k-DOMINATING SETS IN PLANAR GRAPHS WITH APPLICATIONS

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Abstract. A subset of nodes S in a graph G is called *k-dominating* if, for every node u of the graph, the distance from u to S is at most k. We consider the parameter $\gamma_k(G)$ defined as the smallest integer r such that G has a k-dominating set of cardinality r. For planar graphs, we show that for every $\epsilon > 0$ and for every $k \geq (\frac{5}{7} + \epsilon)D$, $\gamma_k(G) = O(1/\epsilon)$. For several classes of planar graphs of diameter D, we show that $\gamma_k(G)$ is bounded by a constant for $k \geq \frac{1}{2}D$. We conjecture that the same result holds for every planar graph. This problem is motivated by the design of routing schemes with compact data structures.

Keywords. Dominating sets, Planar graphs, Compact routing tables.

1 Introduction

Let G be a graph. We denote by $d_G(u,v)$ the distance in G from u to v, and by $\Gamma_k(u)$ the ball of radius k centered at u, i.e., $\Gamma_k(u) = \{v \in V(G) \mid d_G(u,v) \leq k\}$. We extend this notation to every subset $S \subseteq V(G)$, with $\Gamma_k(S) = \bigcup_{u \in S} \Gamma_k(u)$. We say that a subset of nodes $S \subseteq V(G)$ is *k-dominating set* (or, S k-dominates G, or S covers G) if $\Gamma_k(S) = V(G)$. Let $\gamma_k(G)$ denote the cardinality of the smallest k-dominating set of G. Our goal is to bound $\gamma_k(G)$ for various planar graphs G and values of k.

To illustrate the properties of the parameter $\gamma_k(G)$, let us consider a tree T of diameter D. It is clear that if $k \geq \frac{1}{2}D$, then $\gamma_k(T) = 1$. It suffices to consider the center of T. On the other hand, if $k < \frac{1}{2}D$ then there exists some n-node tree T_0 of diameter D for which $\gamma_k(T_0) \geq 2(n-1)/D$. For instance, consider T_0 composed of a star $K_{1,p}$ with each edge subdivided into $q-1$ nodes, where $n = pq+1$, $D = 2q$ and $\gamma_{q-1}(T_0) = p$. Every two leaves of this tree are at distance $2q$, thus p nodes of T_0 are required to $(q-1)$-dominate all the leaves.

More generally, for $k \geq \frac{1}{2}D$, every n-node graph G of diameter D satisfies $\gamma_k(G) < \sqrt{n(1+\ln n)}$. To see this, we use a dual characterization of a k-dominating set as a set S which *hits* the collection $\{\Gamma_k(u) \mid u \in V(G)\}$, i.e., such that $S \cap \Gamma_k(u) \neq \emptyset$ for every $u \in V(G)$. Now, note that for $k \geq \frac{1}{2}D$, $\Gamma_k(x) \cap \Gamma_k(y) \neq \emptyset$ for any two nodes x and y. Thus every set $\Gamma_k(x)$ is a k-dominating set for G. Hence, either there exists a node x such that $|\Gamma_k(x)| < \alpha$, where $\alpha = \sqrt{n(1+\ln n)}$, and we are done, or $|\Gamma_k(x)| \geq \alpha$ for every node x. In the latter case, the claim follows by a result of Lovász [6] about cover sets, which states that there exists a k-dominating set of G of size $\beta < n(1+\ln n)/\min_x |\Gamma_k(x)| \leq \sqrt{n(1+\ln n)}$.

For planar graphs, it is known that the size of a dominating set (i.e., 1-dominating set) is bounded by 3 if $D = 2$, and by 10 if $D = 3$, cf. [7]. Thus, for $k \geq \frac{1}{2}D$, and every planar graph G, $\gamma_k(G) \leq 3$ if $D = 2$, and $\gamma_k(G) \leq 10$ if $D = 3$.

A recent result concerns planar triangulations. A triangulation of the plane is a planar graph with an embedding on the plane such that each face, except maybe the outer-face, is a triangle. In [1] it is shown that every planar triangulation G in which every internal node (i.e., that does not belong to the outer face) has degree at least 6, satisfies $\gamma_k(G) \leq 2$, again for $k \geq \frac{1}{2}D$.

Section 2 presents our main results. For planar graphs, we show that for every $\epsilon > 0$ and for every $k \geq (\frac{5}{7} + \epsilon)D$, $\gamma_k(G) = O(1/\epsilon)$. For outerplanar graphs of diameter D, we show that $\gamma_k(G) \leq 2$ for $k\ ge\frac{1}{2}D$. We conjecture that $\gamma_k(G)$ is bounded by a constant for every planar graph of diameter D for $k \geq \frac{1}{2}D$.

The motivation for studying this parameter stems from the design of routing schemes with compact data structures. In particular, it is shown in Section 3 that for every graph G of diameter D and every $k \geq 0$, G has an interval routing scheme [8, 11] with dilation (i.e., the length of the longest route) at most $D + k$ and compactness (i.e., the number of intervals by arc) at most $\frac{1}{2}\gamma_k(G) + 1$. Moreover, the compactness upper bound can be reduced to $\frac{1}{4}\gamma_k(G) + o(\gamma_k(G))$, if $\gamma_k(G)/\log n \to +\infty$. This result improves (by a multiplicative factor) and generalizes compactness vs. dilation trade-offs of [5] and of [10]. Our result also implies that for every constant $\epsilon > 0$, every planar graph of diameter D has an interval routing scheme with dilation at most $\left\lceil (\frac{12}{7} + \epsilon)D \right\rceil$ and constant compactness.

2 Plane Graphs

2.1 Preliminaries

Hereafter, we assume that G is of diameter D. The following basic property is important for the remaining of the paper.

Proposition 1 *Let $S \subset V(G)$, let $k \geq \frac{1}{2}(D-1)$, and let $u, v \notin \Gamma_k(S)$. Then, every shortest path between u and v does not cross any node of S.*

Proof. Assume that there is some shortest path from u to v that contains a node $w \in S$. Then, $d_G(u,v) = d_G(u,w) + d_G(w,v)$. Since $u, v \notin \Gamma_k(S)$, $u, v \notin \Gamma_k(w)$, thus $d_G(u,w) \geq k+1$ and $d_G(v,w) \geq k+1$ that implies that $d_G(u,v) \geq 2k+2 \geq D+1$, a contradiction. □

Let $(\mathcal{E}, d)$ be a metric space (i.e., with d satisfying the triangle inequality). A subset $S \subseteq \mathcal{E}$ is *connected* if there exists a path between any two pair of points of S. A *separator* of a space $(\mathcal{E}, d)$ is a subset $S \subseteq \mathcal{E}$ such that $\mathcal{E} \setminus S$ is composed of two or more connected components. A separator in the graph G is a separator of the space $(V(G), d_G)$.

