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ON FOUR COLORING PROBLEMS[1]

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Abstract. Coloring problems include some of the most classical combinatorial problems (such as map coloring problems or edge coloring of cubic graphs). But the area is alive and well beeing refreshed by a constant stream of new problems and solutions of old ones.

In this paper we complement this by a survey of four particular problems which are all related to the notion of a homomorphism. This forms one of the most recent approaches to coloring problems.

Keywords. Chromatic number, Planar graphs, Homomorphism, Gap, Density, Quasiordering, Oriented chromatic number, Game chromatic number.

1 Introduction

This paper was written for the traditional Spring School of the Combinatorial Seminar at Charles University which was held in April 1999 in Borová Lada and in Finsterau. In 1999 this school was organized jointly with Humboldt Universitaet Berlin, Universitaet Bonn and Université Bordeaux I. Teachers from these schools took part in the meeting. The text tries to provide a study text for (undergraduate) students and it should serve as a background for the discussions and lectures at Spring School.

Some additional material and some complementary information can be found in the following articles:

[7] O.V. Borodin, A.V. Kostochka, J. Nešetřil, A. Raspaud and E. Sopena: On the maximum average degree and the oriented chromatic number of a graph, *Discrete math.*, to appear.

[24] A.V. Kostochka, E. Sopena and X. Zhu. Acyclic and oriented chromatic numbers of graphs, *J. Graph Theory* **24**, n° 4 (1997), 331–340.

[36] J. Nešetřil: Structural combinatorics - Graph homomorphisms and their use, to appear.

[38] J. Nešetřil, A. Raspaud and E. Sopena: Colorings and girth of oriented planar graphs, *Discrete Math.* **165/166** (1997), 519–530.

[40] J. Nešetřil, C. Tardif: Duality Theorems for Finite Structures (characterizing gaps and good characterizations), KAM-DIMATIA Series 98-407 (submitted).

[41] J. Nešetřil, C. Tardif: Density, KAM-Dimatia Series 98-415, to appear in *Proceedings of Štiřín Castle meeting 1997*, AMS (1999).

[48] E. Sopena: The chromatic number of oriented graphs, *J. Graph Theory* **25** (1997), 191–205.

[49] E. Sopena: Oriented graph colouring, to appear in *Proc. Fifth Czech-Slovak Symposium on Combinatorics, Graph Theory, Algorithms and Applications*, Prague, 1998.

[53] X.Zhu : A Survey on Hedetniemi's Conjecture, *Taiwanese J. Math.* **2** (1998), 1−24.

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[54] X. Zhu: Circular chromatic number: a survey, to appear in *Proc. Fifth Czech-Slovak Symposium on Combinatorics, Graph Theory, Algorithms and Applications*, Prague, 1998.

For general terminology and introduction we refer to

[5] B. Bollobás: *Modern Graph Theory*, Graduate Texts in Mathematics, vol. 184, Springer (1998).

[29] L. Lovász: *Combinatorial problems and exercises*, North-Holland, 2nd Edition, 1993.

[30] J. Matoušek and J. Nešetřil: *Invitation to Discrete Mathematics*, Oxford University Press, 1998.

The reference

[21] T.R. Jensen and B. Toft: *Graph Coloring Problems*, Wiley-Interscience Series in Discrete Mathematics and Optimization, 1995.

deals exclusively with graph coloring problems; the interested reader will find there a lot of interesting research problems!

This paper is organised as follows:

Contents

The problems stated as questions in the text (there are 58 of them) are mostly (with some exceptions) routine problems which serve to improve understanding. Problems in sense of research problems are stated as problems. We tried to make the paper selfcontained (for the benefit of students which are coming with different background) so the first part reviews the material from the (almost) very beginning.

1.1 Chromatic number

Given an undirected graph G, with vertex set $V(G)$ and edge set $E(G)$, a *k-coloring* of G is a partition of $V(G)$ into k classes such that no edge is contained in any of the classes. The classes, say $V_1, \ldots, V_k$, of the partition are also called *color classes* (see Figure 1).

