On the oriented chromatic number of graphs with given excess

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Abstract

The excess of a graph G is defined as the minimum number of edges that must be deleted from G in order to get a forest. We prove that every graph with excess at most k has chromatic number at most $\frac{1}{2}(3+\sqrt{1+8k})$ and that this bound is tight. Moreover, we prove that the oriented chromatic number of any graph with excess k is at most $k+3$, except for graphs having excess 1 and containing a directed cycle on 5 vertices which have oriented chromatic number 5. This bound is tight for $k \leq 4$.

Key words: graph coloring, graph homomorphism, oriented graph coloring, Betti number.
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1 Introduction

For every graph G, we denote by $V(G)$ the set of vertices of G, by $E(G)$ the set of edges or arcs of G, by $v(G)$ the number of vertices of G, by $e(G)$ the number of edges of G and by $c(G)$ the number of connected components of G. The number $v(G)$ will be called the *order* of G. All the graphs we consider have no multiple edges and no loops. An *oriented graph* is a digraph having no opposite arcs. An oriented graph may be thought of as an *orientation* of the underlying undirected graph obtained by giving to each edge one of the

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two possible orientations. Therefore, we shall denote by $\vec{G}$ an oriented graph which is an orientation of some undirected graph G.

Let G be an undirected graph. We shall denote by $d_G(u)$ the *degree* of the vertex u in G. This notation will be shortened to $d(u)$ whenever the graph G is clear from the context. A *directed path* in an oriented graph $\vec{G}$ is an oriented path such that all its arcs have the same direction. Similarly, a *directed cycle* will be an oriented cycle such that all its arcs have the same direction. The number of edges (or arcs) of a path (resp. a cycle) is called the *length* of this path (resp. cycle).

A graph G is said to be *biconnected* if it is connected and the deletion of any vertex in G preserves the connectivity. A *biconnected component* in G is a maximal biconnected subgraph of G. A graph G is said to be *2-edge-connected* if it is connected and the deletion of any edge in G preserves the connectivity. A *2-edge-connected component* in G is a maximal 2-edge-connected subgraph of G.

Let $\vec{G}$ be a oriented graph. An *oriented k-coloring* of $\vec{G}$ is a mapping c of $V(\vec{G})$ to a set C of k elements, called *colors*, satisfying:

(i) $\forall\ uv \in E(\vec{G})$, $c(u) \neq c(v)$,

(ii) $\forall\ uv, wt \in E(\vec{G})$, $c(u) = c(t) \Longrightarrow c(v) \neq c(w)$.

Note that condition (ii) implies in particular that if xyz is a directed path in $\vec{G}$, then $c(x) \neq c(y) \neq c(z) \neq c(x)$. For instance, any oriented coloring of the directed cycle $\vec{C}_5$ of length five must use 5 colors.

Let $\vec{G}$ and $\vec{H}$ be two oriented graphs. A *homomorphism* of $\vec{G}$ to $\vec{H}$ is a mapping f from $V(\vec{G})$ to $V(\vec{H})$ such that for every arc xy in $E(\vec{G})$, $f(x)f(y)$ is an arc in $E(\vec{H})$. We shall write $\vec{G} \longrightarrow \vec{H}$ whenever there exists a homomorphism of $\vec{G}$ to $\vec{H}$. In that case, we shall say that $\vec{G}$ is *$\vec{H}$-colorable* and that the homomorphism f is a *$\vec{H}$-coloring* of $\vec{G}$.

It is not difficult to observe that a $\vec{H}$-coloring of an oriented graph $\vec{G}$ is an oriented k-coloring of $\vec{G}$ where $k = v(\vec{H})$. Similarly, every oriented k-coloring using a set C of colors can be viewed as a homomorphism to some oriented graph having the set C as set of vertices. Therefore, the target graph $\vec{H}$ will often be referred to as a *color-graph* and its vertices as colors. In the following, we shall indifferently speak about k-colorings or $\vec{H}$-colorings depending on the context.

The *oriented chromatic number* of an oriented graph $\vec{G}$, denoted by $\vec{\chi}(\vec{G})$, is defined as the smallest k such that $\vec{G}$ has an oriented k-coloring or, equivalently, as the minimum order of an oriented graph $\vec{H}$ such that $\vec{G} \longrightarrow \vec{H}$. This notion is extended to undirected graphs as follows: the oriented chromatic number of an undirected graph G, denoted by $\vec{\chi}(G)$, is defined as the maximum of the oriented chromatic numbers of its orientations. Similary, the oriented chromatic number of a family of graphs (or oriented graphs) is defined as the maximum of the oriented chromatic numbers of its members.

