and a collection of mappings denoted by elements of Λ such that w is a shortest pattern of any black hole of this collection.*

Proof.

(1) $\to$ (2). Let $w = x_0x_1 \ldots x_{N-1}$ (indices are considered modulo N, in order that w could be treated as a cyclic word). We take $V = \{0, 1, \ldots, N-1\}$. Elements of V are also considered modulo N. For any subset $X \subseteq V$, we shall denote by $X + i$ the set $\{x + i \mid x \in X\}$ (addition modulo N), the cyclic shift of X.

To define the mappings, we fix a number k, to be specified later. We shall define the mappings in such a way that every set of k consecutive elements of V (interval of length k) shall be a black hole having some cyclic shift of w as a pattern.

For each $v \in V$, consider the subword $x_{v-k+1} \ldots x_{v-1}x_v$ of the cyclic word w. For any $x \in \Lambda$, let i and j be the first and the last, respectively, of the indices within this interval for which $x_i = x_j = x$. Define $x(\{v\}) = \{j+1, \ldots, i+k\}$. If none of the letters of this interval is equal to x, we set $x(\{v\}) = V$.

We always have that $v + 1 \in x(\{v\})$; therefore for every subset $A \subseteq V$ we have $|x(A)| \geq |A|$, and if $|x(A)| = |A|$ then $x(A) = A + 1$.

Let $I = \{v, v+1, \ldots, v+l-1\}$ be an arbitrary (cyclic) interval of length $l < N$. Its image $x(I)$ under any mapping $x \in \Lambda$ is again an interval, because the image of each element is an interval and these intervals respectively contain consecutive elements $v+1, \ldots, v+l$. It follows that $x(I) = I + 1$ if and only if $v \notin x(I)$ and $v + l + 1 \notin x(I)$. From the definition of x, we have:

$v \notin x(v)$ if and only if $x_v = x$;

$v + l + 1 \notin x(v + l - 1)$ if and only if $x_{v+l-k} = x$.

Conversely, if these two conditions are satisfied then $x(I) = I + 1$; unless there is a subinterval of I of length k containing no letter x — then $x(I) = V$.

Let A be an arbitrary subset of V. A is the union of disjoint intervals; let their initial vertices be $v_1, \ldots, v_m$, and their lengths $l_1, \ldots, l_m$. From the above remarks we have that $|x(A)| = |A|$ for at most one mapping x, and if x is such then $x_{v_i} = x_{v_i+l_i-k} = x$ for all $i = 1, \ldots, m$.

Thus every interval of length k is a black hole whose pattern is uniquely determined, and this pattern is a cyclic shift of w. On the other hand, if a black hole contains an interval of length $l \neq k$, or two disjoint intervals with initial vertices v_1, v_2, then the word w is periodic with period $|l - k|$, resp. $|v_1 - v_2|$.

Thus, if the word w is not periodic, we can choose $k = 1$, so the only black holes of the resulting collection of mappings will be one-vertex subsets, and their shortest patterns will be cyclic shifts of w. If w is periodic, $w = w_0^p$ for a prime $p > 1$ and a non-periodic word w_0, then we can take $k = |w_0|$, and the resulting collection of mappings will satisfy the required property.

(2) $\to$ (1) Suppose the contrary. Let $w = u^{pq}$ for some word u and $p, q > 1$; suppose that w is a shortest pattern of a black hole, and let A be a black hole of minimum size with pattern w. Consider the sets $A_0 = A,\ A_1 = u(A), \ldots, A_{pq-1} = u^{pq-1}(A)$. These sets are all distinct, and all non-empty. We claim that they are also pairwise disjoint. Indeed, if $B = A_i \cap A_j \neq \emptyset$ then $u(B) \subseteq A_{i+1} \cap A_{j+1}$, etc. (indices taken modulo pq), and $w(B) \subseteq B$ — therefore, some non-empty subset of B is a black hole with pattern w, contrary to our choice of A.

But now we see that the sets

$$X_i = \cup_{j=0}^{q-1} A_{i+pj}$$

for $i = 0, \ldots, p-1$, form an orbit of length p of the mapping u, and so X_0 is a black hole with pattern u^q, contrary to our assumption about w. $\square$

Thus, there are no serious restrictions on shortest patterns of non-nilpotent families, apart from exception of composite powers.

We conclude this note with several open questions.

First, nothing is known about computational complexity of deciding whether a set of mappings is nilpotent or not. It is not even clear whether this problem belongs to NP and/or co-NP.

Second, Propositions 1 and 3 together with Example 3 lead to the following natural conjecture:

Conjecture 5 *For every natural k there exists an exponent $f(k)$ and a constant $c(k)$ such that every nilpotent family of k endomorphisms of the semilattice $(\mathcal{P}(n)^+, \cup)$ of all non-empty subsets of an n-element set has nilpotency class at most $c(k)n^{f(k)}$.*

This conjecture remains open even for $k = 2$.

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