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COLORINGS AND GIRTH OF ORIENTED PLANAR GRAPHS

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Abstract. Homomorphisms between graphs are studied as a generalization of colorings and of chromatic number. We investigate here homomorphisms from orientations of undirected planar graphs to graphs (not necessarily planar) containing as few digons as possible. We relate the existence of such homomorphisms to girth and it appears that these questions remain interesting even if we insist the girth of G is large, an assumption which makes the chromatic number easy to compute. In particular we prove that every orientation of any large girth planar graph is 5-colorable and classify those digraphs on 3, 4 and 5 vertices which color all large girth oriented planar graphs.

Keywords. Graph homomorphisms, Oriented planar graphs, Girth.

1 Introduction and statement of results

Given graphs $G = (V, E)$ and $G' = (V', E')$ a homomorphism from G to G' is any mapping $f : V \to V'$ satisfying

$$[x, y] \in E \implies [f(x), f(y)] \in E'.$$

Here the brackets on both sides of the implication means the same thing : either an edge or an arc. The existence of a homomorphism from G to G' will be denoted by $G \to G'$.

Homomorphisms are clearly related to the chromatic number of undirected graphs (an undirected graph G is k-colorable if and only if there exists a homomorphism from G to K_k, the complete graph on k vertices) and this led to an extensive research (see e.g. [2, 3, 8, 9, 10, 11, 15, 16, 18]). For this reason we will often call colors the vertices of the target graphs (i.e. homomorphic images).

From the homomorphism point of view there is a big difference between directed and undirected graphs. While for undirected graphs the complexity of the problem is well captured by the chromatic number (given a graph G, deciding whether G has a homomorphism to G' is polynomial if and only if G' is bipartite [9]) for directed graphs the problem is much harder and presently seemingly untractable. The present paper is a contribution to this area. We consider (unless otherwise stated) oriented graphs, that is directed graphs not containing two opposite arcs. Oriented graphs are thus orientations of undirected graphs.

Motivated by 3-, 4- and 5-color theorems for (undirected) planar graphs [1, 6, 19], we study similar questions for oriented graphs. It appears that these questions remain interesting even if we insist the girth of G is large, an assumption which makes the chromatic number easy to compute. This is the main motivation of this paper.

More precisely, we consider the following problems :

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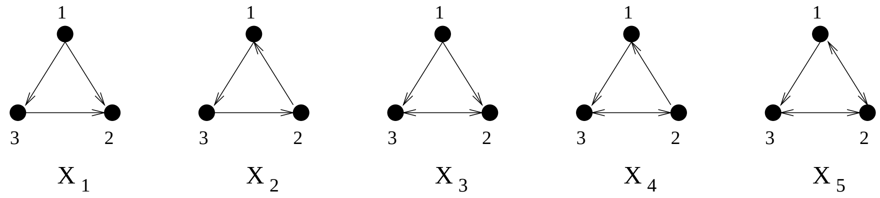


Figure 1: Target digraphs on three vertices

A. The Oriented Coloring Problem : Given an oriented graph $G = (V, A)$, find the smallest number of vertices of an oriented graph $G' = (V', A')$ for which $G \to G'$. This number will be denoted here $\vec{\chi}(G)$ and called the *oriented chromatic number* of G.

Observe that if we consider symmetric digraphs as target graphs this number is then the usual chromatic number. Another motivation stems from a recent paper [17] where it was proved that $\vec{\chi}(G)$ is bounded by a constant for every planar graph G. In fact it was proved that $\vec{\chi}(G) \leq 80$ for every planar graph G and presently this is the best known result.

The oriented coloring problem was further studied in [18] where it was proved that orientations of k-trees and of bounded degree graphs have bounded oriented chromatic number. In particular, the maximum oriented chromatic number of a k-tree is determined up to a $\log k$ factor. Also, examples of planar graphs G with $\vec{\chi}(G) \geq 16$ are presented.