Property 1 as an immediate corollary, keeping in mind that every path joining two nodes of different connected components in $G \setminus S$, for some separator S, has to cross S.

Corollary 2 *Let $k \geq \frac{1}{2}(D-1)$, let S be a separator of G, and let $U = V(G) \setminus \Gamma_k(S)$. Then, there exists a connected component C of $G \setminus S$ such that $U \subseteq V(C)$.*

2.2 Outerplanar Graphs

Recall that a graph H is a *minor* of a graph G if H can be obtained from G by a sequence of zero or more node deletions, edge deletions or edge contractions. Let us denote by $K_{p,q}$ the complete bipartite graph with p nodes in one partition and q nodes in the other one.

We prove the following:

Theorem 3 *If G is a graph with diameter D such that G contains no $K_{2,t+1}$ as a minor then $\gamma_k(G) \leq t$, for every $k \geq \frac{1}{2}D$.*

Proof. Let u and v be any two nodes in G with $d_G(u,v) = D$. Let $V_i = \{x \in V(G) \mid d_G(u,x) = i\}$ for every i, $0 \le i \le D$. We call the *level* of any node x, denoted by $l(x)$, the (unique) subscript i such that $x \in V_i$. Let M be the set of nodes $x \in V_k$ such that there exists a path $xx_1x_2 \ldots x_\ell v$ from x to v satisfying $l(x_i) > k$ for every i, $1 \le i \le \ell$.

We first claim that the set M has cardinality at most t. Indeed, if M contains $t+1$ distinct nodes then G contains $K_{2,t+1}$ as a minor, a contradiction. Consider the graph H composed of the $t+1$ nodes and the $2(t+1)$ paths linking M to u, v, delete the edges and nodes of $G \setminus H$. Then it is easy to see that by contractions we will obtain a $K_{2,t+1}$.

Let now $P = uu_1u_2 \ldots u_pv$ be any path linking u and v and denote by q the largest subscript such that $l(u_q) = k$. Clearly, $l(u_i) > k$ for every i, $q < i \le p$, and thus $u_q \in M$. Hence every path linking u and v crosses the set M which, therefore, is a separator in G. (In particular, u and v belong to different connected components of $G \setminus M$).

Suppose that there exists a node x in G such that $d_G(x, M) > k$. As observed above, either u and x, or v and x do not belong to the same connected component of $G \setminus M$. In the former case, we get

$$d_G(u,x) \;\ge\; d_G(u,M) + d_G(x,M) \;>\; 2k \ge D$$

and in the latter case,

$$d_G(v,x) \;\ge\; d_G(v,M) + d_G(x,M) \;>\; D - k + k = D\ .$$

Therefore, either $d_G(u,x)$ or $d_G(v,x)$ is strictly greater than D, a contradiction.

We thus get that M k-dominates G, which concludes the proof. □

Since every outerplanar graph does not contain $K_{2,3}$ as a minor, we get in particular the following:

Corollary 4 *If G is an outerplanar graph of diameter D then $\gamma_k(G) \le 2$, for every $k \ge \frac{1}{2}D$.*

2.3 Planar Graphs

We assume from now that G is a plane graph, that is a planar graph with an embedding in $\mathcal{R}^2$. More precisely, the nodes are points of $\mathcal{R}^2$ and edges are simple curves that can cross or meet on the nodes only. A connected subset of $\mathcal{R}^2$ is called a *region* of the plane. Given a subgraph H of G and a point $w \notin H$ (i.e., the point $w \in \mathcal{R}^2$ does not belong to an edge or a node of H), we denote by $\mathrm{reg}(w,H)$ the unique region of $\mathcal{R}^2 \setminus H$ containing w. Note that if H is a tree, $\mathcal{R}^2 \setminus H$ consists of one region only. If $\mathcal{R}^2 \setminus H$ is composed of exactly two non-empty regions, we denote by $\overline{\mathrm{reg}}(w,H)$ the other region such that $\mathcal{R}^2 = \mathrm{reg}(w,H) \cup \overline{\mathrm{reg}}(w,H) \cup H$.

Given an integer $p \ge 3$, a *p-gon* w.r.t. G and k is a subgraph H of G defined by a sequence $(x_1, \ldots, x_p)$ of pairwise distinct nodes, and a sequence $(Q_1, \ldots, Q_p)$ of paths such that the four following conditions hold:

1. Q_i is a shortest path from x_i to $x_{(i \bmod p)+1}$, for every $i \in \{1, \ldots, p\}$;
2. $\mathcal{R}^2 \setminus H$ is composed of at most two non-empty regions;
3. $x_p \notin \Gamma_k(Q_1 \cup \cdots \cup Q_{p-2})$;
4. $d(x_i, x_{(i \bmod p)+1}) > k$, for every $i \in \{1, \ldots, p\}$.

Roughly speaking, a p-gon consists of p shortest paths of length at least $k+1$ possibly sharing some edges. Fig. 1 represents a 4-gon H for a plane graph G of diameter $D = 4$ and with $k = 2$. Note that the x_i's may belong to different regions induced by H.

Given a path P and two nodes of P, u and v, we denote by $P[uv]$ the subpath of P between u and v.

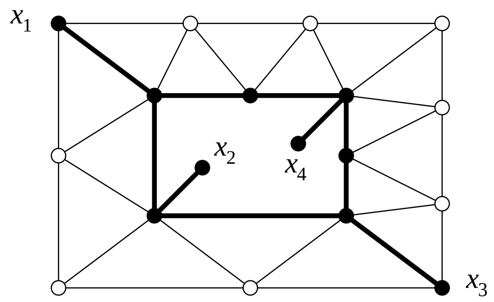

Figure 1: A plane graph G with a 4-gon H for $k = 2$ (bold edges).

Let $Q_1, \ldots, Q_p$ be a set of shortest paths, and, for each $i \in \{1, \ldots, p\}$, let $(z^i_1, \ldots, z^i_{t_i})$ be the sequence of nodes of $P \cap Q_i$ encountered in this order along a walk on P from u to v (possibly $t_i = 0$ if $P \cap Q_i = \emptyset$). The *simplified path of P on $Q_1 \cup \cdots \cup Q_p$* is the path define by

$$S \;=\; P[uz^1_1] \cup Q_1[z^1_1 z^1_{t_1}] \cup P[z^1_{t_1} z^2_1] \cup \cdots \cup Q_i[z^i_1 z^i_{t_i}] \cup P[z^i_{t_i} z^{i+1}_1] \cup \cdots \cup Q_p[z^p_1 z^p_{t_p}] \cup P[z^p_{t_p} v] \ .$$

Roughly speaking, S uses shortcuts of P on Q_1, then shortcuts of P on Q_2, and so on. Note that if P is a shortest path, then S is also a shortest path between u and v. (If $t_i = 0$ for all i, then $S = P$.) Fig. 2 shows the simplified path S of P on $Q_1 \cup Q_2$. Note that S is unique, but other shortcuts are possible: for instance $Q_1[uz^1_2] \cup P[z^1_2 z^2_1] \cup Q_2[z^2_1 v]$.

