On the oriented chromatic number of Halin graphs

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Abstract

An oriented k-coloring of an oriented graph G is a mapping $c : V(G) \to \{1, 2, \ldots, k\}$ such that (i) if $xy \in E(G)$ then $c(x) \neq c(y)$ and (ii) if $xy, zt \in E(G)$ then $c(x) = c(t) \Longrightarrow c(y) \neq c(z)$. The oriented chromatic number $\vec{\chi}(G)$ of an oriented graph G is defined as the smallest k such that G admits an oriented k-coloring. We prove in this paper that every Halin graph has oriented chromatic number at most 9, improving a previous bound proposed by Vignal.

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1 Introduction

We consider oriented graphs, that is digraphs having no loops and no opposite arcs. If G is an oriented graph, we denote by $V(G)$ its set of vertices and by $E(G)$ its set of arcs. If xy is an arc in $E(G)$, we say that y is a *successor* of x and that x is a *predecessor* of y.

An *oriented k-coloring* of an oriented graph G is a mapping $c : V(G) \to \{1, 2, \ldots, k\}$ such that (i) if $xy \in E(G)$ then $c(x) \neq c(y)$ and (ii) if $xy, zt \in E(G)$ then $c(x) = c(t) \Longrightarrow c(y) \neq c(z)$.

With every oriented k-coloring c of G one can associate a digraph H_c, called the *colour-graph* of c, with vertex set $V(H_c) = \{c(x) \mid x \in V(G)\}$ and arc set $E(H_c) = \{c(x)c(y) \mid xy \in E(G)\}$. Thanks to conditions (i) and (ii), H_c is an oriented graph. The oriented k-coloring c can then be viewed as a homomorphism (that is an arc-preserving vertex mapping) from G to H_c. Similarly, every homomorphism of G to an oriented graph H on k vertices can be viewed as a k-coloring of G, using the vertices of H as colours. The oriented coloring problem has been extensively studied these last years [1, 2, 3, 4, 5, 6].

The *oriented chromatic number* $\vec{\chi}(G)$ of an oriented graph G is defined as the smallest k such that G admits an oriented k-coloring or, equivalently, as the minimum number of vertices in an oriented graph H such that G has a homomorphism to H.

From the definition of oriented k-coloring, we get that if xyz is a directed 2-path in G ($xy, yz \in E(G)$) then $c(x) \neq c(y) \neq c(z) \neq c(x)$ for every oriented k-coloring of G. In other words, any two vertices that are linked in G by a directed path of length 1 or 2 must be assigned distinct colours.

Let H be a planar graph and F be its face set. If all the edges on the boundary of some face F_0 (whose vertices are all of degree 3) of F are deleted and a tree with at least three leaves is obtained, then the graph H is called a *Halin graph*. The vertices on F_0 are called *exterior* vertices of H, and the remaining vertices are called *interior* vertices of H.

In [8] Vignal proved that every oriented Halin graph has oriented chromatic number at most 11. She conjectured that the oriented chromatic number of every oriented Halin graph is at most 8. We are going to prove that every oriented Halin graph has oriented chromatic number at most 9, which improves the upper bound obtained by Vignal.

We now introduce some oriented graphs that are used as target graphs in the proof of our main result : the tournament QR_7 constructed from the non zero quadratic residues of 7, defined by $V(QR_7) = \{0, 1, \ldots, 6\}$ and $E(QR_7) = \{ij \mid j - i \equiv 1, 2 \text{ or } 4 \pmod 7\}$, the tournament T_5 defined by $V(T_5) = \{0, 1, \ldots, 4\}$ and $E(T_5) = \{ij \mid j-i \equiv 1 \text{ or } 2 \pmod 5\}$, the circuit on three vertices $\vec{C}_3$ with vertices 1, 2, 3 and arcs 12, 23, 31 and the oriented graph G_9 constructed as follows. Let C_x, C_y and C_z be three circuits on 3 vertices with vertex sets $\{x_1, x_2, x_3\}$, $\{y_1, y_2, y_3\}$, $\{z_1, z_2, z_3\}$ and with arc sets $\{x_1x_2, x_2x_3, x_3x_1\}$, $\{y_1y_2, y_2y_3, y_3y_1\}$, $\{z_1z_2, z_2z_3, z_3z_1\}$ respectively. The graph G_9 is obtained from C_x, C_y and C_z by adding all arcs from every vertex of C_x towards all vertices of C_y, all arcs from every vertex of C_z towards all vertices of C_x, the arc $x_iy_{i+1} \pmod 3$ and the arc $y_ix_{i+1} \pmod 3$ for every $i = 1, 2, 3$ (see Figure 1).