The second problem which we consider is :

B. The Girth Problem : Given an integer $g > 2$, determine the quantity

$$\vec{\chi}(g) \;=\; max\{\ \vec{\chi}(G)\ ;\ G \text{ planar},\ girth(G) = g\ \}$$

We prove the following

Theorem 1

(1) *For every* g, $\vec{\chi}(g) \geq 5$.
(2) $\vec{\chi}(7) > 5$.
(3) *If* $g \geq 16$ *then* $\vec{\chi}(g) = 5$.
(4) *If* $g \geq 11$ *then* $\vec{\chi}(g) \leq 7$.
(5) *If* $g \geq 7$ *then* $\vec{\chi}(g) \leq 12$.
(6) *If* $g \geq 6$ *then* $\vec{\chi}(g) \leq 32$.

We view (3) as yet another 5-color theorem (for high girth planar graphs).

Thus it is clear that for coloring (i.e. homomorphism) of oriented planar graphs of (even) large girth with at most 4 colors one needs some oppositely oriented edges. In this context, we studied in a greater detail celebrated Grötzsch Theorem [6] (see also [19] for a remarkably short proof), which states that every undirected planar graph with no triangle is 3-colorable. Consider the digraphs $X_1, \ldots, X_5$ depicted on Figure 1. We prove :

Theorem 2

(1) *For every* g *and every* $X \in \{X_1, \ldots, X_4\}$, *there exists a graph* $G_{X,g}$ *with girth* g *such that* $G_{X,g} \not\to X$.
(2) *Let* G *be an oriented planar graph with* $girth(G) \geq 16$ *then* $G \to X_5$.
(3) *There exists a graph* G *with girth 8 such that* $G \not\to X_5$.

This result shows in particular that in order to obtain a 3-color theorem for oriented planar graphs with a large girth we need to use a target graph having at least two symmetrical edges.

How many symmetrical edges are needed for 4-color theorem ? Under no girth assumption it is clear that all edges need to be symmetric. The following theorem relates the girth parameter to the

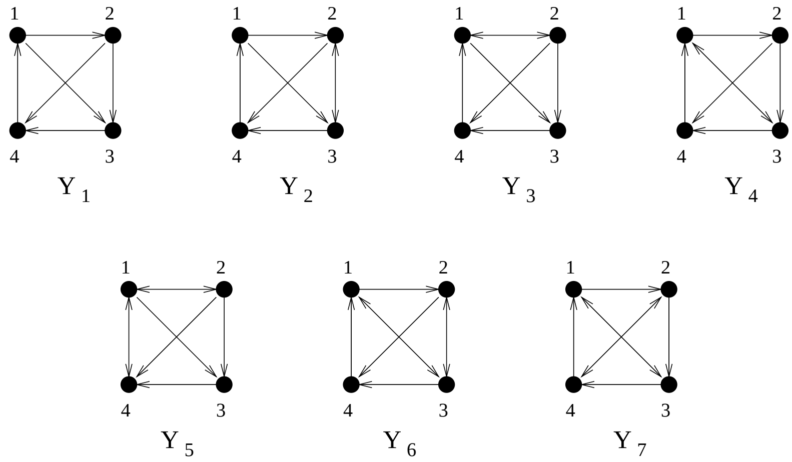


Figure 2: Target digraphs on four vertices

existence of homomorphisms to each digraph with four vertices. Note that the results obtained for a digraph H also hold for the digraph H^{-1}, obtained from H by reversing all the edges. Thus let $Y_1, \ldots, Y_7$ be the digraphs depicted on Figure 2. Then we have :

Theorem 3
(1) *For every g and every $Y \in \{Y_1, Y_2\}$ there exists an oriented planar graph $G_{Y,g}$ with girth g such that $G_{Y,g} \not\to Y$.*
(2) *If G is an oriented planar graph with girth $g \geq 26$ then $G \to Y_3$, $G \to Y_4$ and $G \to Y_5$.*
(3) *If G is an oriented planar graph with girth $g \geq 16$ then $G \to Y_6$.*
(4) *If G is an oriented planar graph with girth $g \geq 11$ then $G \to Y_7$.*

Thus at least one symmetric edge is needed for a high girth oriented 4-color theorem. But this is not sufficient (see Y_2) and two symmetric edges are sufficient in general.