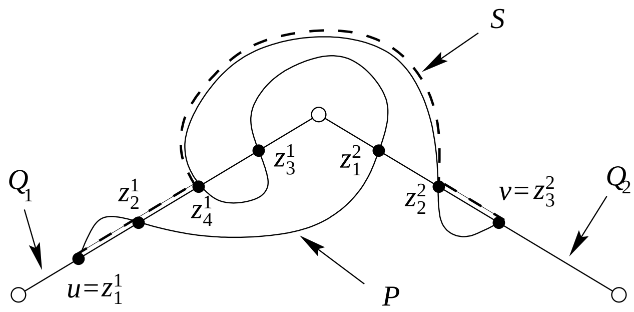

Figure 2: The simplified path S of P on $Q_1 \cup Q_2$.

For notation conveniences, we use $\Gamma_k(H)$ where H is a graph instead of $\Gamma_k(V(H))$.
The next lemma shows an important property of p-gons.

Lemma 5 *Let H be a p-gon of G with $p \le 4$, and with $k \ge \frac{1}{2}(D-1)$. Then, for every $w \notin \Gamma_k(H)$:*

1. *there exists a shortest path A from w to x_p wholly contained in* $\mathrm{reg}(w,H) \cup H$*;*
2. *there exists a shortest path B from w to x_i, $i < p$, wholly contained in* $\mathrm{reg}(w,H) \cup H$.

Moreover, A and B can be chosen such that the graph $H_1 = B \cup (Q_i \cup Q_{i+1} \cup \cdots \cup Q_{p-1}) \cup A$ and the graph $H_2 = B \cup (Q_{i-1} \cup Q_{i-2} \cup \cdots \cup Q_p) \cup A$ form respectively a $(p-i+2)$-gon and an $(i+2)$-gon of G.

Proof. Note that to prove Lemma 5 it suffices to construct a shortest path A from w to x_p, and a shortest path B from w to x_i, $i < p$, that are wholly contained in $\mathrm{reg}(w,H) \cup H$. Indeed, to form the gons H_1 and H_2 we consider the simplified path S of B on A, and replace B by S. Then we check that H_1 and H_2 induced each at most two regions of $\mathcal{R}^2$, and the four conditions of the definition of p-gons follows.

W.l.o.g. we assume that $\mathcal{R}^2 \setminus H$ is composed of two non-empty regions. Indeed, if $\overline{\mathrm{reg}}(w,H) = \emptyset$, then every shortest from w to x_i is wholly contains in $\mathrm{reg}(w,H) \cup H = \mathcal{R}^2$.

Let P be a shortest path from w to x_p. Let u be the closest node of H from w, and assume that $u \in Q_\ell$. If $\ell \in \{p, p-1\}$, then $A = P[wx] \cup Q_\ell[ux_p] \subseteq \mathrm{reg}(w,H) \cup H$, and we are done for A. So, in order to prove Point 1 assume that $\ell \notin \{p, p-1\}$. Consider $S = \bigcup_{i=1}^{p-2} Q_i$. Note that Q_ℓ is contained in S. Now, $w \notin \Gamma_k(S)$ and $x_p \notin \Gamma_k(S)$ (Condition 3 of the definition). By Proposition 1, P between w and x_p cannot intersect S: a contradiction since $1 \le \ell \le p-2$. Thus the path $A = P$ exists, and this completes the proof of Point 1.

To prove Point 2, we proceed by induction on p. We need the following definition: a p-gon H is said *weak* the conditions 3 and 4 does not necessary hold. The *extremity* of a p-gon (weak or not) is the point x_p. For simplicity, a weak 3-gon is called hereafter a *triangle*.

Claim 6 *Let T be a triangle defined by the nodes (x_1, x_2, x_3) and the shortest paths (Q_1, Q_2, Q_3). For every shortest path P between $w \notin T$ and $w' \in \mathrm{reg}(w,T) \cup T$, the simplified path S of P on $Q_1 \cup Q_2 \cup Q_3$, is such that: either $S \subseteq \mathrm{reg}(w,T) \cup T$, or $S \cap T$ contains two nodes u, u' such that $S[wu] \subseteq \mathrm{reg}(w,T) \cup T$, $S[uu'] \subseteq \overline{\mathrm{reg}}(w,T) \cup T$, and $S[u'w'] \subseteq \mathrm{reg}(w,T) \cup T$. Moreover, if $w' = x_i$, then $u \in Q_{(i \bmod p)+1}$ and $S[uu']$ induces at most two regions of $\overline{\mathrm{reg}}(w,T)$.*

Proof. Let P be a shortest path between w and w' simplified on $T = Q_1 \cup Q_2 \cup Q_3$. If $P \subseteq \mathrm{reg}(w,T) \cup T$, we are done with P. Assume that P intersects $\overline{\mathrm{reg}}(w,T)$, and let $z \in \mathcal{R}^2$ be the closest point from w that belongs to $\overline{\mathrm{reg}}(w,T)$. Let u be the node of T just before z from w. By construction, $P[wu] \subseteq \mathrm{reg}(w,T) \cup T$ is a shortest path. Similarly, define $z' \in \mathcal{R}^2$ as the closest point from w' that belongs to $\overline{\mathrm{reg}}(w,T)$, and u' the node of T just before z' from w'. By construction, $P[w'u'] \subseteq \mathrm{reg}(w,T) \cup T$ is a shortest path. Since $P[uu']$ is simplified on T, then we can check that $P[uu']$ does not intersect any point of $\mathrm{reg}(w,T)$, hence $P[uu'] \subseteq \overline{\mathrm{reg}}(w,T) \cup T$. Now, assume that $w' = x_i$. If $u \in (Q_i \cup Q_{(i-1) \bmod p})$, then P would be wholly contained in $\mathrm{reg}(w,T) \cup T$: a contradiction. Thus $u \in Q_{(i \bmod p)+1}$. Moreover, if $P[uu']$ induces more than two regions of $\overline{\mathrm{reg}}(w,T)$, then $P[uu']$ is not simplified on T. □

Claim 7 *Let T be a triangle defined by the nodes (x_1, x_2, x_3) and the shortest paths (Q_1, Q_2, Q_3). Then, for every $w \notin T$, there exists a shortest path wholly contained in $\mathrm{reg}(w,T) \cup T$ between w and x_1, or between w and x_2.*