Oriented colorings have been first considered by Courcelle [3] as a tool for encoding graph orientations by means of vertex labels. Since then, oriented colorings have attracted very much attention and have been studied by several authors (see [2,4–6,8] or [7] for a general overview). In particular, it has been proved in [5] that the family of planar graphs has oriented chromatic number at most 80 and in [8] that there exist oriented planar graphs with oriented chromatic number at least 16. In [6], it has been proved that the family of graphs with degree at most 2 (that is paths and cycles) has oriented chromatic number at most 5. It is not difficult to observe that the family of forests has oriented chromatic number at most 3 (see Section 3).

In this paper, we study the oriented chromatic number of graphs according to the following parameter: the *excess* of a graph G, denoted by $ex(G)$, is defined as the minimum number of edges we have to delete in G in order to get a forest. Therefore, $ex(G) = e(G) - v(G) + c(G)$. This parameter corresponds to the dimension of the *cycle space* of G and is sometimes called in the litterature the *cyclomatic number* [1] or the Betti number of G.

We shall denote by ξ_k the family of graphs with excess k. For instance, the family ξ_0 corresponds to the family of forests. Since every graph with excess $k-1$ is a subgraph of some graph with excess k, we clearly have $\vec{\chi}(\xi_k) \geq \vec{\chi}(\xi_{k-1})$ for every $k > 0$. It has been proved in [2] that by adding one arc to some oriented graph, its oriented chromatic number may be increased by 2 (more precisely, it is proved that for every p, there exists an oriented graph $\vec{H}_p$ such that deleting any arc in $\vec{H}_p$ decreases its oriented chromatic number by 2 and deleting any vertex in $\vec{H}_p$ decreases its oriented chromatic number by p). Therefore, since the family of forests has oriented chromatic number at most 3, we get $\vec{\chi}(\xi_k) \leq 2k + 3$ for every $k \geq 0$.

Our main result shows that this bound can be decreased to $k + 3$ except for orientations of graphs with excess 1 containing a directed cycle on 5 vertices which have oriented chromatic number 5.

This paper is organised as follows. In Section 2 we prove that the (usual) chromatic number of graphs with excess k is at most $\frac{1}{2}(3 + \sqrt{1 + 8k})$ and that this bound is tight. In Section 3, the oriented chromatic number of graphs with excess at most 2 is considered, while the oriented chromatic number of graphs with excess k, $k \geq 3$, is considered in Section 4. Finally, we prove in

Section 5 that our bound is tight for graphs with excess at most 4 and give some general lower bound for the oriented chromatic number of graphs with excess k, $k > 4$.

2 The undirected case

We consider in this section the problem of determining the (usual) chromatic number of graphs with given excess. We prove the following:

Theorem 1 *If G is an undirected graph with excess k then $\chi(G) \le \frac{1}{2}(3 + \sqrt{1+8k})$ and this bound is tight.*

Proof. Let G be a graph with excess k, minimal with respect to the number of vertices, such that $\chi(G) > \frac{1}{2}(3 + \sqrt{1+8k})$. We obviously have

$$v(G) > \frac{1}{2}(3 + \sqrt{1+8k}). \tag{1}$$

If there exists a vertex $u \in V(G)$ with $d(u) \le \frac{1}{2}(3 + \sqrt{1+8k}) - 1 = \frac{1}{2}(1 + \sqrt{1+8k})$ then, according to the minimality of G, there exists a coloring of $G - u$ using $\frac{1}{2}(3 + \sqrt{1+8k})$ colors and this coloring can be extended to G, a contradiction. Therefore,

$$\forall\, u \in V(G), \quad d(u) > \frac{1}{2}(1 + \sqrt{1+8k}). \tag{2}$$

Finally, consider $k = ex(G) = e(G) - v(G) + 1$. From (2) we get

$$\begin{aligned} k &> \frac{1}{2}(\frac{1}{2}(1 + \sqrt{1+8k}))v(G) - v(G) + 1 \\ &= 1 + \frac{1}{4}(\sqrt{1+8k} - 3)v(G) \end{aligned}$$

and from (1)

$$\begin{aligned} k &> 1 + \frac{1}{4}(\sqrt{1+8k} - 3) \times \frac{1}{2}(3 + \sqrt{1+8k}) \\ &= \frac{1}{8}(1 + 8k - 9) + 1 = k, \end{aligned}$$

a contradiction. Therefore, every graph with excess k has chromatic number at most $\frac{1}{2}(3 + \sqrt{1+8k})$.