Definition 1 An *orientation vector* of size n is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\{0, 1\}^n$; let G be an oriented graph and $X = (x_1, x_2, \ldots, x_n)$ be a sequence of pairwise distinct vertices of G. A vertex y of G is said to be an *α-successor* of X if for every i, $1 \leq i \leq n$, we have $\alpha_i = 1 \Longrightarrow x_iy \in E(G)$ and $\alpha_i = 0 \Longrightarrow yx_i \in E(G)$.

Definition 2 We say that a color-graph C satisfies property P_k for some $k > 0$ if for every oriented n-clique subgraph $(c_1, c_2, \ldots, c_n)$ in C with $1 \leq n \leq k$, and every orientation vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of size n, there exists a color c in $V(C)$ which is an α-successor of $(c_1, c_2, \ldots, c_n)$.

Note that every color-graph satisfying property P_k also satisfies property $P_{k'}$ for every $k' < k$. Then we have :

Proposition 3 [7] *The tournement QR_7 satisfies property P_2.*

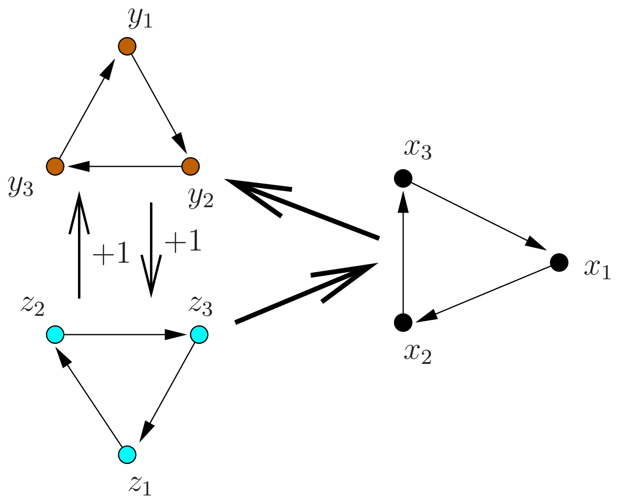

Figure 1: The graph G_9

Proposition 4 [7] *Every oriented tree is C_3-colorable.*

Proposition 5 [7] *Every oriented cycle has a homomorphism to the tournament T_5.*

2 Halin graphs

In this section, we prove that the oriented chromatic number of every oriented Halin graph is at most 9, and that there exists oriented Halin graphs with oriented chromatic number 8.

Theorem 6 *For every oriented Halin graph H, $\vec{\chi}(H) \leq 9$.*

Proof. Let H be an oriented Halin graph with extrior face F_0. We denote by $C(H) = (f_1, f_2, \dots, f_n)$, $n \geq 3$, the cycle induced by the exterior vertices of H, and by T the tree induced by the interior vertices of H. Let p_i denote the father of f_i for every i, $1 \leq i \leq n$.

Thanks to Proposition 4, T is C_x-colorable. Let h' be a homomorphism from T to C_x. We are going to extend h' to a homomorphism h from H to an oriented graph G having at most 9 vertices, containing C_x as an induced subgraph.

We consider two cases according to n :

1. $n \leq 6$.

   Let G be the oriented graph obtained from C_x with $V(G) = \{x_1, x_2, x_3, y_1, y_2, \dots y_n\}$ and $A(G) = A(C_x) \cup \{y_i y_j \mid f_i f_j \in A(H), 1 \leq i, j \leq n\} \cup \{h'(p_i)y_i \mid p_i f_i \in A(H)\} \cup \{y_j h'(p_j) \mid f_j p_j \in A(H)\}$.