Proof. Assume that no shortest paths between w to x_1 or x_2 is wholly contained in $\mathrm{reg}(w,T) \cup T$. For $i \in \{1,2\}$, let $P_i = P_i[wu_i] \cup P_i[u_ix_i]$ be a shortest path between w and x_i as described in Claim 6, i.e., such that $P_i[wu_i] \subseteq \mathrm{reg}(w,T) \cup T$ and $P_i[u_ix_i] \subseteq \overline{\mathrm{reg}}(w,T) \cup T$. By assumption, P_1 and P_2 are not wholly contained in $\mathrm{reg}(w,T) \cup T$. The paths $P_1[u_1x_1]$ and $P_2[u_2x_2]$ must intersect in $z \in \overline{\mathrm{reg}}(w,T) \cup T$. Thus the shortest paths $P_1[wz]$ and $P_2[wz]$ have the same length, and the shortest paths $P_1[u_1z] \cup P_2[zx_1]$ and $Q_2[u_1x_1]$ as well. Thus $P_1[wu_1] \cup Q_2[u_1x_1]$ is a shortest path wholly contained in $\mathrm{reg}(w,T) \cup T$: a contradiction. □

The construction of the path B for a 3-gon H follows from Claim 7. So, let us prove Point 2 for $p = 4$. Let X be a shortest path between x_1 and x_3. Let $u \in X \cap (Q_1 \cup Q_4)$ be the farest node from x_1. Consider Let $z \in \mathcal{R}^2$ be the point of X just after u. Note that $z \notin H$. Let $u' \in X \cap (Q_1 \cup Q_3)$ be the farest node from x_3. Note that the path $X[u'u'] \subseteq \mathrm{reg}(z,H)$. The path $Q_0 = X[x_1u] \cup X[uu'] \cup X[u'x_3]$ is a shortest path between x_1 and x_3 that is wholly contained in $\mathrm{reg}(z,H) \cup H$.

Consider the triangle T defines by the points (x_1, x_2, x_3) and the shortest paths (Q_1, Q_2, Q_0). By Claim 7, there exists a shortest path P_1 between w and x_1 or x_3 wholly contained in $\mathrm{reg}(w,T) \cup T$. W.l.o.g. assume that this path is between w and x_1. (The other case is symetric exchanging the role of x_1 and x_3.) However, we have two cases to consider.

Case 1: Q_0 is in $\mathrm{reg}(w,H) \cup H$, i.e., $\mathrm{reg}(z,H) = \mathrm{reg}(w,H)$.
Observe that $\mathrm{reg}(w,T) \subset \mathrm{reg}(w,H)$. Thus P_1 is also wholly contained in $\mathrm{reg}(w,H) \cup H$. We complete the proof setting $B = P_1$.

Case 2: Q_0 is in $\overline{\mathrm{reg}}(w,H) \cup H$, i.e., $\mathrm{reg}(z,H) = \overline{\mathrm{reg}}(w,H)$.
Consider the triangle T' defines by (x_1,x_3,x_4). W.l.o.g. assume that P_1 is simplified on T'. (Indeed, since $P_1 \subseteq \mathrm{reg}(w,T) \cup T$, the simplified path of P_1 on T' is also wholly in $\mathrm{reg}(w,T) \cup T$.) If P_1 is wholly contained in $\mathrm{reg}(w,H) \cup H$, then we are done setting $B = P_1$. By Claim 6, there exists $a \in P_1$ such that $P_1[wa] \subseteq \mathrm{reg}(w,H) \cup H$, and $P_1[ax_1] \subseteq \mathrm{reg}(w,T') \cup T'$. Now consider the triangle T'' defined by (x_2,x_3,x_1) that T by changing its extremity. By Claim 7 there exists a shortest path between w and x_2 of x_3 wholly contained in $\mathrm{reg}(w,T'') \cup T''$. W.l.o.g. we assume that the path is simplified with H, and is not wholly contained in $\mathrm{reg}(w,H) \cup H$.

Case 2.1: The path is between w and x_2, denoted by P_2.
Let u,u' be the two nodes defined by Claim 6 when applied between on P_2 w.r.t. the triangle T'. Assume $u' \in Q_3$, and consider the closed curve $C = P_2[wu'] \cup Q_3[u'a] \cup P_[aw]$. The path P_2 contains some points in $\mathrm{reg}(x_2,C) \cup C$ and some points in $\overline{\mathrm{reg}}(x_2,C) \cup C$, thus must intersects C in b. Note that $b \notin P_2[wu']$, since P_2 is a shortest path. Note also that $b \notin Q_3[u'a]$, since P_2 is simplified with Q_3. Thus P_2 intersects P_1 in α. Depending on whether $u' \in Q_3[x_4a]$ or not, the path $P_1[w\alpha] \cup P_2[\alpha x_2]$ or the path $P_1[wa] \cup Q_3[au'] \cup P_2[u'x_2]$ is a shortest path between w and x_2 wholly contained in $\mathrm{reg}(w,H) \cup H$. Remains the case, $u' \in Q_4$. We have to consider the shortest path P_3 between w and x_3, that is also assumed not wholly contained in $\mathrm{reg}(w,H) \cup H$, and simplified on H. By Claim 6, let v be the node of T' when considering the triangle T' and the shortest path P_3. The path P_3 and P_2 must intersects, and let β the closest intersection from w. Depending on whether $v \in Q_4[x_4u]$ or not, the path $P_3[wv] \cup Q_4[vu] \cup P_2[ux_2]$ or the path $P_3[w\beta] \cup P_2[\beta x_2]$ is a shortest path between w and x_2 wholly contained in $\mathrm{reg}(w,H) \cup H$.

Case 2.2: The path is between w and x_3, denoted by P_3.
The case is similar.

Therefore, in all the cases, we have constructed a shortest path between w and some x_i, $i < p$, that completes the proof. □

Before proving the main result of the paper, we will also need the following easy lemma.

Lemma 8 *Let Q be a path of q nodes. For every $\lambda \geq 0$, $\gamma_\lambda(Q) = \lceil q/(2\lambda+1) \rceil$.*

Proof. Let A be λ-dominating set of Q of minimal cardinatily. Since a node of A dominates at most $2\lambda+1$ nodes, we have $\gamma_\lambda(Q) \geq q/(2\lambda+1)$. On the other hand; it is possible to split the q nodes of Q into $\lceil q/(2\lambda+1) \rceil$ segments of $2\lambda+1$ consecutive nodes, the last segment may have less nodes. □

2.4 Main Result

The main result we want to prove is:

Theorem 9 *For every planar graph G of diameter D, and for every $k \geq (\frac{5}{7}+\epsilon)D$, for every $\epsilon > 0$, $\gamma_k(G) < 3/\epsilon + 6$.*

Actually, Theorem 9 derives from the following important result:

Theorem 10 *For every planar graph G of diameter D, and for every $k > \frac{5}{7}D - 1$, there exists a connected subgraph H of G such that $V(H)$ is a k-dominating set of G, H is composed of at most 6 shortest paths of G, and such that $|V(H)| \leq 6D - 1$.*

The proof of Theorem 10 will be given after the short proof of Theorem 9. Indeed, using Theorem 10, it suffices to construct a $\lceil \epsilon D \rceil$-dominating set for H of size $O(1/\epsilon)$.