By Theorem 1(3) for an oriented 5-color theorem we do not need any symmetric edge. Below we give the full discussion of this fact for tournaments on five vertices. Let $Z_1, \ldots, Z_5$ be the tournaments depicted on Figure 3 (as stated below, we do not have to consider those tournaments having no 4-cycle). Then the following holds :

Theorem 4
(1) *If Z is a tournament with either no 3-cycle or no 4-cycle then for every g there exists a planar oriented graph $G_{Z,g}$ with girth g such that $G_{Z,g} \not\to Z$.*
(2) *For every g and every $Z \in \{Z_1, Z_2\}$ there exists an oriented planar graph $G_{Z,g}$ with girth g such that $G_{Z,g} \not\to Z$.*
(3) *If G is an oriented planar graph with girth $g \geq 31$ then $G \to Z_3$.*
(4) *If G is an oriented planar graph with girth $g \geq 26$ then $G \to Z_4$.*
(5) *If G is an oriented planar graph with girth $g \geq 16$ then $G \to Z_5$.*
(6) *For every $Z \in \{Z_1, \ldots, Z_5\}$ there exists an oriented planar graph $G_{Z,7}$ of girth 7 such that $G_{Z,7} \not\to Z$.*

In the next section we prove the positive parts of Theorems 1, 2, 3 and 4. In Section 3 we give some examples thus proving the negative parts of the above statements. Section 4 contains some open problems.

One last note regarding oriented coloring of graphs with large girth. Let $G = (V, E)$ be an undirected graph. Subdivide each edge of G by a single vertex and orient the edges such that each new (subdivision) vertex v satisfies $d^+(v) = d^-(v) = 1$. Call the resulting oriented graph G^*. Then we

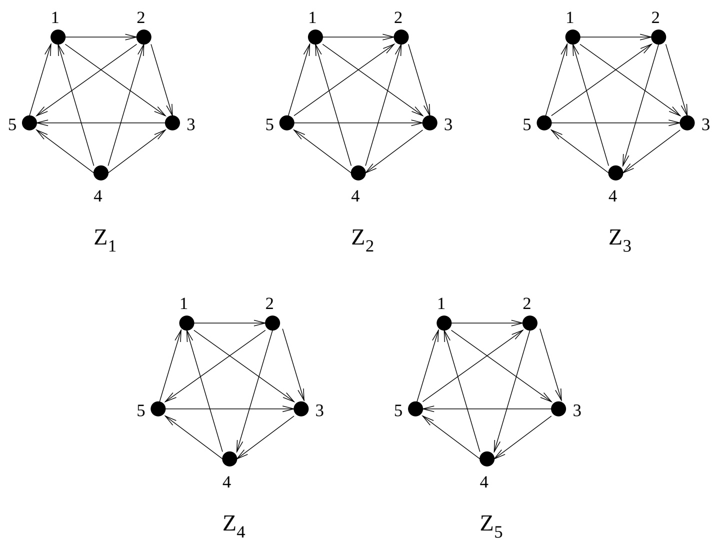


Figure 3: Target tournaments on five vertices

clearly have $\vec{\chi}(G^*) \geq \chi(G)$. Note that the graph G^* is bipartite and that $girth(G^*) \geq 2 \times girth(G)$. Thus assuming a girth 6 for oriented graphs is generally not sufficient for high girth coloring results. We can for instance construct bipartite graphs G^* with high girth and large $\vec{\chi}(G^*)$. Note also that G^* is 2-degenerate (every subgraph of G^* has a vertex with degree at most 2).

2 Upper bounds

Since all the target graphs we will use have no sources and no sinks (that is vertices with in-degree or out-degree zero) every vertex with degree one can be mapped into it. Thus we may assume that all our graphs are orientations of a graph G with minimal degree at least two.

Given an undirected planar graph $G = (V, E)$ we denote an arbitrary orientation of it by $\vec{G}$. Denote by $\overline{V}$ the set of all branching vertices of G (that is vertices with degree at least 3). Clearly we may view G as a subdivision of a graph $\overline{G} = (\overline{V}, \overline{E})$ which we call the *branching graph* of G. Note that $\overline{G}$ is planar and has minimal degree at least 3. As there is a vertex in its dual which has maximum degree 5 we know that some of the faces of $\overline{G}$ have at most 5 incident edges. Now if $girth(G) \geq 5d+1$ then one of the edges of $\overline{G}$ has to be subdivided by d points. Defining the length of a path as the number of edges on it, we thus proved the following :

Lemma 5 *Let G be a subdivision of a branching graph $\overline{G}$, let $girth(G) \geq 5d+1$. Then G contains a path of length $d+1$ all of whose internal vertices have degree 2 in G.*

We will call such a path a *long ear* (of length $d+1$).