Viewing such a schema one of the main applications of coloring – scheduling – suggests itself: given an unlimited supply of lecture rooms, how to schedule the talks so that no conflict in rooms,

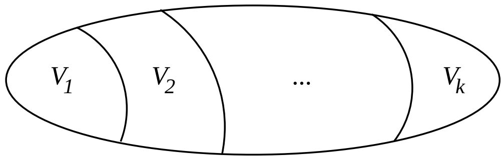


Figure 1: Partition of the vertex set into color classes

topics, lecturers, student mobility, etc. arise? Or: how to assign telecommunication channels to satellites (or cellular phone transmitters) so that overlaps and jamming is as low as possible? Both these applications are not textbooks fairy tales but actively pursued problems.

A related (in a dual sense) notion is the notion of an independent set: a subset $A \subseteq V(G)$ is called *independent* in G if it contains no edge of G. Clearly then any k-coloring is a partition of $V(G)$ into k independent sets (see again Figure 1).

Now this makes possible quantitative analysis: Denote by $\chi(G)$ the minimal k for which there exists a k-coloring of a graph G. $\chi(G)$ is called the *chromatic number* of G. On the other hand let $\alpha(G)$ denote the maximal size of an independent set A in G. $\alpha(G)$ is called the *independence number* of G.

By the above discussion (see also Figure 1) we know that

$$\alpha(G) \cdot \chi(G) \;\geq\; |V| \tag{1}$$

This is trivial but there is more here than meets an eye. If we rewrite this as

$$\chi(G) \;\geq\; \frac{|V|}{\alpha(G)} \tag{2}$$

then this inequality is very important and interesting for several reasons which we are going to explain.

(i) Inequality (2) gives a lower bound for the chromatic number of a graph. In fact, this is one of the very few *lower* bounds for chromatic number. The lower bounds for chromatic number are hard to get and one of the reasons for this may be the phaenomenon known as *good characterization* (or, from the theoretical computer science point of view, the class NP $\cap$ coNP).

The meaning of this is perhaps best to illustrate by the following story: Suppose we want to prove that a certain graph G satisfies $\chi(G) \leq k$. This may be a very difficult task but once a partition into independent sets is found it is easy to verify that this particular partition is indeed a coloring. In other words, the problem:

Input: a graph G,
Question: Does $\chi(G) \leq k$ hold?

is an NP-problem.

Other situation arises if we consider the problem of determining whether $\chi(G) > k$ or not. Here it is not clear how to convince our enemy (or ourselves) that this is indeed so. This lower bound inequality claims that there is no coloring with k colors and that seems to be difficult to see and it is not clear how to verify this easily. In other words the problem $\chi(G) > k$ is not known to be in NP *or* the problem $\chi(G) \leq k$ is not known to be in the complexity class denoted as coNP.

Problems which belong to both NP and coNP are also said to have a *good characterization.* More on this can be read for instance in [44, 47]. Perhaps it is worth to add that in the hands of Edmonds the notion of a good characterization was a cradle of complexity theory.

(ii) The inequality (2) is one of the very few general lower bounds for chromatic number. It was first isolated by P. Erdős in the thirties and we feel justified to call it *Erdős inequality.*

Upper bounds are

(a) $\chi(G) \leq |V(G)| - \alpha(G) + 1$,

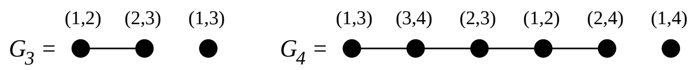


Figure 2: Graphs with $\chi(G_k) = \lceil \log_2 k \rceil$ and $\omega(G_k) = 2$

(b) $\chi(G) \leq \Delta(G) + 1$ (with equality only for odd cycles and complete graphs as you know from Brooks's theorem).

Besides the Erdős inequality the following general lower bound (which goes back to Shannon) is useful and important:

$$\chi(G) \geq \omega(G) \tag{3}$$

where $\omega(G) = \alpha(\overline{G})$ is the maximal size of a complete subgraph in G.

(iii) It is hard to say why the bounds (2) and (3) are important. On one side it is easy to construct examples which show that these bounds are very weak. For example, for the graph $G = K_k + \overline{K_n}$ (that is the disjoint union of a complete graph and a discrete graph), we have $\chi(G) = k$ while $\alpha(G) = n + 1$. Therefore, for large n, $\frac{|V(G)|}{\alpha(G)} < 1 + \varepsilon$ (for any $\varepsilon > 0$).