Proof. The connected subgraph H constructed in Theorem 10 is a family of at most 6 shortest paths, each one of length at most D, and is a k-dominating set for G for $k > \frac{5}{7}D - 1$. Choosing a λ-dominating for each path, we obtain a $(k+\lambda)$-dominating set for G, say B. By Lemma 8, $|B| \leq 6\lceil D/(2\lambda+1) \rceil$.

Let us set $\lambda = \lceil \epsilon D \rceil$. B is therefore a k'-dominating set for G for every $k' \geq \left\lceil \frac{5}{7}D \right\rceil + \lceil \epsilon D \rceil \geq (\frac{5}{7}+\epsilon)D$. We have for $\epsilon > 0$,

$$|B| \;\leq\; 6\left\lceil \frac{D}{2\lambda+1} \right\rceil \;<\; \frac{6D}{2\lceil \epsilon D \rceil + 1} + 6 \;<\; \frac{6D}{2\epsilon D + 1} + 6 \;<\; \frac{3}{\epsilon} + 6$$

that completes the proof of Theorem 9. □

Proof of Theorem 10. Here we present a constructive proof, implying a polynomial algorithm for construction of such k-dominating sets. The construction is composed of three steps. W.l.o.g. we assume hereafter that $k < D$, the result clearly holds for $k = D$.

Step 1: We start assuming that $k \geq \frac{1}{2}(D-1)$. Let x_1 and x_2 be two nodes such that $d_G(x_1, x_2) > k$, and let H be any shortest path connecting them. If $V(H)$ is a k-dominating then we are done: H is connected and $|V(H)| \leq D$. Otherwise, let us show that G has a 3-gon. Let $x_3 \notin \Gamma_k(H)$, and let A (resp. B) be a shortest path between x_3 and x_1 (resp. between x_3 and x_2). A and B are chosen such that $A \cup H \cup B$ induces at most two regions of $\mathcal{R}^2$. This can be done by making that for every $t \in A \cap B$, $A[x_3 t] = B[x_3 t]$, and for every $t' \in A \cap H$ (resp. $t' \in B \cap H$), $A[x_1 t'] = H[x_1 t']$ (resp. $A[x_2 t'] = H[x_2 t']$). Assigning $Q_1 = H$, $Q_2 = B$, and $Q_3 = A$, and because $x_3 \notin \Gamma_k(Q_1)$, we check that $Q_1 \cup Q_2 \cup Q_3$ is a 3-gon of G. We can go to Step 2.

Step 2: G has a 3-gon, say H.
If $V(H)$ is a k-dominating set, then we are done: H is connected and $|V(H)| \leq 3D$. Otherwise, let $w \notin \Gamma_k(H)$. By Lemma 5, G has a $(3-i+2)$-gon and a $(i+2)$-gon. Since $1 \leq i < 3$, G has a 4-gon, and we can go to Step 3.

Step 3: G has a 4-gon, say H.
If $V(H)$ is a k-dominating set, then we are done: H is connected and $|V(H)| \leq 4D$. Let $m = |V(G) \setminus \Gamma_k(H)|$ be the number of nodes not covered by $V(H)$, $m > 0$. Let $w \notin \Gamma_k(H)$. Assume that H is defined by the nodes $(x_1, \ldots, x_4)$ and the shortest paths $(Q_1, \ldots, Q_4)$. By Lemma 5, let A be the shortest path between w and x_4 that is wholly contained in $\mathrm{reg}(w, H) \cup H$, and let B be the second shortest path between w and x_i, $i < 4$, that is wholly contained in $\mathrm{reg}(w, H) \cup H$. Let $H_1 = B \cup Q_i \cup \cdots \cup Q_3 \cup A$ be the $(6-i)$-gon and let $H_2 = B \cup Q_{i-1} \cup \cdots \cup Q_4 \cup A$ be the $(i+2)$-gon. H_1 and H_2 are 3, 4, or 5-gon.

Let $H' = H_1 \cup H_2$. If $V(H')$ is a k-dominating set, then we are done: H' is connected and $|V(H')| \leq 6D - 1$. Let $m' = |V(G) \setminus \Gamma_k(H')|$, $m' > 0$, and let $w' \notin \Gamma_k(H')$. Note that $m' \leq m - 1$ because $V(H')$ covers $\Gamma_k(H)$ and also w ($V(H) \subset V(H')$). H_1 and H_2 defines at most three regions: $\overline{\mathrm{reg}}(w, H)$, and $\mathrm{reg}(w, H)$ that is split by H_2 in two sub-regions: $R_1 = \mathrm{reg}(z, H_2) \setminus (\overline{\mathrm{reg}}(w, H) \cup H)$ and $R_2 = \overline{\mathrm{reg}}(z, H_2)$, where $z \in V(H_1) \setminus V(H_2)$ (such a node exists). See Fig. 3. On this picture, $i = 2$, H_1 is drawn with bold edges, and H_2 is obtained by following the path w, x_2, x_3, x_4, w. Note that on Fig. 3 two incident paths (like A and B) may be partially merged. The proof anyway is not based on this particular example.

By Corollary 2, and considering H, $w' \notin \overline{\mathrm{reg}}(w, H)$, since w and w' must belong to the same region. So, assume that $w' \in R_j$, for $j = 1$ or 2. Note that $R_j = \mathrm{reg}(w', H_j)$.

Case 1: H_j is a 3-gon or a 4-gon.
In this case, we prove that $\Gamma_k(H_j) = \Gamma_k(H')$. Clearly, $\Gamma_k(H_j) \subseteq \Gamma_k(H')$. By Corollary 2 H_j covers (and thus H' as well since $H_j \subset H'$) all the points of $\overline{\mathrm{reg}}(w', H_j)$. Assume there is $w'' \in R_j$ such that $w'' \in \Gamma_k(H')$ but $w'' \notin \Gamma_k(H_j)$. This implies that there exists $u \in V(H') \setminus V(H_j)$ such that $w'' \in \Gamma_k(u)$. Any shortest path from u to w'' must intersect H_j in a node v because H_j is a separator. Thus $w'' \in \Gamma_k(v)$, proving that $w'' \in \Gamma_k(H_j)$: a contradiction. Thus $\Gamma_k(H_j) = \Gamma_k(H')$, and thus the number of nodes uncovered by H_j is $m' \leq m - 1$. Depending on whether H_j is a 3-gon or a 4-gon we can go to Step 2 or Step 4, and by induction the number of uncovered nodes will decrease up to 0.