Recall that the circulant (directed) graph $G(n; a_1, a_2, \ldots, a_k)$ is the graph whose vertex set is $\mathbb{Z}_n$ and whose arcs are those pairs (x, y) such that $\exists\, i,\ 1 \leq i \leq k,\ y - x \equiv a_i \pmod{n}$. Circulant graphs are clearly transitive. We shall use (for Theorem 1) the circulant tournaments $T_5 = G(5; 1, 2)$ (also depicted as Z_5 on Figure 3) and $T_7 = G(7; 1, 2, 3)$. These tournaments have the following properties :

Lemma 6 *Let P_3 be an arbitrary oriented path of length 3 with end-vertices a and b. For every pair x, y of (not necessarily distinct) vertices of T_7 there exists a homomorphism $f : P_3 \to T_7$ such that $f(a) = x$ and $f(b) = y$.*

Lemma 7 *Let P_4 be an arbitrary oriented path of length 4 with end-vertices a and b. For every pair x, y of (not necessarily distinct) vertices of T_5 there exists a homomorphism $f : P_4 \to T_5$ such that $f(a) = x$ and $f(b) = y$.*

Lemmas 6 and 7 can be derived from the following more general statement :

Proposition 8 *Let $G_{n,d}$ be the circulant graph $G(n;1,2,\ldots,d)$. Then the end-vertices of every oriented path P of length at least $\frac{n-1}{d-1}$ can be mapped by a homomorphism $P \to G_{n,d}$ to any pair of (not necessarily distinct) vertices of $G_{n,d}$.*

Proof. Let P be an oriented path of length $t \geq \frac{n-1}{d-1}$ with vertices $a_0, a_1, \ldots, a_t$. As $G_{n,d}$ is a transitive graph we may assume that a_0 maps to 0. Considering just the initial part P_i of P with vertices $a_0, a_1, \ldots, a_i$ we denote by A_i the possible images of the vertex a_i under a homomorphism $f_i : P_i \to G_{n,d}$. We clearly have $|A_1| = d$. It then suffices to prove that for each $i \leq t$, $|A_i| = i(d-1)+1$ since we will then have $|A_t| = n$. However it is clear that the set A_1 is formed by d consecutive (modulo n) integers (either $\{1,2,\ldots,d\}$ if $(a_0,a_1) \in P$ or $\{n-1,n-2,\ldots,n-d\}$ if $(a_1,a_0) \in P$). And assuming that A_i, $i < t$, is a set of consecutive integers then A_{i+1} is again a set of consecutive integers of length $|A_i| + (d-1)$. □

Note that the circulant tournament $G(7;1,2,4)$ also satisfies the claim of Lemma 6. This tournament has been used in [18] where it is proved that it is a homomorphic image of any oriented outerplanar graph. Moreover this tournament is optimal (and unique) since there exist oriented outerplanar graphs with oriented chromatic number 7.

Let us now turn to the proof of our statements.

Proof of Theorem 1(3,4).
(3) Let $G = (V,E)$ be a planar graph of girth at least 16. We prove by induction on $|V|$ that $G \to Z_5$. By Lemma 5 G contains a long ear P of length at least 4, with end-vertices a and b. For the graph $G' = G \setminus P$ we can use the induction hypothesis and so get a homomorhism $f' : G' \to Z_5$. Put $x = f'(a)$, $y = f'(b)$ and apply lemma 7 to get a homomorphism $f : P \to Z_5$. Clearly f' and f may be combined to get a homomorphism $G \to Z_5$. Thus $\vec{\chi}(G) \leq 5$ and using Theorem 1(1) we get the desired result.