Another combinatorial classique is the existence of graphs G with $\chi(G) = k$ and $\omega(G) = 2$ (that is containing no triangle). Consider for instance the following (important) example (see Figure 2). Let $X = \{1, 2, \ldots, n\}$ and let G_n be the graph defined as follows:

$$V(G_n) = \{\ (i,j)\ ;\ 1 \leq i < j \leq n\ \},\quad E(G_n) = \{\ \{(i,j),(k,l)\}\ ;\ i < j = k < l\ \}.$$

One can check that

$$(a)\ \omega(G_n) = 2, \qquad (b)\ \chi(G_n) = \lceil \log_2 n \rceil.$$

(To prove that $\chi(G_n)$ is unbounded follows directly from Ramsey's Theorem.)

Question 1
(i) Prove (a) and (b).
(ii) Do you see the above mentioned Ramsey argument?

For an oriented graph G define the *line graph* $L(G)$ of G as follows: $V(L(G)) = E(G)$ and $((u,v),(x,y)) \in E(L(G))$ if and only if $v = x$ (see Figure 3).

Question 2
Prove that if G (we mean here the undirected underlying graph of G) has chromatic number k then $L(G)$ has chromatic number at least $\lceil \log_2 k \rceil$.

Denote by $L^{(k)}(G)$ the graph which we obtain by iterating k times the line graph construction: $L^{(k)}(G) = L(L(\ldots(L(G))\ldots))$.

Question 3
Prove $\chi(L^{(k)}(G)) \geq \lceil \log^{(k)} \chi(G) \rceil$.

Question 4
(i) Give a combinatorial meaning of the graphs $L^{(k)}(G)$ (in terms of subgraphs of graph G).
(ii) Prove that $L(T_n) = G_n$ where T_n is the transitive tournament with n vertices and G_n was defined above.
(iii) What is a shortest cycle in $L^{(k)}(G)$? What is a shortest odd cycle in $L^{(k)}(G)$ for acyclic G ?

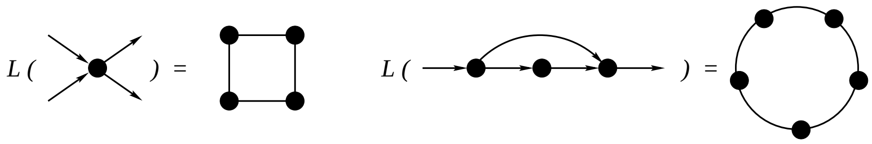


Figure 3: Sample line graphs

Despite of all this evidence it is perhaps surprising that the inequalities (1), (2) and (3) lead to important classes of graphs. Perhaps the most widely studied such class is the class of perfect graphs: a graph G is *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. In this extreme case, inequalities (2) and (3) are linked by the following result of Lovász [28]:

Theorem 1 (Perfect Graph Theorem)
For a graph G the following statements are equivalent:
(i) G is perfect,
(ii) the complement $\overline{G}$ of G is perfect,
(iii) any induced subgraph G' of G satisfies $\omega(G') \cdot \alpha(G') \geq |V(G')|$.

The most celebrated and challenging problem in this area is the following so-called *perfect graph conjecture*, proposed by C. Berge in 1960:

Conjecture 2
A graph G is perfect if and only if neither G nor its complement $\overline{G}$ contains an induced cycle of length at least 5.

Question 5
List at least 7 (of course 7!) naturally defined classes of perfect graphs.

(*iv*) Another reason why Erdős inequality is important is the fact that for almost every graph this bound is tight. This was shown by Erdős that for the random graph $\mathbf{G}_n$ on n vertices (where every edge independently appears with probability 1/2), we have $\omega(\mathbf{G}_n) = 2\log_2 n$. For chromatic number we have that $\chi(\mathbf{G}_n)$ is of order $\frac{n}{2\log_2 n}$. The "2" here was a major achievement by Bollobas [4] (see also Alon and Spencer [1] for a textbook on the probabilistic method.)

But this leads to a whole new area and it is out of the scope of this paper.

1.2 Planar graphs

During more than one century the Four-Color Conjecture (4CC) was certainly one of the most popular unsolved problems of mathematics. Maybe the whole field of graph theory has grown out of this fertile problem which certainly has had a seminal role in modern (yes, applied) mathematics.