   It is easy to check that the mapping $h : V(H) \longrightarrow V(G)$ defined by

$$\begin{cases} h(x) = h'(x) & \text{for all } x \in V(T), \\ h(f_i) = y_i & \text{for all } i, \ 1 \leq i \leq n \end{cases}$$

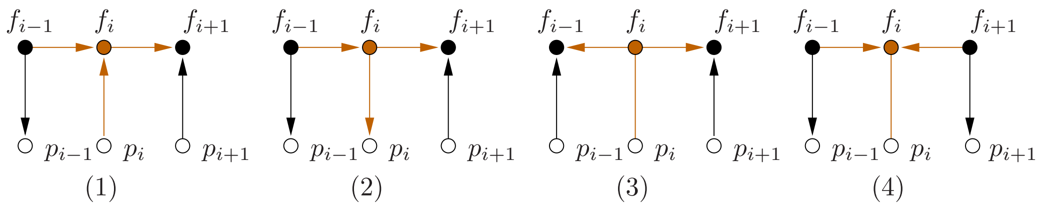

Figure 2: Configurations for the case 2.(b)

is a homomorphism. So, $\vec{\chi}(H) \leq |G| \leq 9$.

2. $n \geq 7$.

   We consider three sub-cases :

   (a) For every i, $1 \leq i \leq n$, $p_i f_i \in A(H)$ or for every i, $1 \leq i \leq n$, $f_i p_i \in A(H)$.
   Since $C(H)$ is a cycle, by Proposition 4 $C(H)$ is T_5-colorable. Let g be a homomorphism from $C(H)$ to T_5. We consider the mapping $h : V(H) \longrightarrow V(G)$, defined by :

   $$h(x) = \begin{cases} h'(x) & \text{if } x \in V(T), \\ g(x) & \text{if } x \in V(C(H)) \end{cases}$$

   where G is the oriented graph obtained from C_x and T_5 by adding all the arcs from every vertex of C_x towards every vertex of T_5, if $p_i f_i \in A(H)$ for every i, $1 \leq i \leq n$, and all the arcs from every vertex of T_5 towards every vertex of C_x, if $f_i p_i \in A(H)$ for every i, $1 \leq i \leq n$.
   In both cases h is clearly a homomorphism from H to G. Therefore, $\vec{\chi}(H) \leq |G| = 8$.

   (b) H contains one of the configurations depicted in Figure 2 (in the configurations (3) and (4) of Figure 2 the edge $p_i f_i$ can be oriented in any direction).
   Since $C(H) \setminus f_i$ is a path, according to Proposition 4 there exists a homomorphism g from $C(H) \setminus f_i$ to the circuit $\vec{C}_3$. We consider the mapping $h : V(H \setminus f_i) \rightarrow V(G_9)$ defined by

   $$\begin{cases} h(x) = h'(x) & \text{if } x \in V(T), \\ h(f_j) = y_{g(f_j)} & \text{if } j \neq i \text{ and } p_j f_j \in A(H), \\ h(f_j) = z_{g(f_j)} & \text{if } j \neq i \text{ and } f_j p_j \in A(H). \end{cases}$$

   It is easy to check that h is a homomorphism from $H \setminus f_i$ to G_9. Let x_i and x'_i be the unique successor and the unique predecessor of $h'(p_i)$ in C_x respectively. By setting $h(f_i) = x_i$ (respectively $h(f_i) = x'_i$) if $p_i f_i \in A(H)$ (respectively if $f_i p_i \in A(H)$), h can be extended to a homomorphism from H to G_9.

   (c) n is even and for every i, $1 \leq i \leq n/2$, $p_{2i-1} f_{2i-1}, f_{2i} p_{2i} \in A(H)$.
   Two subcases arise :

      i. There exists an i such that $f_{i-1} f_i, f_{i+1} f_i \in A(H)$ or $f_i f_{i-1}, f_i f_{i+1} \in A(H)$.
      We can suppose without loss of generality that $p_i f_i \in A(H)$.

Since $C(H) \setminus f_i$ is a path, there exists a homomorphism g of $C(H) \setminus f_i$ to $\vec{C}_3$ such that $g(f_{i+1}) = 1$. We consider the mapping $h : V(H \setminus f_i) \to V(G_9)$ defined by

$$\begin{cases} h(x) = h'(x) & \text{if } x \in V(T), \\ h(f_j) = y_{g(f_j)} & \text{if } j \neq i \text{ and } p_j f_j \in A(H), \\ h(f_j) = z_{g(f_j)} & \text{if } j \neq i \text{ and } f_j p_j \in A(H). \end{cases}$$