(4) This can be proved analogously by using Lemmas 5 and 6. □

Proof of Theorem 2(2).
It suffices to check that for every oriented path P of length 3 and for every pair x, y of (not necessarily distinct) vertices of X_5 there exists a homomorphism $f : P \to X_5$ such that $f(a) = x$ and $f(b) = y$. Lemma 5 then implies the desired result. □

Proof of Theorem 3(2,3,4).
$(2,3)$ This is obtained in the same way as before, by considering oriented paths of length 4 (Y_6) and 6 (Y_3, Y_4 and Y_5).

(4) The graph Y_7 on Figure 2 is in fact the circulant graph $G(4;1,2)$. Hence, combining Lemma 5 and Proposition 8 we get the desired result. □

Proof of Theorem 4(3,4,5).
The result concerning Z_5 has already been established in the proof of Theorem 1(3). For Z_3 and Z_4 we still use the same technique by considering all paths of length respectively 7 and 6. □

It remains to establish the upper bounds in Theorem 1(5,6). For this we use a different technique. Recall that an acyclic k-coloring of an undirected graph is a coloring which uses k colors and such that every cycle uses at least three colors. The following has been proved in [17] :

Lemma 9 [17] *For every k, there exists an oriented graph H_k such that for any orientation $\vec{G}$ of an undirected graph G with an acyclic k-coloring there exists a homomorphism $\vec{G} \to H_k$. Moreover, the size of H_k is $k \times 2^{k-1}$.*

In [4] Borodin proved that any planar graph has an acyclic 5-coloring. By Lemma 9 we thus obtain that $\vec{\chi}(3) \leq 80$ [17]. Any planar graph with girth $g \geq 11$ contains a long ear of length at least 3 and thus can be acyclically 3-colored. Borodin, Kostochka and Woodall [5] recently proved that every planar graph with girth $g \geq 6$ (resp. $g \geq 7$) can be acyclically 4-colored (resp. acyclically 3-colored). This gives the bounds stated in Theorem 1(5,6). Relationships between oriented colorings and acyclic colorings have been considered in [14] where it was proved that a family $\mathcal{F}$ of undirected graphs is acyclically colorable using a bounded number of colors if and only if all the orientations of the graphs in $\mathcal{F}$ have oriented chromatic number bounded by some constant.

3 Lower bounds

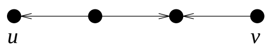
In this section we will construct some sample planar graphs thus proving the negative statements of Theorems 1, 2, 3 and 4.

Proof of Theorem 1(1,2).
(1) Suppose that there exists a tournament T_4 on 4 vertices such that every oriented planar graph G with sufficiently large girth has T_4 as a homomorphic image. Since any (directed) cycle having p vertices with $p \equiv 1$ or $2 \pmod 3$ cannot be 3-colored, T_4 contains a directed 4-cycle. Thus T_4 is the tournament Y_1 depicted on Figure 2.
For every g we now construct an oriented planar graph $G_{T_4,g}$ such that $G_{T_4,g} \not\to T_4$, thus leading to a contradiction. Let P_g be the oriented path on $\lfloor \frac{g}{2} \rfloor$ vertices whose edges have alternatively forward and backward direction and let u and v denote its end-vertices :

The graph $G_{T_4,g}$ is then constructed as follows : let $x_1 x_2 \ldots x_p$ be a directed cycle on $p \geq g$ vertices, $p \equiv 1$ or $2 \pmod 3$. To every vertex x_i attach two copies of P_g by identifying the two u-vertices with x_i and adding an edge linking the two v-vertices. The graph $G_{T_4,g}$ thus obtained has clearly girth g or $g+1$. Moreover, for every homomorphism $f : G_{T_4,g} \to T_4$, one vertex x_i at least satisfies $f(x_i) = 4$. It is then easy to check that the two v-vertices of the paths attached to x_i are mapped to the same color, namely 1 or 4 depending on the parity of $\lfloor \frac{g}{2} \rfloor$. Since these two vertices are joined by an edge we obtain the desired contradiction and the result follows.