To see that this bound is tight, consider the complete graph K_n on n vertices with $n = \frac{1}{2}(3 + \sqrt{1+8k})$. This graph has chromatic number $\frac{1}{2}(3 + \sqrt{1+8k})$

and excess

$$\begin{aligned} ex(K_n) &= \frac{1}{2}n(n-1) - n + 1 = \frac{1}{2}(n^2 - 3n + 2) \\ &= \frac{1}{8}(3+\sqrt{1+8k})^2 - \frac{3}{4}(3+\sqrt{1+8k}) + 1 \\ &= \frac{1}{8}(9+1+8k+6\sqrt{1+8k}) - \frac{9}{4} - \frac{3}{4}\sqrt{1+8k} + 1 \\ &= \frac{10}{8} + k + \frac{6}{8}\sqrt{1+8k} - \frac{9}{4} - \frac{3}{4}\sqrt{1+8k} + 1 = k. \end{aligned}$$

This concludes the proof. ∎

3 Oriented graphs with excess $k \leq 2$

In this section, we consider the problem of determining the oriented chromatic number of graphs with excess 0, 1 and 2.

For a graph G, we define the *pruned graph* of G, denoted by $P(G)$, as the (unique, up to isomorphism) graph obtained from G by repetedly deleting isolated vertices and vertices of degree 1 until the obtained graph is an isolated vertex or a graph with minimum degree 2. The pruned graph of an oriented graph is defined in a similar way. Note that both the initial graph and its pruned graph have the same excess.

An oriented graph will be called a *good* oriented graph if it has no source nor sink (that is every vertex has at least one incoming arc and at least one outcoming arc). The next observation will be extensively used in the sequel:

Observation 2 *Let $\vec{G}$ be an oriented graph. If $\vec{H}$ is a good oriented graph such that $P(\vec{G}) \longrightarrow \vec{H}$, then $\vec{G} \longrightarrow \vec{H}$.*

To see that, let $v_1, v_2, \ldots, v_\ell$ denote the ordered sequence of vertices that have been deleted from $\vec{G}$ in order to get $P(\vec{G})$ and let f be a homomorphism of $P(\vec{G})$ to $\vec{H}$. One can extend f to a homomophism of $\vec{G}$ to $\vec{H}$ by mapping vertices $v_\ell, v_{\ell-1}, \ldots, v_1$ (in that order) to vertices of $\vec{H}$ as follows: an isolated vertex is mapped to any vertex of $\vec{H}$, a vertex of degree 1 is mapped according to the image of its unique neighbor, which is always possible since $\vec{H}$ is good.

A graph with excess 0 is clearly a forest and its pruned graph is thus an isolated vertex. Since the smallest good oriented graph is the directed cycle $\vec{C}_3$ on 3 vertices and since the directed path on 3 vertices has oriented chromatic number 3, we get:

Theorem 3 [6] $\vec{\chi}(\xi_0) = 3$. *Moreover, every oriented graph with excess 0 has a homomorphism to $\vec{C}_3$.*

A graph with excess 1 is a graph whose pruned graph is a cycle. Let $\vec{T}_5$ be the tournament defined by $V(\vec{T}_5) = \{0, 1, \ldots, 4\}$ and $ij \in E(\vec{T}_5)$ if and only if $j - i \equiv 1$ or $2 \pmod 5$. The tournament $\vec{T}_5$ is clearly vertex-transitive and has the following property: for every oriented path $\vec{P} = uvw$ of length 2 and any two distinct vertices a and b in $V(\vec{T}_5)$ there exists a homomorphism $\varphi : \vec{P} \longrightarrow \vec{T}_5$ with $\varphi(u) = a$ and $\varphi(w) = b$ [6]. The following has been proved in [6]:

Proposition 4 **[6]** *Every oriented cycle has a homomorphism to the tournament $\vec{T}_5$.*

This result can be strenghtened as follows:

Proposition 5 *Every oriented cycle except the directed cycle on 5 vertices has oriented chromatic number at most 4.*

Proof. Let $u_1 u_2 \ldots u_n$ denote the vertices of an oriented cycle $\vec{C}$ on n vertices.

Suppose first that $\vec{C}$ is not a directed cycle. Without loss of generality, we can then suppose that $u_1 u_2$ and $u_1 u_n$ are two arcs in $\vec{C}$. By Theorem 3, we know that the oriented path $u_2 u_3 \ldots u_n$ has a homomorphism to $\vec{C}_3$. By assigning to u_1 a fourth color, we get a homomorphism of $\vec{C}$ to some oriented graph on 4 vertices.