The story began in 1852 when the claim *"Every planar map must be 4-colorable!"* flashed across the mind of Francis Guthrie while coloring a map of England... His brother communicated this conjecture to his teacher [2], Augustus De Morgan, who then told other mathematicians about this problem. The first written reference to this problem appeared in the *Proceedings of the London Mathematical Society* in a paper by Cayley in 1878. The next year, Kempe published the first "proof" of the conjecture. In 1890, Heawood pointed out the error in Kempe's argument but, using Kempe's idea [3], proved that five colors are sufficient for coloring any planar map.

[2] Here is a personal message to all students attending the 1999 Spring School: please, do not hesitate to ask for any strange question, even if the answer seems to be difficult (or seemingly easy).

[3] This is the celebrated Kempe chains argument which you probably know from school.

The story of this problem is very rich and crosses a lot of famous mathematicians. The interested reader can read more (and a lot) about it in Ore [43] or Thomas [50].

In 1976, Kenneth Appel and Wolfgang Haken, from the University of Illinois, announced a complete (computer-aided) proof of the 4CC [2], based on a method introduced by Heesch in 1969. The basic idea of the so-called method of "reducible configurations" is the following. A configuration (that is a subgraph with some vertices with prescribed degrees) C is *reducible* if every planar graph G containing it is 4-colorable whenever $G - C$ is 4-colorable. In other words, no minimal counter-example to the 4CC can contain a reducible configuration. A set of configurations is *unavoidable* if every planar graph has to contain at least one of these configurations. To prove the unavoidability of such a set the key point is to define some *discharging rules* which lead to a contradiction with Euler's formula if the graph does not contain any of the unavoidable configurations. This technique, on a very simple example, will be illustrated in Section 3 (see the proof of Theorem 30).

Now, for proving the 4CC, it is enough to exhibit an unavoidable set of reducible configurations! In their proof, Appel and Haken used an unavoidable set of over 1900 reducible configurations and made use of more than 300 discharging rules (this clearly explains why they needed to use some computer!). This original proof was verified and simplified (and in fact reproved from the beginning) in 1997 by Robertson, Sanders, Seymour and Thomas [46] who used an unavoidable set of 'only' 633 reducible configurations and 'only' 32 discharging rules (however the proof still relies on computer checking...).

In this text we shall just need elementary properties of planar graphs. But one should remark that 4CC was responsible for a lot of theory related to planar graphs and graphs in general (this is also documented in the accompanied text by Jarik Nešetřil and André Raspaud: Dualities, Flows and Cycle Covers).

At this place let us mention one aspect which is important for our further discussion: we say that a graph H with vertices $\{v_1, ...v_n\}$ is *minor* of a graph G if and only if G contains a subgraph G' and vertex disjoint connected subgraphs $G_1, \dots, G_n$ of G' such that v_i and v_j are joined by an edge in H *if and only if* there is an edge between G_i and G_j in G'. This is denoted by $H \preceq G$.

Complicated isn't it? However this definition is natural and it is possible to say that it captures the mixed nature (i.e. geometrical, algebraic and combinatorial aspect) of graphs very well. One can also define minor relation more concisely and for example to generate minor relation $\preceq$ by iterating the edge deletion and edge contraction operations.

It is important that the minor operation preserves topological properties of graphs: if G is planar then any of its minors is planar too. Thus such ("minor closed") properties we can characterize by forbidden minors. Of course we can then restrict only to those forbidden minors which are minimal or which are mutually not related by minor relation $\preceq$.

And here comes a surprise (we would like to say major surprise); any set of planar graphs which are minor incomparable is finite. And not only that: according to Robertson and Seymour project and a grand series of papers[4] we have

Theorem 3
Any set of minor incomparable graphs is finite.

This result is certainly the most complicated result quoted in this paper. Relations with the property stated in the theorem were studied intensively and we shall discuss this later in this paper (see Subsection 2.3).

The problems we shall discuss in Sections 3 and 4 are related to planar graph coloring.

1.3 Homomorphisms

Graph theory receives its mathematical motivation connection from the two main areas of mathematics: algebra and geometry (topology and the graph notions stood at the birth of algebraic topology).