(2) According to Theorems 1(1) and 4(1) it suffices to consider the tournaments on five vertices depicted on Figure 3. For Z_1 and Z_2 the result will be proved later (see the proof of Theorem 4(2)). We now construct for every tournament $Z \in \{Z_3, Z_4, Z_5\}$ an oriented planar graph $G_{Z,7}$ of girth 7 such that $G_{Z,7} \not\to Z$. The three corresponding constructions are given below.

Construction of $G_{Z_3,7}$: Let P be the following oriented path :



The graph $G_{Z_3,7}$ is then obtained as follows : let $x_1, \ldots, x_7$ be a directed 7-cycle. To every vertex x_i attach five copies of P by identifying the u-vertices with x_i. Denote by $v_1, \ldots, v_5$ the corresponding v-vertices and add all the edges (v_i, v_{i+1}), $1 \leq i < 5$. Let now f be any homomorphism from $G_{Z_3,7}$ to Z_3. Since all the directed 3-cycles in Z_3 contain color 4, at least one vertex x_i is mapped by f to color 4. It is easy to check that the corresponding v-vertices must then be assigned colors 1, 2, 3 or 5. But the 4-tournament induced by these colors is transitive so that one cannot color all the five v-vertices.

Construction of $G_{Z_4,7}$: Let P_3 denote the directed path of length 3 and u, v denote its end-vertices. The graph $G_{Z_4,7}$ is constructed as follows : let $x_1, \ldots, x_7$ be a directed 7-cycle. To every vertex x_i attach four copies of P_3 by identifying the u-vertices with x_i (denote by $v_1, \ldots, v_4$ the

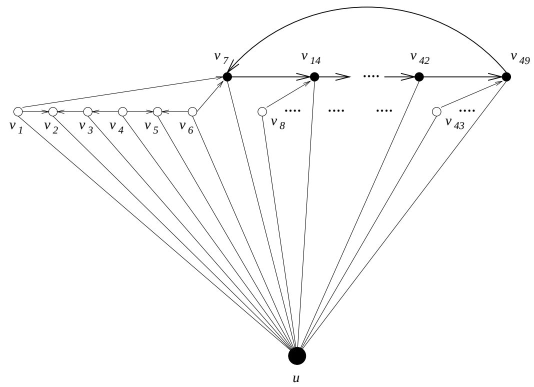


Figure 4: Construction of $G_{Z_5,7}$

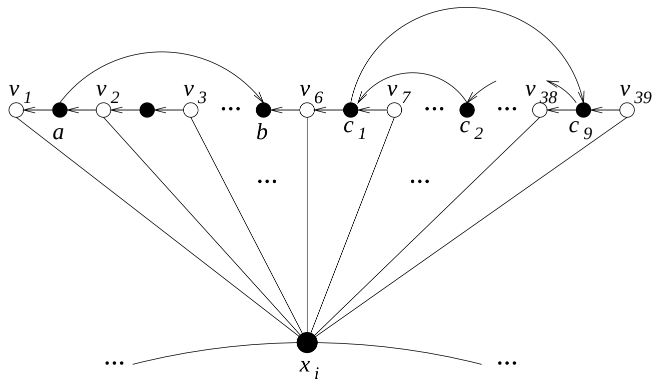


Figure 5: Construction of $G_{X_5,8}$

four corresponding v-vertices) and four copies of P_3 by identifying the v-vertices with x_i (denote by $u_1, \ldots, u_4$ the four corresponding u-vertices). Add then the edges (v_i, v_{i+1}) and (u_i, u_{i+1}) for every i, $1 \leq i < 4$. Let now f be any homomorphism from $G_{Z_4,7}$ to Z_4. Since all the directed 3-cycles in Z_4 use colors 2 or 3 at least one of the vertices x_i is mapped by f to color 2 or 3. If x_i is mapped to color 2 (resp. to color 3) then all the corresponding u-vertices (resp. v-vertices) have to be mapped to colors 2, 3 or 4 (resp. to colors 1, 2 or 3). But in both cases these 3 colors induce a transitive 3-tournament so that the corresponding u- or v-vertices cannot be colored.