Suppose now that $\vec{C}$ is the directed cycle on $n \neq 5$ vertices. The result is obvious for $n \leq 4$. For $n \geq 6$, we can partition the cycle into directed paths of length 3 or 4 since every such n can be expressed as a sum of 3's and 4's. By coloring $1, 2, 3$ the paths of length 3 and $1, 2, 3, 4$ the paths of length 4 we obtain a homomorphism of $\vec{C}$ to the oriented graph on four vertices $\{1, 2, 3, 4\}$ with arcs 12, 23, 34, 31 and 41. Hence, the oriented chromatic number of $\vec{C}$ is at most 4. ∎

It is not difficult to observe that in the proof of Proposition 5 one can obtain in each case a homomorphism to some good tournament of order 4. (The target graphs all contain a directed 3-cycle and a fourth vertex of degree at most 2, so that one can add some arc to get the desired tournament). Therefore we get, considering that the directed cycle on 5 vertices has oriented chromatic number 5:

Theorem 6 $\vec{\chi}(\xi_1) = 5$. *Moreover, every oriented graph with excess 1 has a homomorphism to $\vec{T}_5$ and every oriented graph with excess 1 containing no directed 5-cycle has oriented chromatic number at most 4 and admits a homomorphism to some good tournament on 4 vertices.*

We now consider the case of graphs with excess 2 and prove the following:

Theorem 7 $\vec{\chi}(\xi_2) = 5$. *Moreover, every oriented graph with excess 2 admits a homomorphism to some good tournament of order 5.*

Proof. Let G be a graph with excess 2 and $\vec{G}$ be any orientation of G. According to Observation 2, it is enough to consider the case when $G = P(G)$. We have three cases to consider, according to the form of the graph G:

(1) *G has two connected components, each being a cycle.*

Both cycles are $\vec{T}_5$-colorable by Proposition 4 so that $\vec{G}$ is $\vec{T}_5$-colorable.

(2) *G is connected and made of two cycles C_1 and C_2 linked by a path of length $\ell \geq 0$.*

Let u denote the vertex of C_2 belonging to the path. The graph $\vec{G} - \vec{C}_2 + u$ has excess 1 and, by Theorem 6, is $\vec{T}_5$-colorable. Let f be a homomorphism of $\vec{G} - \vec{C}_2 + u$ to $\vec{T}_5$ and g be a homomorphism of $\vec{C}_2$ to $\vec{T}_5$ (such a homomorphism exists by Proposition 4). Since $\vec{T}_5$ is vertex-transitive, the homomorphism g can be chosen such that $f(u) = g(u)$. Combining f and g, we get a homomorphism of $\vec{G}$ to $\vec{T}_5$.

(3) *G is connected and made of two cycles C_1 and C_2 having t consecutive edges in common, $t \geq 1$.*

Let u denote one of the two vertices of G with degree 3. The graph $\vec{G} - u$ is a tree and, by Theorem 3, has a homomorphism to $\vec{C}_3$. Denote by f such a homomorphism and by u_1, u_2 and u_3 the 3 neighbors of u in $\vec{G}$. If all arcs linking u to its neighbors have the same direction, we assign u some fourth color, thus extending f to a homomorphism of $\vec{G}$ to some oriented graph on 4 vertices. Otherwise, one arc linking u to some u_i has a direction opposite to the two others. In that case, we can extend f by assigning u some fourth color and recoloring u_i with some fifth color. We thus get some homomorphism of $\vec{G}$ to some oriented graph on 5 vertices. In both cases, the target graph can be completed in order to get some good oriented graph.

In each case, G has a homomorphism to some good oriented graph of order at most 5, which concludes the proof. ∎

4 Oriented graphs with excess $k \geq 3$

In this section, we consider the problem of determining the oriented chromatic number of graphs with excess k, $k \geq 3$. We start by two useful lemmas.

Lemma 8 *Let G be a 2-edge-connected graph with excess k. For every $u \in V(G)$, $ex(G - u) \leq k - \lceil \frac{d(u)}{2} \rceil$.*

Proof. Let $d = d(u)$ and $A = \{x_1, x_2, \ldots, x_d\}$ be the set of all neighbours of u in G. For every $x_i \in A$ there exists at least some $x_j \in A$, $j \neq i$ such that x_i and x_j belong to the same connected component of $G - u$. Therefore $G - u$ has at most $\lfloor \frac{d}{2} \rfloor$ connected components. Then:

$$ex(G-u) = e(G) - d - v(G) + 1 + c(G-u)$$

$$\leq e(G) - v(G) + 1 - (d - \lfloor \frac{d}{2} \rfloor) = k - \lceil \frac{d}{2} \rceil.$$

∎

Lemma 9 *Let G be a 2-edge-connected graph with $ex(G) > 1$.*

(a) If there is a cycle in G containing exactly one vertex with degree at least 3 and if all other cycles in G contain at least two vertices of degree at least 3, then G contains a vertex u of degree at least 3 such that $G-u$ is connected.