It is easy to check that h is a homomorphism $H \setminus f_i$ to G_9.
We suppose first that $f_{i-1}f_i, f_{i+1}f_i \in A(H)$. Let x_i be the unique successor of $h'(p_i)$ in C_x. By setting $h(f_i) = x_i$, h can be extended to a homomorphism from H to G_9.
We now suppose that $f_i f_{i-1}, f_i f_{i+1} \in A(H)$. In this case, since we have $h(f_{i+1}) = z_1$. If $h(f_{i-1}) = z_1$ by setting $h(f_i) = y_3$, h can be extended to a homomorphism from H to G_9. If $h(f_{i-1}) = z_2$ by setting $h(f_i) = y_1$, h can be extended to a homomorphism from H to $G_9 + y_1 z_1$. If $h(f_{i-1}) = z_3$ by setting $h(f_i) = y_3$, h can be extended to a homomorphism from H to $G_9 + y_3 z_3$.

ii. $C(H)$ is a circuit.
Let us suppose that $f_i f_{i+1} \in A(H)$ for every i, $1 \leq i \leq n$. We consider three subcases according to n :

A. $n \equiv 0 \pmod 3$. The mapping $g : V(C(H)) \to V(\vec{C}_3)$ defined by $g(f_i) = i \pmod 3$ for all f_i, $1 \leq i \leq n$, is clearly a homomorphism and it is easy to check that the mapping $h : H \to G_9$ defined by

$$\begin{cases} h(x) = h'(x) & \text{if } x \in V(T), \\ h(f_i) = y_{g(f_i)} & \text{if } p_i f_i \in A(H), \\ h(f_i) = z_{g(f_i)} & \text{if } f_i p_i \in A(H), \end{cases}$$

is a homomorphism.

B. $n \equiv 1 \pmod 3$. In this case, the mapping defined by $g(f_i) = i \pmod 3$ for all f_i, $1 \leq i \leq n-1$, is a homomorphism from $C(H) \setminus f_n$ to $\vec{C}_3$ and it is easy to check that the mapping h defined by

$$\begin{cases} h(x) = h'(x) & \text{if } x \in V(T), \\ h(f_i) = y_{g(f_j)} & \text{if } i \neq n \text{ and } p_j f_j \in A(H), \\ h(f_j) = z_{g(f_j)} & \text{if } i \neq n \text{ and } f_j p_j \in A(H), \end{cases}$$

is a homomorphism from $H \setminus f_n$ to G_9 such that $h(f_1) = y_1$ and $h(f_{n-1}) = y_3$. By setting $h(f_n) = z_1$, h can be extended to a homomorphism from H to $G_9 + z_1 y_1$.

C. $n \equiv 2 \pmod 3$. In this case, the mapping g defined by $g(f_i) = i \pmod 3$ for every f_i, $1 \leq i \leq n-5$, is a homomorphism from $C(H) \setminus \{f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_n\}$ to $\vec{C}_3$ and it is easy to checked that the mapping h defined by

$$\begin{cases} h(x) = h'(x) & \text{if } x \in V(T), \\ h(f_i) = y_{g(f_i)} & \text{if } p_j f_j \in A(H) \text{ and } 1 \leq i \leq n-5, \\ h(f_j) = z_{g(f_j)} & \text{if } f_j p_j \in A(H) \text{ and } 1 \leq i \leq n-5, \end{cases}$$

is a homomorphism from $H \backslash \{f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_n\}$ to G_9 such that $h(f_1) = y_1$ and $h(f_{n-5}) = y_3$. By setting $h(f_{n-4}) = z_1$, $h(f_{n-3}) = y_1$, $h(f_{n-2}) = z_2$, $h(f_{n-1}) = y_2$ et $h(f_n) = z_3$, h can be extended to a homomorphism from H to $G_9 + \{z_1y_1, z_2y_2\}$.

(d) There exists an i, $1 \leq i \leq n$, such that $f_{i-1}p_{i-1}$, p_if_i, $p_{i+1}f_{i+1} \in A(H)$ or $p_{i-1}fi-1$, f_ip_i, $f_{i+1}p_{i+1} \in A(H)$.

We can suppose without loss of generality that for some i, $1 \leq i \leq n$, $f_{i-1}p_{i-1}$, p_if_i, $p_{i+1}f_{i+1} \in A(H)$ and we get $i = 1$. So, we suppose f_np_n, p_1f_1, $p_2f_2 \in A(H)$.