Case 2: H_j is a 5-gon.
Note that in this case the shortest path B is between w and x_1 or x_3, otherwise H_j would be a 3-gon or a 4-gon. W.l.o.g. assume B is between w and x_1 (the other case is symmetric by exchanging

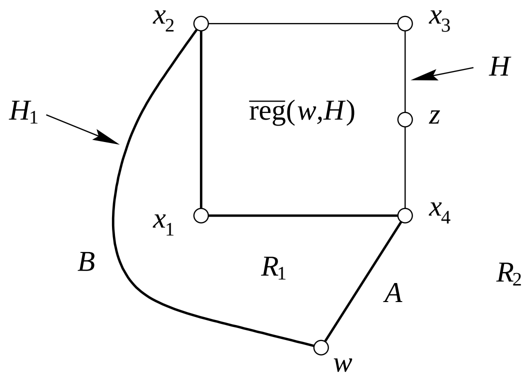

Figure 3: A node $w' \notin \Gamma_k(H')$ has to belong to R_1 or R_2.

node names x_1 and x_3). Assume there is a path B' between w and x_2 that is wholly contains in $\text{reg}(w,H) \cup H$. Then, we can remove the path B constructed while applying Lemma 5, and rename B' into B. Then the resulting gons H_1 and H_2 when new simplified paths A and B are considered consists of 4-gons, and we can conclude by Case 1. Similarly, if there is a path A' between w and x_3 that is wholly contains in $\text{reg}(w,H) \cup H$, then we can remove path A and rename A' into A. The resulting 4-gons resulting when considering new simplified paths A and B, allow to go again to Case 1.

We are left when all shortest paths from w to x_2 and from w to x_3 are not wholly contained in $\text{reg}(w,H) \cup H$.

Let X be a shortest from w to x_2, and let Y be a shortest from w to x_3 (W.l.o.g. we assume that X and Y are simplified on H). Thus X and Y are not wholly contained in $\text{reg}(w,H) \cup H$. So, X must intersect Q_3 or Q_4, and Y must intersect Q_1 or Q_4. Let u be the farest node from w such that $u \in X \cap (Q_3 \cup Q_4)$, and let v be the farest node from w such that $v \in Y \cap (Q_1 \cup Q_4)$.

Case 2.1: $u \notin Q_4$ or $v \notin Q_4$.

Similarly to the proof of Lemma 5, X and Y intersects $\alpha \in \overline{\text{reg}}(w,H)$. Assume $u \notin Q_4$. Then, $X[wu]$ is wholly contained in $\text{reg}(w,H) \cup H$ and intersects Q_3 leading to x_3. Thus $X[wu] \cup Q_3[ux_3]$ is a shortest path wholly contained in $\text{reg}(w,H) \cup H$: a contradiction with the definition of Y. The case $v \notin Q_4$ is symmetric: the path $Y[wv] \cup Q_1[vx_2]$ is a shortest path wholly contained in $\text{reg}(w,H) \cup H$: a contradiction with the definition of X. Thus Case 2.1 is not possible.

Case 2.2: $u, v \in Q_4$.

Then we are left with a configuration similar to the one depicted on Fig. 4.

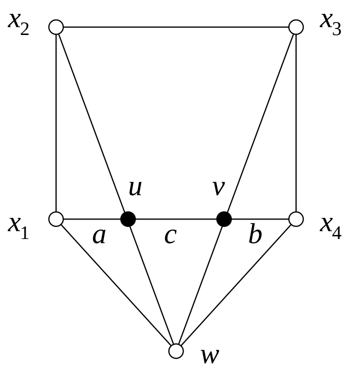

Figure 4: The final configuration.

We now assume that $k > \frac{5}{7}D - 1$. Let $a = d_G(x_1, u)$, $b = d_G(v, x_4)$, and $c = d_G(u, v)$. W.l.o.g. we assume that $a \leq b$. Note that since $a + c + b \leq D$, we have $a \leq \frac{1}{2}(D - c)$. Note also that we have $d_G(u, x_2) \leq D - k - 1$ and $d_G(v, x_3) \leq D - k - 1$ because $w \notin \Gamma_k(u)$ and $w \notin \Gamma_k(v)$.

If $c \geq \frac{1}{7}D$, then

$$\begin{aligned} d_G(x_1,x_2) &\leq a + d_G(u,x_2) \leq \frac{1}{2}(D-c) + D - k - 1 && (1)\\ &< \frac{1}{2}\left(D - \frac{1}{7}D\right) + D - \frac{5}{7}D + 1 - 1 = \frac{5}{7}D \ . && (2) \end{aligned}$$

It follows that $d_G(x_1,x_2) < \frac{5}{7}D < k+1$: a contradiction since H is a 4-gon, forcing to have $d_G(x_1,x_2) > k$.

If $c \leq \frac{1}{7}D$, then

$$\begin{aligned} d_G(x_2,x_3) &\leq d_G(x_2,u) + c + d_G(v,x_3) \leq 2D - 2k - 2 + \frac{1}{7}D && (3)\\ &< 2D - 2\left(\frac{5}{7}D - 1\right) - 2 + \frac{1}{7}D = \frac{5}{7}D \ . && (4) \end{aligned}$$

It follows that $d_G(x_2,x_3) < \frac{5}{7}D < k+1$: a contradiction since H is a 4-gon, forcing to have $d_G(x_2,x_3) > k$.

Hence the Case 2.2 is impossible. Therefore, we have proved Theorem 10. □

2.5 Conjecture and Worst-Case

Note that Theorem 10 allows to show:

Corollary 11 *For planar graph G of diameter D, and every $k \geq \frac{1}{2}D$,*

- *If $D = 2$, $\gamma_k(G) \leq 3$ (tight);*
- *If $D = 3$, $\gamma_k(G) \leq 10$;*
- *If $D = 4$, $\gamma_k(G) \leq 23$;*
- *If $D = 7$, $\gamma_k(G) \leq 41$.*

Proof. As already mentioned in the introduction, the two first results come from [7]. Note that there is a planar graph G of diameter 2 with $\gamma_1(G) = 3$. Actually, the case $D = 3$ comes from the fact (proved in [7]) that $\gamma_1(G) \leq 10$.

Theorem 10 gives $\gamma_k(G) \leq 6D - 1$ for every $k > \frac{5}{7}D - 1$. For $D = 4$, $k > \frac{5}{7}D - 1 = \frac{13}{7}$. Thus that $\gamma_2(G) \leq 23$.