(b) If every cycle in G contains at least two vertices of degree at least 3, then G contains two distinct vertices u_1 and u_2 of degree at least 3 such that $G-u_1$ and $G-u_2$ are both connected.

Proof. Let G be a graph satisfying the conditions of the lemma. Since $ex(G) > 1$, G contains at least two vertices of degree at least 3. If the graph G is biconnected, then every vertex u in G is such that $G-u$ is connected and the result follows.

Otherwise, consider the graph $B(G)$, whose vertices are the blocks (that is the biconnected components) of G and such that there is an edge linking two such blocks if and only if they share a common vertex. It is well-known that the graph $B(G)$ is a tree. Since G is 2-edge-connected, every leaf of $B(G)$ contains a cycle. Since G is not biconnected, $B(G)$ contains at least two leaves.

In case (a), there is at least one leaf in $B(G)$, say B_0, containing a cycle having at least two vertices of degree at least 3. Let b_0 denote the unique vertex of B_0 belonging to some other block. We can therefore choose a vertex u of degree at least 3, $u \neq b_0$, such that $G-u$ is connected.

In case (b), every leaf in $B(G)$ has the same property as B_0. Since $B(G)$ has at least two leaves, we can find two distinct vertices u_1 and u_2 of degree at least 3 such that both $G-u_1$ and $G-u_2$ are connected. ∎

We are now able to prove our main result:

Theorem 10 $\forall\ k \geq 2$, $\vec{\chi}(\xi_k) \leq k+3$. *Moreover, every oriented graph with excess $k \geq 2$ admits a homomorphism to some good tournament of order $k+3$.*

Proof. We prove the theorem by indution on k. By Theorem 7, the result holds for $k = 2$. Suppose now that the result holds for every ℓ, $2 \leq \ell \leq k-1$ and let G be a graph with excess $k > 2$. According to Observation 2 it is enough to consider the case when $G = P(G)$.

Let $\vec{G}$ be any orientation of G. We consider five cases (for every case i, we assume that none of the cases $1, 2, \ldots, i-1$ occurs):

(1) *$\vec{G}$ contains a cycle with a vertex u of degree 2 having either two incoming arcs or two outcoming arcs.*

We clearly have $ex(G-u) = k-1$. Therefore, by the induction hypothesis, $\vec{G} - u$ has a homomorphism f to some good tournament of order $k+2$. By assigning u some $(k+3)$-*th* color, we can extend f to a homomorphism of $\vec{G}$ to some good tournament of order $k+3$.

(2) *$\vec{G}$ contains a cycle with a vertex u of degree 2 having one incoming arc and one outcoming arc. Moreover, the 2 neighbors of u are linked by a directed path of length 1 or 2 which does not contain the vertex u.*

Again we have $ex(G-u) = k-1$. Therefore, by the induction hypothesis, $\vec{G} - u$ has a homomorphism f to some good tournament of order $k+2$. Since the two neighbors of u, say r and s, are linked by a directed path of length 1 or 2, we have $f(r) \neq f(s)$. Therefore, by assigning u some $(k+3)$-*th* color, we can extend f to a homomorphism of $\vec{G}$ to some good tournament of order $k+3$.

(3) *$\vec{G}$ has a cycle containing an induced directed path $uu_1u_2u_3v$ of length 4 (vertices u_1, u_2 and u_3 have degree 2 in $\vec{G}$).*

By deleting the vertices u_1, u_2 and u_3, we get a graph $\vec{G}'$ with excess $k-1$ which has a homomorphism f to some good tournament $\vec{H}_{k+2}$ of order $k+2$. Let $f(u) = x$ and $f(v) = y$. We have three subcases to consider. In each subcase, we show how to extend f to a homomorphism of $\vec{G}$ to some good tournament of order $k+3$ by using a $(k+3)$-*th* color α.

3.1. $x = y$. The mapping f can be extended by setting $f(u_1) = z$, $f(u_2) = \alpha$ and $f(u_3) = z'$, where $xz, z'x \in E(\vec{H}_{k+2})$.