Since $C(H) \setminus f_1$ is a path, there exists a homomorphism g from $C(H) \setminus f_i$ to C_3 such that $g(f_2) = 1$. We consider the mapping $h : V(H \setminus f_1) \rightarrow G_9$ defined by

$$\begin{cases} h(x) = h'(x) & \text{if } x \in V(T), \\ h(f_j) = y_{g(f_j)} & \text{if } j \neq 1 \text{ and } p_jf_j \in A(H), \\ h(f_j) = z_{g(f_j)} & \text{if } j \neq 1 \text{ and } f_jp_j \in A(H). \end{cases}$$

It is easy to check that h is a homomorphism $H \setminus f_1$ to G_9 such that $h(f_2) = y_1$. Since $f_np_n \in A(H)$, $h(f_n) \in V(C_z)$.

We have four subcases according to the orientation of the edges f_nf_1 and f_1f_2 :

i. $f_nf_1, f_1f_2 \in A(H)$.
In this case, H contains the Configuration (1) of Figure 2.

ii. $f_1f_n, f_2f_1 \in A(H)$.
If $h(f_n) = z_3$; by setting $h(f_1) = y_2$, h can be extended to a homomorphism from H to G_9.
If $h(f_n) = z_2$; by setting $h(f_1) = y_2$, h can be extended to a homomorphism from H to $G_9 + y_2z_2$.
If $h(f_n) = z_1$; let j, $2 \leq j \leq n-1$, be the largest integer such that $p_if_i \in A(H)$ for every i, $1 \leq i \leq j$. In this case, we can suppose that for every i, $2 \leq i \leq j$, $f_if_{i-1} \in A(H)$ (otherwise H contains the configuration (3) of Figure 2). We consider now the edge f_jf_{j+1} ; if $f_{j+1}f_j \in A(H)$, we have the configuration (1) of Figure 2 (take $f_i = f_j$, $f_{i+1} = f_{j-1}$ and $f_{i-1} = f_{j+1}$), which corresponds to case 2.(b). If $f_jf_{j+1} \in A(H)$, let $h(f_j) = y_k$; by setting $h(f_{j+1}) = z_k$ and $h(f_m) = y_{g(f_m)-1}$ (respectively $h(f_m) = z_{g(f_m)-1}$) if $p_mf_m \in A(H)$ (respectively, if $f_mp_m \in A(H)$), $j+2 \leq m \leq n$, we can have $h(f_n) = z_3$. By setting $h(f_1) = y_2$, h can be extended to a homomorphism from H to $G_9 + y_kz_k$.

iii. $f_1f_n, f_1f_2 \in A(H)$.
If $h(f_n) = z_1$; by setting $h(f_1) = y_3$, h can be extended to a homomorphism from H to G_9.
If $h(f_n) = z_3$; by setting $h(f_1) = y_3$, h can be extended to a homomorphism from H to $G_9 + y_3z_3$.
If $h(f_n) = z_2$; let j, $2 \leq j \leq n-1$, the largest integer such that $p_if_i \in A(H)$ for every i, $1 \leq i \leq j$. We consider two subcases :

A. There exists an integer r, $3 \leq r \leq j$, such that $f_rf_{r-1} \in A(H)$.
In this case, we have $f_if_{i-1} \in A(H)$ for every i, $r+1 \leq i \leq j$ (otherwise H contains the configuration (3) of Figure 2). We consider now the

edge f_jf_{j+1} ; if $f_{j+1}f_j \in A(H)$, we have the configuration (1) of Figure 2 (take $f_i = f_j$, $f_{i+1} = f_{j-1}$ and $f_{i-1} = f_{j+1}$), which corresponds to case 2.(b). If $f_jf_{j+1} \in A(H)$, let $h(f_j) = y_k$; by setting $h(f_{j+1}) = z_k$ and $h(f_m) = y_{g(f_m)-1}$ (respectively $h(f_m) = z_{g(f_m)-1}$) if $p_mf_m \in A(H)$ (respectively, if $f_mp_m \in A(H)$), $j+2 \leq m \leq n$, we get $h(f_n) = z_1$. By setting $h(f_1) = y_3$, h can be extended to a homomorphism from H to $G_9 + y_kz_k$.