[4]See N. Robertson and P. Seymour: from *Graph minors. I: Excluding a forest* (1983), to *Graph minors. XV: Giant steps* (1996), published in *J. Combin. Theory, Ser. B.*

Consequently various operations and comparisons for graphs stress either its algebraic part (e.g. various products) or geometrical part (e.g. contraction, subdivision). It is only natural that the key place in the modern graph theory is played by (fortunate) mixtures of both approaches as exhibited best by the various modifications of the notion of graph minor. However from the algebraic point of view perhaps the most natural graph notion is the notion of a homomorphism:

Definition 4 (Homomorphism)
Given two graphs G and G' a *homomorphism* f of G to G' is any mapping $f : V(G) \to V(G')$ which satisfies the following condition:

$$(1)\quad [x, y] \in E(G) \text{ implies } [f(x), f(y)] \in E(G').$$

The condition (1) should be understood as follows: on both sides of the implication one considers the same type of edges (undirected or directed). The analogous definitions give the notions of the homomorphisms for hypergraphs (set systems) and relational systems.

The existence of a homomorphism from G to H will be denoted by $G \to H$, in this case we also say that *G is homomorphic to H*, the non-existence of such a homomorphism by $G \not\to H$ and in such a case we say that *G fails to be homomorphic to H*. If G is homomorphic to H and also H is homomorphic to G then we say that G and H are *homomorphism equivalent* (or simply *hom-equivalent*) and we denote this by $G \sim H$.

Homomorphism is an algebraic notion which in graph theory founds its way to problems related to products, reconstruction and chromatic polynomials, just to name a few. Our approach here is motivated by the chromatic number connection expressed by the following observation which holds for every undirected graph G

$$G \to K_k \text{ if and only if } \chi(G) \leq k. \tag{4}$$

Proceeding a bit further let $< G, H >$ denotes the set of all homomorphisms from G to H and $|G, H|$ denote the size of the set $< G, H >$. The number $|G, K_k| = | < G, K_k > |$ is usually denoted by the symbol $f(G, k)$ from the following reason: for every graph G there exists a unique real polynomimal $f(x)$ with integer coefficients such that for every positive integer k holds $f(k) = f(G, k)$. This polynomial is called the *chromatic polynomial* of the graph G.

For instance, the chromatic polynomial of the graph $K_4 - e$ is $x^4 - 5x^3 + 8x^2 - 4x$ (which tells that there exist no 1-coloring, no 2-coloring, six 3-colorings, etc. of $K_4 - e$).

Question 6
(i) Prove the *deletion contraction equality* $f(G, k) = f(G + e, k) + f(G.e, k)$ for every pair $e = \{x, y\}$, $x \neq y$, $e \notin E(G)$ ($G + e$ means to add edge e, $G.e$ means to contract e that is to identify x and y).
(ii) Use (i) to prove that f is the evaluation of a real polynomial.

Question 7
(i) Determine the chromatic polynomial for any tree.
(ii) Give examples of graph parameters determined by $f(G, k)$.

Another definition makes study of graph homomorphisms easier: a *core* of a graph G is any minimal subgraph of G which is hom-equivalent to G. We also say that G is a core if G is not hom-equivalent to any proper subgraph of G.

It is easy to see that up to an isomorphism the core of every finite graph is uniquely determined: if G_1 and G_2 are two subgraphs of G which form a core of G then let $f_i : G \to G_i$, $i = 1, 2$, be the corresponding homomorphisms. However then the mapping $f_1 \circ f_2$ restricted to the set $V(G_1)$ maps

$V(G_1)$ to itself and thus it is injective. Similarly the mapping $f_2 \circ f_1$ restricted to the set $V(G_2)$ is an injective mapping of $V(G_2)$ to itself. Thus both f_i's restricted to the set $V(G_i)$ are isomorphisms of G_1 and G_2.

Somehow less obvious (but not difficult) is to answer the following:

> **Question 8**
> Prove that for every vertex-transitive finite graph G all the subgraphs of G which form a core of G are vertex-transitive. (A graph G is *vertex transitive* if for any pair of its distinct vertices x, y there exists an automorphism of G which maps x to y.)

