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ON THE MINIMUM NUMBER OF EDGES GIVING MAXIMUM ORIENTED CHROMATIC NUMBER[1]

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Abstract. We show that the minimum number of edges in a graph on n vertices with oriented chromatic number n is $(1+o(1))n\log_2 n$.

Keywords. Oriented chromatic number.

In 1995, in a conversation with the French member of the set of the authors of this note, Pál Erdős asked about the minimal number of edges a graph on n vertices with oriented chromatic number n can have. During the conference on the Future of Discrete Mathematics in the cosy but fruitful atmosphere of the Štiřín Castle we found an elementary answer to this question which we present below.

We consider first an extremal question closely related to Erdős's problem. A colouring of the vertices of a graph G whose edges are initially coloured with k colours is *admissible*, if every colour class spans an independent set and each pair of colour classes is joined by edges of one colour only. After Alon and Marshall [1], we define the *k-chromatic number $\chi_k(G)$ of G* as the minimal number ℓ such that for any colouring of the edges of G with k colours there exists an admissible ℓ-colouring of the vertices of G. Finally, by $f_k(n)$ we denote the minimum number of edges in a graph G on n vertices for which $\chi_k(G)=n$. We shall show that for any fixed k the function $f_k(n)$ grows roughly as $n\log_k n$.

Theorem 1 *For a fixed $k\geq 2$ and n large enough*

$$n(\log_k n - 4\log_k\log_k n - 5) \leq f_k(n) \leq \lceil\log_k n\rceil(n-\lceil\log_k n\rceil)\ .$$

The argument we use in the proof of Theorem 1 in the case when $k=2$ can be easily employed to answer Erdős's original question. Recall (see [2–4]) that the oriented chromatic number of a graph G is defined as the smallest number ℓ, such that for every orientation $\vec{G}$ of G there exists a tournament $\vec{T}(\vec{G})$ on ℓ vertices such that $\vec{G}$ can be homomorphically embedded into $\vec{T}(\vec{G})$.

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Theorem 2 *Let $g(n)$ be the smallest number for which there exists a graph G with n vertices, $g(n)$ edges, and oriented chromatic number n. Then, for large enough n, we have*

$$n(\log_2 n - 4\log_2\log_2 n - 5) \le g(n) \le \lceil \log_2 n\rceil (n - \lceil \log_2 n\rceil)\ .$$

Proof of Theorem 1. In order to see the upper bound for $f_k(n)$ observe that the complete bipartite graph on n vertices, with bipartition (W', W''), where $|W'| = \lceil \log_k n\rceil$ and $|W''| = n - \lceil \log_k n\rceil$, has k-chromatic number n. Indeed, label elements of W' by $1, \dots, \lceil \log_k n\rceil$ and elements of W'' by $0, 1, \dots, n - \lceil \log_k n\rceil - 1$. Now colour the edge $\{i', i''\}$, where $i' \in W'$, $i'' \in W''$, with the jth colour, $j = 0, 1, \dots, k-1$, if j appears at the i'th position in the expansion of i'' in the k-ary system. Note that for each pair of vertices $x', y' \in W'$ there exists $z'' \in W''$ such that the edges $\{x', z''\}$ and $\{y', z''\}$ are coloured with different colours; it is enough to take z'' which has different digits at positions x' and y'. Similarly, for every $x'', y'' \in W''$ one can find a position z' at which the digits of the k-ary expansions of x'' and y'' differ; then the colours of the edges $\{x'', z'\}$ and $\{y'', z'\}$ must differ as well. Hence, every admissible colouring of the vertices of such a coloured graph must use different colours for different vertices, i.e. the k-chromatic number of the graph equals n.

The proof of the lower bound for $f_k(n)$ is slightly less immediate. Let G be a graph with n vertices and

$$e(G) \le n(\log_k n - 4\log_k \log_k n - 5)$$

edges, which are coloured with k colours. We need to show that for each such colouring there exists an admissible colouring of the vertices of G which uses only $n-1$ colours.

Our argument will be based on the following observation (see Tuza [5]).

Claim. *If the edges of the complete graph K_n are covered by a family of k-partite graphs $G_1, \dots, G_m$, of $r_1, \dots, r_m$ vertices, respectively, then $\sum_i r_i \ge n\log_k n$.*