B. For all r, $2 \leq r \leq j$, $f_{r-1}f_r \in A(H)$.
We consider the edge f_jf_{j+1} ; if $f_jf_{j+1} \in A(H)$, let $h(f_j) = y_k$; by setting $h(f_{j+1}) = z_k$ and $h(f_m) = y_{g(f_m)-1}$ (respectively $h(f_m) = z_{g(f_m)-1}$) if $p_mf_m \in A(H)$ (respectively, if $f_mp_m \in A(H)$), $j+2 \leq m \leq n$, we get $h(f_n) = z_1$. By setting $h(f_1) = y_3$, h can be extended to a homomorphism from H to $G_9 + y_kz_k$. If $f_{j+1}f_j \in A(H)$ we consider the two following subcases :

- There exists an integer s, $j+2 \leq s \leq n-1$ such that $f_{s-1}p_{s-1}$, $p_sf_s \in A(H)$.
  Since $f_{s-1}p_{s-1} \in A(H)$, $h(f_{s-1}) \in V(C_z)$; let $h(f_{s-1}) = z_k$, we then set $h(f_s) = y_k$. We consider now the edge $f_{s-1}f_s$. If $f_{s-1}f_s \in A(H)$, by setting $h(f_m) = y_{g(f_m)-1}$ (respectively $h(f_m) = z_{g(f_m)-1}$) if $p_mf_m \in A(H)$ (respectively, if $f_mp_m \in A(H)$), $s+1 \leq m \leq n$ we get $h(f_n) = z_1$. By setting $h(f_1) = y_3$, h can be extended to a homomorphism from H to $G_9+\{z_ky_k\}$. If $f_sf_{s-1} \in A(H)$, by setting $h(f_m) = y_{g(f_m)+1}$ (respectively $h(f_m) = z_{g(f_m)+1}$) if $p_mf_m \in A(H)$ (respectively, if $f_mp_m \in A(H)$), $s+1 \leq m \leq n$ we get $h(f_n) = z_3$. By setting $h(f_1) = y_3$, h can be extended to a homomorphism from H to $G_9 + \{y_kz_k, y_3z_3\}$.
- For all s, $j+1 \leq s \leq n$, $f_sp_s \in A(H)$.
  We consider the edge $f_{j+1}f_{j+2}$. We suppose first that $f_{j+2}f_{j+1} \in A(H)$; in that case, we have the configuration (2) of Figure 2 (take $f_i = f_{j+1}$, $f_{i-1} = f_{j+2}$ and $f_{i+1} = f_j$), which corresponds to case 2.(b). We suppose now that $f_{j+1}f_{j+2} \in A(H)$. We have $f_mf_{m+1} \in A(H)$ for every m, $j+1 \leq m \leq n-1$ (otherwise, H contains the configuration (4) of Figure 2). In fact, we have in that case : $p_if_i \in A(H)$ for $i = 1, 2, \ldots, j$, $f_ip_i \in A(H)$ for $i = j+1, \ldots, n$, $f_if_{i+1} \in A(H)$ for $i \in \{1, \ldots, n-1\} \setminus j$ and $f_{j+1}f_j$, $f_1f_n \in A(H)$ (see Figure 3).
  We first show that the graph $H' = H \setminus \{f_j\}$ is QR_7-colorable. Since the tournament QR_7 contains the circuit C_x as a subgraph, the tree $T + \{f_np_n\}$ is QR_7-colorable. Let h be a homomorphism from $T + \{f_np_n\}$ to QR_7. Since every vertex of QR_7 has three predecessors and three successors, we can suppose that $h(f_n) \neq h(p_1)$. Since $h(f_n) \neq h(p_1)$, thanks to Proposition 3 there exists a color in $V(QR_7)$ for f_1. We now color successively the vertices $f_2, \ldots, f_j$. Every vertex f_i, $1 \leq i \leq j$, has two neighbors, f_{i-1} and p_i, which are already QR_7-colored (they can have the same color). Since $f_{i-1}f_i$ and $p_if_i \in A(H)$, thanks to Proposition 3, there always exists a color