3.2. $x \neq y$ and $xy \in E(\vec{H}_{k+2})$. The mapping f can be extended by setting $f(u_1) = y$, $f(u_2) = z$ and $f(u_3) = \alpha$ where $yz \in E(\vec{H}_{k+2})$.

3.3. $x \neq y$ and $yx \in E(\vec{H}_{k+2})$. If there exists two vertices z and w in $\vec{H}_{k+2}$ such that $w \neq y$ and $xz, zw \in E(\vec{H}_{k+2})$, f can be extended by setting $f(u_1) = z$, $f(u_2) = w$ and $f(u_3) = \alpha$. Otherwise, since $\vec{H}_{k+2}$ has at least 4 vertices, there exists two vertices z and w' in $\vec{H}_{k+2}$ such that $xz, w'z \in E(\vec{H}_{k+2})$ and $w' \notin \{x, y, z\}$. In that case, f can be extended by setting $f(u_1) = \alpha$, $f(u_2) = w'$ and $f(u_3) = z$, with $zy \in E(\vec{H}_{k+2})$.

In all these three subcases, f has a homomorphism to some color graph that can clearly be completed to obtain a good tournament of order $k+3$.

(4) *G contains a vertex u of odd degree, $d(u) = 2t+1$, such that all edges*

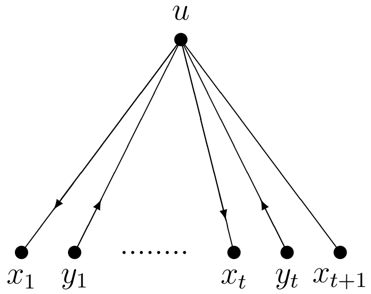
incident to u in G are contained in some cycle.

We have two subcases to consider:

4.1. *$ex(G-u) \neq 1$ or $\vec{G}-u$ does not contain the circuit $\vec{C}_5$ as a subgraph.*

Since u has degree $2t+1$ and all the edges incident to u are contained in some cycle, u belongs to at least $t+1$ cycles. Now by deleting u, the excess decreases by at least $t+1$ and so we have $ex(G-u) \leq k-t-1$. According to the induction hypothesis we thus have $\vec{\chi}(\vec{G}-u) \leq k+2-t$. Let f be a homomorphism of $\vec{G}-u$ to some good tournament of order $k+2-t$. Note that in order to extend f to a homomorphism of $\vec{G}$ to some good tournament of order $k+3$ we have $k+3-(k+2-t) = t+1$ free colors.

It is not difficult to see that the configuration that requires the largest number of new colors is the following (the undirected edge may be oriented in any direction):



where for every i, $1 \leq i \leq t$, $f(x_i) = f(y_i)$ and for every i, j, $1 \leq i < j \leq t+1$, $f(x_i) \neq f(x_j)$.

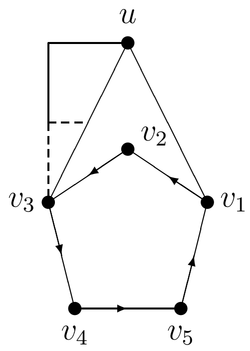
In that case, we can extend f by recoloring the vertices $y_1, y_2, \ldots, y_t$ with t distinct new colors and by assigning to u the $(t+1)$-*th* free color. The so-obtained color graph can clearly be completed to obtain a good tournament of order $k+3$.

4.2. *$ex(G-u) = 1$ and $\vec{G}-u$ contains a directed cycle $\vec{C}_5$.*

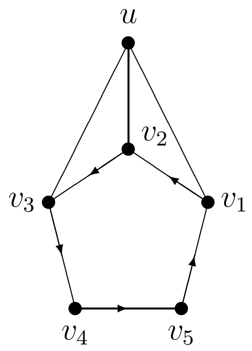
We consider two cases according to the degree of the vertex u:

4.2.1. $d(u) = 3$.

In this case only one of the two following configurations is possible (an undirected edge stands for any oriented path, a dashed edge for an optional oriented path), the other possibilities having been considered in the previous cases:



(a) $d(v_1) = 3$, $d(v_3) = 3$ or 4



(b) $d(v_1) = d(v_2) = d(v_3) = 3$

Configuration (a). Consider the graph $\vec{G} - v_1$ and let w denote the third neighbour of v_1 (we may have $w = u$). This graph has excess 1 and, by Proposition 4, has a homomorphism f to T_5. Since the vertices v_2 and v_5 have degree 1 in $\vec{G} - v_1$, the homomorphism f can be chosen such that $f(v_2) \neq f(v_5) \neq f(w)$ if v_1w is an arc in $\vec{G}$ or $f(v_5) \neq f(v_2) \neq f(w)$ if wv_1 is an arc in $\vec{G}$ (recall that every vertex in T_5 has 2 successors and 2 predecessors). By assigning to v_1 some 6-*th* color, f can be extended to a homomorphism of $\vec{G}$ to some oriented graph of order 6.