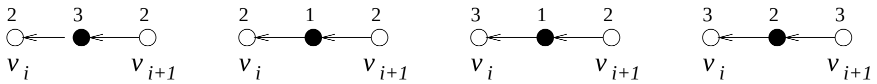
Construction of $G_{Z_5,7}$ *:* This construction is illustrated by Figure 4. Let P_3 denote the directed path of length 3 and u, v denote its end-vertices. Take 49 copies of P_3, identify all the u-vertices and denote by $v_1, v_2, \ldots, v_{49}$ the corresponding v-vertices. Then add edges in order to construct the directed 7-cycle $v_7, v_{14}, \ldots, v_{7j}, \ldots, v_{49}$. Finally, for every j, $1 \leq j \leq 7$ add the edges (v_{7j-6}, v_{7j-5}), (v_{7j-4}, v_{7j-5}), (v_{7j-3}, v_{7j-4}), (v_{7j-3}, v_{7j-2}), (v_{7j-1}, v_{7j-2}), (v_{7j-1}, v_{7j}), and (v_{7j-6}, v_{7j}). Let us now prove that there is no homomorphism from the graph $G_{Z_5,7}$ thus obtained to Z_5. Let f be such a homomorphism. Since Z_5 is vertex-transitive we may assume that $f(u) = 1$. Then all the v-vertices have to be assigned colors 1, 2, 4 or 5. Due to the 7-cycle linking the vertices v_{7j}, $1 \leq j \leq 7$ at least one of them, say v_{7k}, has to be assigned color 4. It is then easy to check that the 7-cycle on vertices $v_{7k-6}, v_{7k-5}, \ldots, v_{7k}$ cannot be colored by using only colors 1, 2, 4 or 5.

This concludes the proof. $\square$

Proof of Theorem 2(1,3).

(1) The result for graphs X_1 and X_2 is already stated in Theorem 1(1). The graph X_3 contains a 2-cycle but no 3-cycle so no directed cycle having an odd number of vertices can be mapped to X_3. For every g let us now construct an oriented planar graph $G_{X_4,g}$ of girth g such that $G_{X_4,g} \not\to X_4$. Let P_g denote the path of length $\lfloor \frac{g}{2} \rfloor$ whose edges have alternatively forward and backward direction and u, v denote its end-vertices (as depicted in the proof of Theorem 1(1)). Let $x_1, x_2, \ldots, x_g$ be a directed cycle on g vertices. To every vertex x_i attach two copies of P_g by identifying the two u-vertices with x_i and adding an edge linking the two v-vertices. Let now f be any homomorphism from the graph $G_{X_4,g}$ thus obtained to X_4. At least one vertex x_i must be assigned color 3. It is easy to check that the two corresponding v-vertices have also to be assigned the same color (3 or 2 according to the parity of $\lfloor \frac{g}{2} \rfloor$), a contradiction since they are linked by an edge.

(3) We will construct a graph $G_{X_5,8}$ (see figure 5) such that $G_{X_5,8} \not\to X_5$. Let P_3 be the directed path of length 3 and u, v denote its end-vertices. The graph $G_{X_5,8}$ is then constructed as follows : let $x_1 x_2 \ldots x_9$ be a directed cycle of length 9. To every vertex x_i, $1 \le i \le 9$, we attach 39 copies of P_3 by identifying all the u-vertices with vertex x_i. Let $v_1, v_2, \ldots, v_{39}$ denote the corresponding v-vertices. Between every two vertices v_i and v_{i+1} we then add a path of length 2 having two backward edges (those additional "middle" vertices are drawn as full circles in figure 5). Let a (resp. b) denote the middle vertex lying between v_1 and v_2 (resp. v_5 and v_6). We then add an edge from a to b. We finally add a directed 9-cycle connecting all the middle vertices lying between v_6 and v_7, v_{10} and v_{11}, …, v_{38} and v_{39}. We will denote by $c_1, \ldots, c_9$ those middle vertices linked by the 9-cycle.
Let us now prove that $G_{X_5,8} \not\to X_5$. Since $x_1 x_2 \ldots x_9$ is a directed 9-cycle, every homomorphism f from $G_{X_5,8}$ to X_5 has to use all the three colors 1, 2 and 3. Hence there is at least one vertex x_i with $f(x_i) = 3$. It is then easy to check that all the v-vertices have to be mapped to colors 2 or 3. The only possibilities for coloring the middle vertices are thus the following :