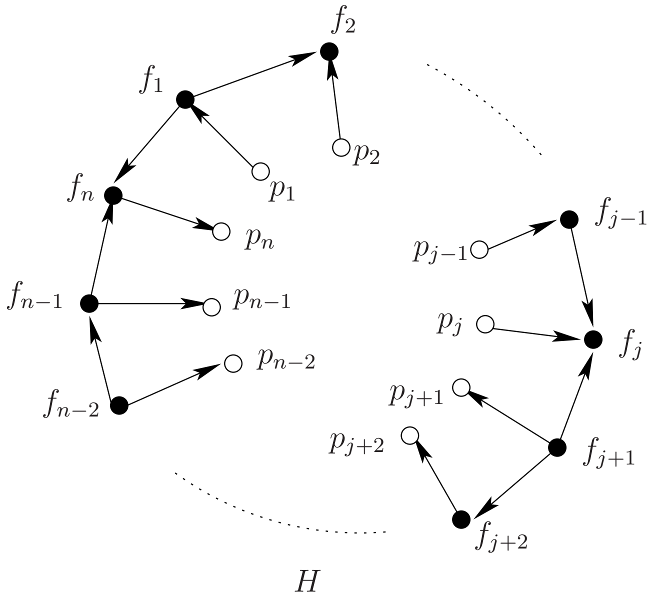

Figure 3: Configuration for the subcase 2.(d).iii.B

in QR_7 for f_i. We now color successively the vertices $f_{n-1}, \ldots, f_{j+2}$. Every vertex f_i, $j+2 \leq i \leq n-1$, has two neighbors, f_{i+1} et p_i, which are already QR_7-colored (they can have the same color). Since $f_i f_{i+1}$ and $f_i p_i \in A(H)$, thanks to Proposition 3, there exists a color in QR_7 for f_i.

Hence, there exists a QR_7-coloring h of H'. By setting $h(f_{j+1}) = c_8$ for an additional color c_8, we get $\vec{\chi}(H) \leq 9$.

iv. $f_1 f_n, f_2 f_1 \in A(H)$.

If $h(f_n) = z_1$, by setting $h(f_1) = y_2$, h can be extended to a homomorphism from H to G_9.

If $h(f_n) = z_2$, by setting $h(f_1) = y_2$, h can be extended to a homomorphism from H to $G_9 + z_2 y_2$.

If $h(f_n) = z_3$, let j, $2 \leq j \leq n-1$, be the largest integer such that for every i, $1 \leq i \leq j$, $p_i f_i \in A(H)$. In this case, we can suppose that $f_i f_{i-1} \in A(H)$ for all i, $2 \leq i \leq j$ (otherwise H contains the configuration (3) of Figure 2). We consider now the edge $f_j f_{j+1}$; if $f_{j+1} f_j \in A(H)$, we have the configuration (1) of Figure 2 (take $f_i = f_j$, $f_{i+1} = f_{j-1}$ and $f_{i-1} = f_{j+1}$), which corresponds to case 2.(b) and if $f_j f_{j+1} \in A(H)$, we have the case 2.(d).iii (take $f_1 = f_j$, $f_2 = f_{j-1}$ and $f_n = f_{j+1}$).

We thus found in all cases a homomorphism from H to some oriented graph with at most 9 vertices. Thus, $\vec{\chi}(H) \leq 9$. ∎

Concerning the lower bound of the oriented chromatic number of the family of Halin graphs, we have :

Proposition 7 *There exists oriented Halin graphs with oriented chromatic number at least 8.*

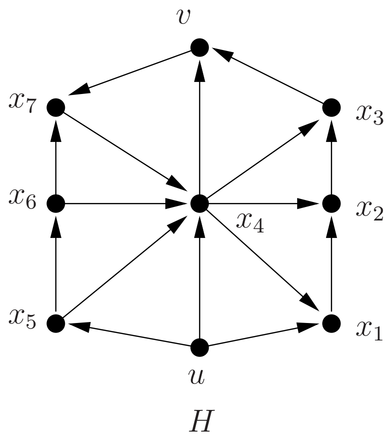

Figure 4: A Halin graph with oriented chromatic number 8

Proof. We consider the graph H depicted on Figure 4. We show that $\vec{\chi}(H) = 8$.

Let c be an oriented k-coloring of H. Let us first notice that for all i, j, $1 \leq i < j \leq 7$, the two vertices x_i and x_j are linked by a directed path of length at most 2. Thus, for all i, j, $1 \leq i < j \leq 7$, the vertices x_i and x_j must be colored with two distinct colors. So, we have $k \geq 7$. Let $i = c(x_i)$, $i = 1, 2, \ldots, 7$. If $k = 7$, we necessarily have $c(u) = 7$ (there is a directed path of length at most 2 linking u and each of $x_1, x_2, \ldots, x_6$). Similarly there is a directed path of length at most 2 linking v to each of $x_2, x_3, \ldots, x_7$ and thus $c(v) = 1$. By considering the arcs vx_7 and ux_1, we get a contradiction and thus $k \geq 8$. ∎

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