As an algebraic concept it is perhaps not surprising that the homomorphisms are related to several operations defined for graphs. Perhaps most natural of these is the *direct product* of graphs (which is sometimes called *categorial product* or even *tensor product*). As we shall not use any other product we shall simply use the word *product*. It is defined as follows:

Definition 5 (Direct product of graphs)
Suppose G_1 and G_2 are simple finite graphs. The product $G_1 \times G_2$ of G_1 and G_2 has vertex set $V(G_1 \times G_2) = \{(x_1, x_2) : \ x_1 \in V(G_1) \text{ and } \ x_2 \in V(G_2)\}$ and edge set $E(G_1 \times G_2) = \{[(x_1, x_2)(y_1, y_2)] : [x_i, y_i] \in E(G_i), \text{ for } i = 1, 2\}$.

The mapping $\pi_i : V(G_1 \times G_2) \to V(G_i)$ defined by $\pi_i(x_1, x_2) = x_i$, $i = 1, 2$, is a homomorphism $G_1 \times G_2 \to G_i$ and it is called *i-th projection*. As is well known from any course introducing categories (or theory of structures), projections determine product uniquely up to an isomorphism. We have the following:

Theorem 6
Let G be a graph with two homomorphisms $f_i : G \to G_i$, $i = 1, 2$. Assume that for any graph H with two homomorphisms $g_i : H \to G_i$, $i = 1, 2$, there exists a unique homomorphism $h : H \to G$ such that $g_i = f_i \circ h$, $i = 1, 2$. Then H is isomorphic to $G_1 \times G_2$. Explicitely, and more strongly, there exists an isomorphism $\iota : G_1 \times G_2 \to G$ such that $f_i \circ \iota = \pi_i$, $i = 1, 2$.

This theorem may be visualized by the following diagrams:

$$\begin{tikzcd} H \arrow[rr, dashed, "h"] \arrow[dr, "g_1"'] \arrow[drrr, "g_2" near start] & & G \arrow[dl, "f_1"' near start] \arrow[dr, "f_2"] & \\ & G_1 & & G_2 \end{tikzcd} \qquad \begin{tikzcd} G \arrow[rr, dashed, "iso"] \arrow[dr, "f_1"'] \arrow[drrr, "f_2" near start] & & G_1 \times G_2 \arrow[dl, "\pi_1"' near start] \arrow[dr, "\pi_2"] & \\ & G_1 & & G_2 \end{tikzcd}$$

We shall not prove this theorem as you probably heard it somewhere. If not, try to do it, it is a good exercise (in the style which is dominating this paper; and it is not evident, there is something to prove).

Given an n-coloring f (considered as a homomorphism $f : G \to K_n$) of the graph G we obtain that $f \circ \pi_G$ is an n-coloring of the product $G \times H$. Therefore, $\chi(G \times H) \leq \chi(G)$. Similarly, we have $\chi(G \times H) \leq \chi(H)$, and hence $\chi(G \times H) \leq min\{\chi(G), \chi(H)\}$. Hedetniemi's conjecture asserts that the equality holds here for all graphs G and H. This is one of the oldest conjectures in the area and surprisingly little progress has been achieved so far (see e.g. [53]):

Problem A1 (Hedetniemi's conjecture)

Is it true that $\chi(G \times H) = min\{\chi(G), \chi(H)\}$ for all pairs of graphs G and H?

Denote by $f(n)$ the following function: $f(n) = min\{\chi(G \times H);\ \chi(G) = \chi(H) = n\}$. The above conjecture then states that $f(n) = n$ for every n. How little is known about this illustrates the fact that the following (seemingly much weaker problem) is open as well:

Problem A2

Is the function f unbounded?

Presently the best result here is that either the function is bounded by 9 or it is unbounded, see [53].

From the point of view of this paper both problems **A1** and **A2** are related to study of homomorphisms $G \times H \to K_k$. An alternative approach to this conjecture may be based on the following idea: let us fix a graph G; then for any graph H the homomorphism $f : G \times H \to K_k$ may be viewed as a family of mappings $\{f_y; y \in V(H)\}$ defined by $f_y(x) = f(x, y)$. These mappings $f_y : V(G) \to V(K_k)$ are not necessarily homomorphisms, yet these are not arbitrary mappings and as a collection they have several properties. These properies are expressed by the following notion of *exponential graph* (sometimes called *power graph*) ${K_k}^G$ which is a particular case of *map object* (or *exponentiation*) defined in category theory. As is usual we give two definitions: one describing the graph ${K_k}^G$ from inside (i.e. exlicitely describing the vertices and edges) and one definition by means of homomorphisms (i.e. outside or implicite definition). This is similar to the situation when we defined product. This time we shall start with the homomorphical definition (i.e. categorical definition).