Therefore if a vertex v_i is assigned color 3 then all the vertices v_j, $j < i$ must be also assigned color 3. The 9-cycle $c_1, c_2, \ldots, c_9$ must use the color 3. Then so do all the vertices $v_1, v_2, \ldots, v_6$, which implies that both vertices a and b are assigned the same color 2, a contradiction since there are linked by an edge. □

Proof of Theorem 3(1).
For Y_1 the result is a consequence of Theorem 1(1). For every g let us now construct an oriented planar graph $G_{Y_2,g}$ of girth g such that $G_{Y_2,g} \not\to Y_2$. Let P_g denote the path of length $\lfloor \frac{g}{2} \rfloor$ whose edges have alternatively forward and backward direction and u, v denote its end-vertices (as depicted in the proof of Theorem 1(1)). Let $x_1, x_2, \ldots, x_{2k+1}$ be a directed cycle on $2k+1$ vertices, $2k+1 \ge g$. To every vertex x_i attach two copies of P_g by identifying the two u-vertices with x_i and adding an edge linking the two v-vertices. Let now f be any homomorphism from the graph $G_{Y_2,g}$ thus obtained to Y_2. Since $x_1, \ldots, x_{2k+1}$ is an odd cycle, at least one vertex x_i must be assigned color 4. Then the two corresponding v-vertices have also to be assigned the same color (4 or 1 according to the parity of $\lfloor \frac{g}{2} \rfloor$), a contradiction since they are linked by an edge. □

Proof of Theorem 4(1,2,6).
(1) It is sufficient here to remark that if Z does not contain a 3-cycle (it is then a transitive tournament) then no directed cycle can be mapped to Z. If Z does not contain a 4-cycle (it obviously neither contains a 5-cycle) then no directed cycle on p vertices, $p \equiv 1$ or $2 \pmod 3$, can be mapped to Z.

(2) We construct for every g an oriented planar graph $G_{Z_1,g}$ such that $G_{Z_1,g} \not\to Z_1$. Consider the graph G_g used in proof of Theorem 1(1). Since G_g cannot be mapped to Y_1 every homomorphism $f : G_g \to Z_1$ has to use color 5. The graph $G_{Z_1,g}$ is then obtained from G_g by associating with every vertex x of G_g a new vertex v_x and an edge directed from v_x to x. Since color 5 has incoming degree

0 in Z_1 we are done.
For Z_2 it is also sufficient to consider the graph G_g used in proof of Theorem 1(1). Every cycle in Z_2 uses color 3. Thus in every homomorphism $f : G_g \to Z_2$ there is at least one vertex x_i with color 3. The two v-vertices of the paths attached to x_i are then mapped to the same color (3 or 4 according to the parity of $\lfloor \frac{g}{2} \rfloor$) which leads to a contradiction since they are linked by an edge.

(6) This result is already stated in Theorem 1(2). □

4 Discussion and open problems

1. The main open problem is to narrow the difference between the lower and upper bounds for our extremal function $\vec{\chi}(g)$.

2. The complexity of coloring by small digraphs was considered by Bang-Jensen, Hell and McGillivray [3]. In particular they proved that coloring by any semicomplete graph (that is a digraph which arises from a tournament by the addition of some arcs) with 3 and 4 vertices containing 2 cycles is an NP-complete problem. This supports their general conjecture that the existence of H-coloring is NP-complete for a semicomplete graph H if and only if H contains at least two cycles. However this does not apply to the problems considered in this paper as we consider problems restricted to planar graphs of large girth. As shown for instance by the proofs of Theorem 1(3) and Theorem 3(4) these problems can be in P even under the existence of many cycles in the target graph (Z_5 and Y_7 respectively).

3. Given a class $\mathcal{K}$ of graphs we say that a graph H is *universal* if $G \to H$ for any graph $G \in \mathcal{K}$. Thus $\vec{\chi}(G)$ is bounded for $\mathcal{K}$ if and only if there exists a universal graph H. Universal graphs were studied e.g. in [8, 12, 18]. In a forthcoming paper we will discuss classes of planar graphs for which there exists a planar universal graph.

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