Configuration (b). The graph $\vec{G}$ contains 3 cycles that cannot be alltogether directed cycles. Therefore, there is at least one vertex v_i, $1 \leq i \leq 3$, such that $\vec{G} - v_i$ has no $\vec{C}_5$. By choosing such a v_i instead of u, we go back to case 4.1.

4.2.2. $d(u) \geq 5$.

In this case, $ex(G) \geq 4$. If $G - u$ is connected, then $d(u) - 2 \geq \lceil \frac{d(u)}{2} \rceil$ colors are free so that one can extend any oriented coloring of $\vec{G} - u$ to an oriented coloring of $\vec{G}$ as it was done in case 4.1. Otherwise, there exists some vertex v in $\vec{C}_5$ such that $d(v) \geq 3$, $G - v$ is connected and $ex(G - v) > 1$. By choosing v instead of u, we again go back to case 4.1.

(5) Suppose now that none of the previous cases occurs. Since G is not a forest, there exists in G a 2-edge-connected component G^* such that G^* and $G - G^*$ are linked by at most one edge. If G^* is a connected component of G, or if G itself is 2-edge-connected (in that case $G^* = G$), then there is no edge linking G^* and $G - G^*$. Otherwise, we denote by w the unique vertex of G^* adjacent to some vertex of $G - G^*$.

We first claim that $ex(G^*) > 1$. Suppose to the contrary that $ex(G^*) = 1$, which means that G^* is a cycle (recall that we have $G = P(G)$). Since case 1 does not occur, $\vec{G}^*$ must be a directed cycle. Since case 2 does not occur, $\vec{G}^*$ must be a directed cycle of length at least 5 but, in that case, case 3 necessarily occurs, a contradiction.

We now claim that G^* is an even graph (that is every vertex in G^* has an even degree). Since case 4 does not occur, the only vertex that can have an odd degree in G is the vertex w. Since every graph has an even number of vertices with an odd degree, the vertex w has necessarily an even degree in G^* and we are done.

We finally claim that if w exists and $d_{G^*}(w) = 2$, then all cycles in G^* contain at least two vertices of degree at least 4, except possibly one cycle containing w which, in that case, contains a vertex of degree at least 4, otherwise all cycles in G^* contain at least two vertices of degree at least 4. Suppose to the contrary that G^* contains a cycle of length ℓ having $\ell - 1$ vertices (distinct from w if it exists) of degree 2. As discussed before, this cycle must be a directed cycle in $\vec{G}^*$ since case 1 does not occur. Since case 2 does not occur we necessarily have $\ell \geq 5$ and, since case 3 does not occur, $\ell < 4$, a contradiction.

Therefore, by applying Lemma 9 to G^*, we get that G^* contains at least one vertex u of degree at least 4, $u \neq w$ if w exists, such that $G^* - u$ is connected. Clearly, G and $G - u$ have the same number of connected components. Therefore,

$$ex(G - u) = e(G - u) - v(G - u) + c(G - u)$$

$$= e(G) - d(u) - v(G) + 1 + c(G) = ex(G) - d(u) + 1.$$

We have now two cases to consider:

5.1. *$ex(G - u) \neq 1$ or $\vec{G} - u$ does not contain a directed cycle $\vec{C}_5$.*

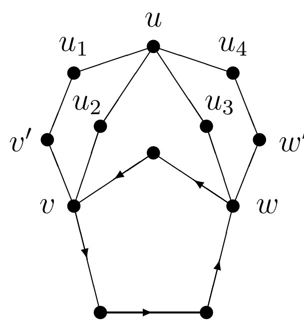
According to the induction hypothesis, we have $\vec{\chi}(\vec{G} - u) \leq k - d(u) + 4$. Then every homomorphism of $\vec{G} - u$ to some good tournament of order $k - d(u) + 4$ can be extended to a homomorphism of $\vec{G}$ to some good tournament of order $k + 3$ by using the same technique as in case 4.1, since we have $d(u) - 1 > \lceil \frac{d(u)}{2} \rceil$ free colors.