Let us partition the vertices of G into two classes W' and W'', where W' consists of all vertices of G of degree at least $\log_k^2 n$. Then

$$|W'| \le \frac{2e(G)}{\log_k^2 n} < \frac{2n}{\log_k n}\ ,$$

and so $|W''| \ge n(1 - 2/\log_k n)$. Now, for every vertex $w' \in W'$ we define a complete k-partite graph $G_{w'}$, choosing as the ith set of the k-partition of $G_{w'}$ the set of all vertices of W'' which are connected to w' by edges of the ith colour. Thus, the total number of vertices of $G_{w'}$ is the same as the number of neighbours of w' in W''. Furthermore, let H'' be the graph with vertex set W'', in which two vertices are adjacent if they lie within distance two in the subgraph $G[W'']$ induced by W'' in G. Note that every graph F contains a k-partite subgraph F' such that the maximum degree of the graph obtained from F by removing the edges of F' is at most $\lfloor \Delta(F)/k\rfloor$. Thus, since the maximum degree $\Delta(H'')$ of H'' is bounded from above by $\Delta^2(G[W'']) \le \log_k^4 n$, H'' can be decomposed into

$$m'' \le 1 + \log_k(\log_k n)^4 = 4\log_k\log_k n + 1$$

k-partite subgraphs $H_1, \dots, H_{m''}$. Observe also that the sum of the orders of all graphs $\{G_{w'}\}_{w'\in W'}$ and $\{H_i\}_{i=1}^{m''}$ can be bounded from above by

$$\begin{aligned}
&\sum_{w'\in W'} |G_{w'}| + \sum_{i=1}^{m''} |H_i| \\
&\le e(G) + |W''|(4\log_k\log_k n + 1) \\
&\le |W''|\Big(\frac{\log_k n - 4\log_k\log_k n - 5}{1 - 2/\log_k n} + 4\log_k\log_k n + 1\Big) \\
&\le |W''|\Big\{\big(\log_k n - 4\log_k\log_k n - 5\big)\Big[1 + \frac{2}{\log_k n} + O\Big(\frac{1}{\log_k^2 n}\Big)\Big] + 4\log_k\log_k n + 1\Big\}
\end{aligned}$$

$$< |W''|(\log_k n - 1) < |W''| \log_k |W''| \,.$$

Hence, the Claim implies that there exists a pair of vertices $\{x'', y''\}$ which appears as an edge in none of the graphs $\{G_{w'}\}_{w' \in W'}$ and $\{H_i\}_{i=1}^{m''}$, i.e. vertices x'' and y'' are not adjacent, they have no common neighbours in W'', and for each of their common neighbours $z' \in W'$ both edges $\{x'', z'\}$ and $\{y'', z'\}$ are coloured with the same colour. Thus, the colouring in which x'' and y'' are coloured with the same colour, while all other vertices of G receive different colours, is admissible. Consequently, $\chi_k(G) \le n - 1$ and the assertion follows. □

Proof of Theorem 2. Note that a graph G on n vertices has oriented chromatic number n if and only if in some orientation $\vec{G}$ of G each pair of non-adjacent vertices of G is connected by a directed path of length two. Now to estimate $g(n)$ from above take the 2-coloured complete bipartite graph with bipartition (W', W'') used in the proof of Theorem 1 to get an upper bound for $f_2(n)$; the edges coloured with the first colour direct from W' to W'', for the edges of the second colour choose the other direction.

Similarly, to get a lower bound for $g(n)$ divide the set of vertices of a graph G into two classes W' and W'', where W' consists of all vertices of degree at least $\log_2^2 n$. Then one can mimic the proof of Theorem 1 and show that in order to orient all edges in such a way that all non-adjacent vertices of W'' are joined by a directed path of length two, G must have at least $n(\log_2 n - 4 \log_2 \log_2 n - 5)$ edges. □

We conclude with a few words on the expected behaviour of $f_k(n)$ and $g(n)$. It is tempting to conjecture that the upper bounds given in Theorems 1 and 2 are close to the truth and $f_k(n) = n \log_k n + O(n)$ and $g(n) = n \log_2 n + O(n)$, where the term $O(n)$ depends on the arithmetic properties of n. On the other hand, we should mention that the elementary bipartite construction we used to get the upper bound for $f_k(n)$ is far from being best possible. For instance, one can modify it slightly by deleting one vertex from the smaller set of the bipartition at the same time adding a perfect matching to the larger of the sets; the graph obtained in such a way has k-chromatic number $n - 1$ and, typically, less than $\lceil \log_k(n-1) \rceil (n - 1 - \lceil \log_k(n-1) \rceil)$ edges. Furthermore, if we have a family of graphs $G_1, G_2, \ldots, G_r$, such that G_i has n_i vertices and k-chromatic number n_i, then the graph $1 + \sum_{i=1}^r G_i$ obtained by taking disjoint copies of $G_1, G_2, \ldots, G_r$ and adding to it one more vertex of degree $\sum_{i=1}^r n_i$ has the largest possible k-chromatic number as well. Hence, one can take small "extremal" graphs and build out of them graphs with a large k-chromatic number, which also improve the upper bound for $f_k(n)$ given in Theorem 1. The same observation applies for the oriented chromatic number; the Reader can easily provide examples of sparse small extremal graphs for this problem with n vertices and less than $\lceil \log_2 n \rceil (n - \lceil \log_2 n \rceil)$ edges, and describe a recursive construction for obtaining such an extremal graph out of two smaller ones. However, to find an exact guess for the structure of the extremal graphs does not seem to be easy for us.

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