Now, consider $D = 7$. Remark that in the proof of Theorem 10, either the Case 2.2 does not occur, and the result is proved for $k \geq \frac{1}{2}(D-1) = 3$ (thus proving that $\gamma_3(G) \leq 6D - 1 = 41$), or the Case 2.2 does occur. In the latter case, Eq. (1) and (3) imply that $d_G(x_1,x_2)$ or $d_G(x_2,x_3) < \frac{5}{7}D$, that is $d_G(x_1,x_2)$ or $d_G(x_2,x_3) \leq 4$. Since $d_G(x_1,x_2)$ and $d_G(x_2,x_3)$ must be $> k$, we get a contradiction from $k = 4$. Thus $\gamma_4(G) \leq 6D - 1 = 41$ as required. □

We leave open the problem to bound $\gamma_k(G)$ for $k \geq \frac{1}{2}D$, and we conjecture:

Conjecture 12 *For every planar graph G of diameter D, and for every $k \geq \frac{1}{2}D$, there exists a constant c_0 such that $\gamma_k(G) \leq c_0$.*

Here we show that $c_0 \geq 4$.

Theorem 13 *For every even diameter $D \geq 8$, there exists a planar graph G of diameter D, such that for $\gamma_{D/2}(G) = 4$.*

Proof. Let $D = 2t$, $t \geq 4$. The graph denoted by G_t is composed of: (1) two nodes A and B called *poles*; (2) $2t$ disjoint paths $P_1, \ldots, P_{2t}$ called *meridians*, each of length $2t$, joining A to B; (3) a cycle C of length $2t$, called *equator*, joining the middle nodes of each meridians and with exactly one node of degree two between two consecutive meridians. (See Fig. 5 for $t = 4$).

We recall that for a cycle C_{2t} of length $2t$, $\gamma_{D/2}(C_{2t}) = 2$. Hence taking the nodes A,B, it is clear that all the nodes of the meridians are covered, but the 2-nodes of the equator are note covered. To complete, we have to take two opposite nodes u, v of the equator. $\{A, B, u, v\}$ is then a $D/2$-dominating set of G_t. It follows that $\gamma_{D/2}(G_t) \leq 4$.

Now if we take two opposite nodes different from A and B belonging to two distinct meridians we will not cover all the nodes of the $2t$ meridians. In this case we will need more than 4 nodes ($t \geq 4$) to cover all the nodes of the meridians. This completes the proof.

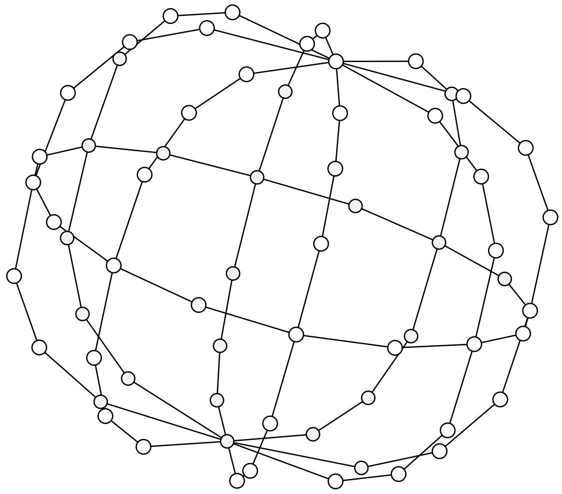

Figure 5: The planar graph embedded on a sphere for $D = 8$ such that $\gamma_4(G) = 4$.

□

3 Application to Routing with Compact Tables

A point-to-point communication network is modeled as a graph $G = (V, E)$, where the set of nodes represent the processors of the network and every pair of two opposite arcs represents a bidirectional communication link. A *routing scheme* R is a distributed algorithm whose role is to deliver messages between nodes of the network. The routing scheme consists of certain *distributed data structures* in the network, and a *delivery protocol*, which can be invoked in any node u with two parameters: a *routing label* $\mathcal{L}(v)$ of the destination node v, and the message's information field. The message is delivered to v via a sequence of transmissions determined uniquely by the distributed data structure, i.e., at each intermediate node along the route, the routing scheme decides the next neighbor to which the message should be forwarded. The length of the route traversed by a message from u to v in the graph G according to the routing scheme R is denoted by $\rho_R(u, v)$. The *dilation* of a routing scheme R is the maximal route length of a path traversed by a message, formally $\max_{u \neq v}\{\rho_R(u, v)\}$.

An *interval routing scheme* R on G is a routing scheme consisting of a pair $(\mathcal{L}, \mathcal{I})$, generated in the preprocessing step, where $\mathcal{L}$ is a *node-labeling*, $\mathcal{L} : V \to \{1, \ldots, n\}$, and $\mathcal{I}$ is an *arc-labeling*, $\mathcal{I} : E \to 2^{\mathcal{L}(V)}$, that satisfy the following condition. For any node u, the collection of sets that label all the outgoing arcs of u forms a partition of the name range (possibly excluding u itself). Formally, for every $u \in V$ (hereafter E_u denotes the set of arcs incident to u),

1. $\bigcup_{e \in E_u} \mathcal{I}(e) \cup \mathcal{L}(u) = \{1, \ldots, n\}$,

2. $\mathcal{I}(e_1) \cap \mathcal{I}(e_2) = \emptyset$ for every two distinct arcs $e_1, e_2 \in E_u$.

The delivery protocol is defined as follows. Given a destination node v, set the first header to be $h = \mathcal{L}(v)$. Also, for every node u, receiving a message with header h, first check if $\mathcal{L}(u) = h$, and end the routing protocol if equality (the message is arrived at destination). If not, then send the message and the same header h on the output port corresponding to the unique arc (u, w) such that $h \in \mathcal{I}(u, v)$. Namely, the message is sent on the arc which is labeled by a set that contains the destination label.

Note that any interval routing scheme can be implemented by classical routing tables. The main difference is in the coding of the table.

Given an integer n and a subset $I \subseteq \{1, \dots, n\}$, define the *compactness of I w.r.t. n*, denoted $c_n(I)$, as the smallest integer k such that I can be represented by the union of k intervals $[a, b]$ of consecutive integers from $\{1, \dots, n\}$, with n and 1 being considered as consecutive (cyclically). The *compactness* of an interval routing scheme $R = (\mathcal{L}, \mathcal{I})$ on G is the maximum, over all arcs $e \in E$, of the compactness $c_n(\mathcal{I}(e))$ of the set $\mathcal{I}(e)$ labeling e. The *total compactness* is the sum $\sum_{e \in E} c_n(\mathcal{I}(e))$.