Definition 7 (Map object)
Let G and F be graphs. A graph P together with homomorphism $e : G \times P \to F$ is said to be *power graph* with *evaluation map* e if the following condition is satisfied:

> *For every graph H and every homomorphism $f : G \times H \to F$ there exists unique homomorphism $f^{\#} : H \to P$ such that $f = e \circ (id_G \times f^{\#})$.*

$$
\begin{tikzcd}
G \times F^G \arrow[dr, "e"] & \\
 & F \\
G \times H \arrow[uu, "(id_G \times f^{\#})"] \arrow[ur, "f"'] &
\end{tikzcd}
$$

By $(id_G \times f^{\#})$ we denoted here the mapping $V(G) \times V(H) \to V(G) \times V(P)$ defined by $(x, y) \to (x, f^{\#}(y))$. $(id_G, f^{\#})$ is clearly a homomorphism $G \times H \to G \times P$.

Similarily as for the direct product it follows:

> **Question 9**
> Let P with evaluation map e and P' with evaluation map e' be power graphs of graphs G and F. Then graphs P and P' are isomorphic. More precisely there exists a graph isomorphism $\iota : P \to P'$ such that $e' \circ (id_G \times \iota) = e$. Prove this assertion.

The mapping $f^{\#}$ which appears in the definition of power graph is sometimes called *name of f* mapping. As the power graph P is (up to an isomorphism; even up to "commuting isomorphism") uniquely determined by G and F we can denote P by F^G.

Now we shall give an explicit definition of the graph power:

Definition 8 (Power graph)
Let G and F be graphs. The power graph F^G is defined as follows: $V(F^G)$ is the set of all mappings $V(G) \to V(F)$ and we let $\{f, g\} \in E(F^G)$ if and only if $\{f(x), g(y)\} \in E(F)$ whenever $\{x, y\} \in E(G)$.

Now we have

Theorem 9
Let F, G be graphs. Then the power graph F^G constructed in the Definition 7 satisfies Definition 8 with the evaluation map $e : V(G) \times V(F^G) \to V(F)$ defined by $e(x, f) = f(x)$ (of course this justifies the term; this is the *evaluation mappping. Thus both definitions define, up to a commuting isomorphism, the same graph).*

Proof.
By Question 9 we know that it suffices to prove that the power graph satisfies the properties of a map object (i.e. Definition 2). Let graphs F, G be given, let F^G be graph constructed in the Definition 8. First, we prove that the mapping $e : V(G) \times V(F^G) \to V(F)$ defined by $e(x, f) = f(x)$ is a homomorphism $G \times F^G \to F$. Towards this end, let $\{(x, f), (x', f')\}$ be an edge of $G \times F^G$. This means that $\{x, x'\} \in E(G)$ and also $\{f(x), f'(x')\} \in E(F)$ (see the definition $\{f, f'\} \in E(F^G)$). But this means that $\{e(x, f), e(x', f')\} = \{f(x), f'(x')\} \in E(F)$ as desired.

Secondly, we prove the second part of the definition of "map-object". Thus let H be any graph together with a homomorphism $f : G \times H \to F$. Define a mapping $f^\# : V(H) \to V(F^G)$ by $(f^\#(y))(x) = f(x, y)$ (thus $f^\#(y)$ is a mapping $V(G) \to V(F) \in V(F^G)$). Obviously $f = e \circ (id_G \times f^\#)$ as this is exactly how we defined the mapping $f^\#$. We have to only check that $f^\#$ is a homomorphism $H \to F^G$. This is indeed so: let $\{y, y'\} \in E(H)$. We shall show $\{f^\#(y), f^\#(y')\} \in E(F^G)$. Towards this end, let $\{x, x'\} \in E(G)$. Then $\{(f^\#(y))(x), (f^\#(y'))(x')\} = \{f(x, y), f(x', y')\} \in E(F)$ which was to be proved. □

Question 10
Define powers of directed graphs and map objects for directed graphs and check that both definitions are defining the same graph.