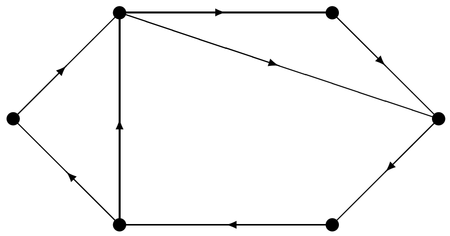
5.2. *$ex(G - u) = 1$ and $\vec{G} - u$ contains a directed cycle $\vec{C}_5$.*

We then consider two subcases:

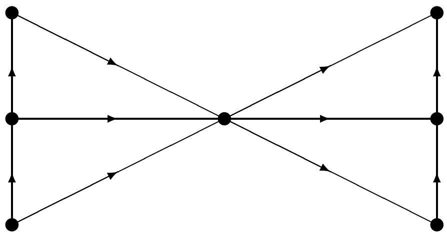
5.2.1. $d(u) = 4$.

In that case, $ex(G) = 4$. It is not difficult to check that either one can choose the vertex u among the vertices of $\vec{C}_5$ in such a way that this situation does not happen, or $\vec{G}$ has the following form (an undirected edge may be oriented in any direction):





(a) An oriented graph with excess 3 and oriented chromatic number 6



(b) An oriented graph with excess 4 and oriented chromatic number 7

Fig. 1. Proof of Proposition 11

In that case, we know by Theorem 6 that $G - u$ has a homomorphism f to the good tournament $\vec{T_5}$. We then have two free colors, say α and β. If a and b are respectively the colors of vertices v and w, one can extend f to a homomorphism of $\vec{G}$ to some good tournament on 7 vertices by coloring the vertex u with some color $c \in V(\vec{T_5})$, distinct from a and b, and recoloring the vertices u_i, $1 \leq i \leq 4$, by α (resp. β) if $(u, u_i) \in E(\vec{G})$ (resp. if $(u_i, u) \in E(\vec{G})$). Note that the paths uv and uw are directed and that the vertices v' and w' can be recolored with some color in $V(\vec{T_5})$ distinct from c (there exist two choices for coloring these two vertices by a color in $V(\vec{T_5})$ [6]).

5.2.2. $d(u) \geq 6$.

In this case, $ex(G) \geq 6$. Every homomorphism of $\vec{G} - u$ to $\vec{T_5}$ can be extended to a homomorphism of $\vec{G}$ to some good tournament of order $d(u) + 3$ by using the same technique as in case 4.1, since we have $d(u) - 2 > \lceil \frac{d(u)}{2} \rceil$ free colors.

This concludes the proof. ∎

5 Lower bounds

We have given in Theorems 3, 6 and 7 tight bounds for the oriented chromatic number of graphs with excess 0, 1 and 2. We prove first that the bounds given by Theorem 10 are also tight for graphs with excess 3 and 4:

Proposition 11 *There exist graphs with excess 3 (resp. 4) and oriented chromatic number 6 (resp. 7).*

Proof. Examples of such graphs are depicted in Figure 1 (in both cases, any two vertices are linked by a directed path of length at most 2, so that the oriented chromatic number equals the number of vertices). ∎

Concerning a general lower bound for graphs with excess k, we have the following:

Theorem 12 *For every $n \geq 1$ and every k, $k \geq 3.2^{n-1}(2n-3)+4$, $\vec{\chi}(\xi_k) \geq 3.2^n - 1$.*

Proof. Let $\vec{G}_1 = \vec{C}_5$ and $\vec{G}_n$, $n \geq 2$, be the oriented graph obtained by taking two disjoint copies of $\vec{G}_{n-1}$ and a new vertex u_n linked by adding all the arcs from the vertices of the first copy towards u_n and all the arcs from u_n towards the vertices of the second copy.

This sequence of oriented graphs satisfies:

$$\begin{cases} ex(\vec{G}_1) = 1 \text{ and } \vec{\chi}(\vec{G}_1) = 5 \\ \forall\ n \geq 2,\ ex(\vec{G}_n) = 2(ex(\vec{G}_{n-1}) + \vec{\chi}(\vec{G}_{n-1}) - 1) \text{ and } \vec{\chi}(\vec{G}_n) = 2\vec{\chi}(\vec{G}_{n-1}) + 1 \end{cases}$$

Solving this system, we obtain:

$$ex(\vec{G}_n) = 3.2^{n-1}(2n-3)+4,\ n \geq 1, \text{ and } \vec{\chi}(\vec{G}_n) = 3.2^n - 1.$$

Therefore :

$$\forall\ k \geq 3.2^{n-1}(2n-3)+4,\ \vec{\chi}(\xi_k) \geq 3.2^n - 1.$$

∎

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