Because interval routing schemes can be implemented by routing tables, we deal with the compactness of a routing table. Intuitively, smaller compactness and degrees imply smaller routing tables. The interval routing strategy presented in [8], based on routing on a minimum spanning tree, has compactness 1 (for every graph of diameter D) and dilation at most $2D$. It is known that there are worst-case graphs for which every routing table of compactness 1 has dilation at least $2D - 3$ [9]. It is also known that every graph has routing table of compactness $\sqrt{n \ln n} + O(1)$ with dilation at most $\lceil 1.5D \rceil$ [5], whereas there are worst-case graphs for which every routing table of compactness k has dilation at least $\lfloor 1.5D \rfloor - 1$ for every $k = \Omega(\frac{n}{D \log(n/D)})$ [2]. Other results can be found on the recent survey [3].

Here we show a trade-off between dilation and compactness.

Theorem 14 *Let G be a graph of diameter D with n nodes and m edges. Let $k \geq 0$, and let $t = \gamma_k(G)$. Then, G has a routing table with dilation at most $D + k$, compactness at most $\frac{1}{2}t + 1$, and total compactness at most $tn + 2m$. Moreover, if $t / \log n \to +\infty$, then the compactness can be reduced to $\frac{1}{4}t + o(t)$.*

Proof. Let $\{x_1, \dots, x_t\}$ be a k-dominating set in G of minimal cardinality $t = \gamma_k(G)$. Construct a partition of the graph nodes into t pairwise disjoint connected regions $R_i \subseteq \Gamma_k(x_i)$ around each x_i, such that each region R_i consists of nodes closest to x_i (breaking tie arbitrarily) and the union of the regions covers all the nodes. Obviously the radius of each region is at most k.

For every x_i, construct a shortest path spanning tree T_i for G. Let $\hat{T}_i$ be the restriction of T_i to the region R_i, i.e., $\hat{T}_i = T_i \cap R_i$.

Partition the range of integers $[1, n]$ into t contiguous intervals I_i of size $|R_i|$, for $i \in \{1, \dots, t\}$. For each region R_i, assign each node $v \in R_i$ a distinct label $\mathcal{L}(v)$ from the range I_i, in DFS order, starting at x_i. Use these labels to define an interval routing scheme on $\hat{T}_i$ as in [8]. The compactness is 2 (instead of 1) because a cyclic interval in I_i has to be represented by two sub-intervals of I_i. For every two nodes $u, w \in R_i$, this scheme yields a shortest route on $\hat{T}_i$ (albeit perhaps not shortest in G).

In addition, for every $1 \leq i \leq t$ and for every node $w \notin R_i$, add the interval I_i to the arc connecting w to its parent in T_i.

As the regions are disjoint, it is easy to verify that for every node v, all the intervals assigned to the arcs of v are pairwise disjoint. Overall, at most $\frac{1}{2}t + 1$ intervals were assigned to each arc because every subset of at most t intervals has no more than $\frac{1}{2}t$ non contiguous intervals (one more sub-interval is required for the own region of v). In fact, exactly $t + \deg(v)$ intervals altogether ($\deg(v) + 1$ intervals for R_i and $t - 1$ for all the others) are assigned to the arcs of a node v of degree $\deg(v)$. Thus the total compactness is $tn + 2m$.

Observe that the resulting route between nodes belonging to the same region has length at most $2k \leq D + k$. As for routing from a node $u \in R_i$ to a node $w \in R_j$, $j \neq i$, note that the first segment of the route, until reaching R_j, proceeds along a shortest path from u to x_j, and once entering R_j, the remaining segment of the route follows a shortest path. Hence the total length of the route cannot exceed $d_G(u,x_j)+d_G(x_j,w)\le D+k$.

Actually, the bound on the compactness can be slightly reduced to $\frac{1}{4}t+o(t)$, for t large enough, using the result of [4]. This result shows that, for every t, there exists a permutation π of $\{1,\dots,t\}$ such that the compactness of every set F of a family $\mathcal{F}$ of subsets of $\{1,\dots,t\}$, with $|\mathcal{F}|<\exp(t/2)/t$, satisfies:

$$c_t(\pi(F)) \ < \ \frac{1}{4}t+\frac{1}{4}\sqrt{2t\ln(|\mathcal{F}|n)}$$

where $\pi(F)$ denotes the set $\{\pi(x)\mid x\in F\}$. Let $\mathcal{I}(e)$ be the set of labels assigned to arc e by the previous interval routing scheme. Set $\mathcal{F}=\{F(e)\mid e\in E(G)\}$, where $F(e)=\{i\mid I_i\subseteq\mathcal{I}(e)\}$ corresponds to the set of region's indices contained in $\mathcal{I}(e)$. Note that the range of labels of each region is wholly contained in $\mathcal{I}(e)$ (excepted for v's region). Since $\bigcup_{j=1}^{t}I_j=[1,n]$, we have that $c_n(\mathcal{I}(e))\le c_t(F(e))+1$. Observe that the relative order of the ranges can be chosen arbitrary: all the labels of some I_j can be chosen larger or smaller than all the label of any other range. So, indices of I_j's ranges can be permuted by some permutation π of $[1,t]$ in order to decrease $c_t(\pi(F(e)))$. By [4], if $|\mathcal{F}|<\exp(o(t))/t$, there exists π_0 such that $c_t(\pi_0(F(e)))<\frac{1}{4}t+o(t)$, for every $F(e)\in\mathcal{F}$. Note that $|\mathcal{F}|=2m$. So, the condition $|\mathcal{F}|<\exp(o(t))/t$ can be replaced by $t/\log n\to+\infty$, as $m=O(n^2)$. □

This result can be seen as a generalization of [5], originally proved for $k=\left\lceil\frac{1}{2}D\right\rceil$. In this paper it was showed that every graph G satisfies $\gamma_k(G)\le\lceil\sqrt{n\ln n}\,\rceil$, and the compactness they derived was $\lceil\sqrt{n\ln n}\,\rceil+1$. Thus Theorem 14 improves by an asymptotic factor 4 the compactness bound obtained by [5]. The same kind of construction has been also implicitly used in [10]. Precisely, it is shown in [10] that every graph G satisfies $\gamma_k(G)\le\sqrt{n}$ for $k\ge\frac{2}{3}D$. Thus, every graph has routing table of compactness at $\frac{1}{4}\sqrt{n}+o(\sqrt{n})$ for dilation $\left\lceil\frac{5}{3}D\right\rceil$. (The original compactness result of [10] was $\lceil\sqrt{n}\,\rceil+1$.)

Combining Theorem 9 and Theorem 14, we obtain:

Corollary 15 *Let $\epsilon>0$ be an arbitrary constant. Every planar graph has a routing table with dilation at most $\left\lceil(\frac{12}{7}+\epsilon)D\right\rceil$, and constant compactness.*

Note that Conjecture 12 for $c_0=4$ implies that every planar graphs of diameter D enjoys a routing table with dilation at most $\lceil 1.5D\rceil$ and compactness at most 3.

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