In fact one can define the "map-objects" by means of the above definition for every "structure", or category. Map objects however need not exist, categories where they do exist for any pair of objects (like F and G above) are called *cartesian closed categories.*

Let us return to our problem on chromatic number of product: does $\chi(G \times H) = min\{\chi(G), \chi(H)\}$ hold? As a corollary to the above *definition* of F^G when applied to the graph $F = K_k$ we have

Corollary 10
Given two graphs G and H. The following are equivalent:
(i) The product $G \times H$ is k-colorable.
(ii) H admits a homomorphism to K_k^G.
(iii) G admits a homomorphism to K_k^H.

Proof.
$H \to K_n^G$ if and only if $G \times H \to K_k$. □

Thus in order to settle the conjecture it suffices (well, it is equivalent) to prove the following seemingly weaker questions as it asks, for every fixed graph G, to color only one graph!

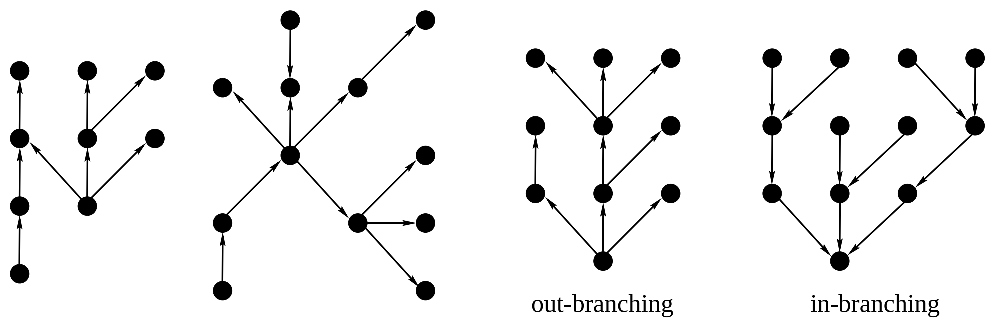


Figure 4: Sample oriented trees

Question 11
Prove that the following statements are quivalent:
(i) If $\chi(G) > k$, then K_k^G is k-colorable;
(ii) $\chi(G \times H) = min\{\chi(G), \chi(H)\}$ for all pairs of graphs G and H.

There are other paradoxes related to Problems **A1** and **A2**. Very little is known despite of all efforts. The conjecture is trivial for $k = 1$, can you prove it for $k = 2$? It is also known that it holds for $k = 3$ (and this is harder, see El-Zahar and Sauer [10], see also Zhu [53]). It seems that power graphs and algebra of homomorphisms is suited to the study of the problem (see recent work of B. Larose and C. Tardif).

2 A tree problem

Our problem is very easy to state:

Problem B1 (First version)

Given two trees T_1 and T_2 such that T_1 is homomorphic to T_2 but T_2 fails to be homomorphic to T_1, does there exist a tree T such that T_1 is homomorphic to T and T is homomorphic to T_2 while T_2 fails to be homomorphic to T and T fails to be homomorphic to T_1?

Using the arrow notation introduced above we get the following form:

Problem B2 (Second version)

Given trees T_1 and T_2 satisfying $T_1 \to T_2 \not\to T_1$ does there exist a tree T such that $T_1 \to T \to T_2$ while $T_2 \not\to T \not\to T_1$ (somehow we tend to write arrows in one way only).

This still looks complicated and a suspicion that this is unnecessary so is still in place. Aren't we describing in many words a rather simple situation? Well, the trees T_1 and T_2 are trees thus they do not contain a cycle and hence they have chromatic number either 1 or 2 (as they are surely bipartite). Consequently there exists exactly one pair (T_1, T_2) when T_1 is homomorphic to T_2 while T_2 fails to be homomorphic to T_1, isn't it? (Well only one pair up to homomorphic equivalence, this pair is (K_1, K_2); perhaps we could list (K_0, K_1) too..) This is so easy that this does not deserve to be called a problem.

So which trees do we mean here? We mean *oriented trees*. An oriented tree is any orientation of an (undirected) tree. Examples of oriented trees are depicted in Figure 4.

You certainly have seen some special oriented trees which have special names like branching, or in- (out-) branching (see also Figure 4).

Question 12
Do you remember how is spanning in-branching related to a description of all eulerian tours in the connected balanced oriented graphs? To start with, can you prove that every balanced connected oriented graph has a spanning in-branching? (In a balanced graph G the in-degree $d_G^-(x)$ equals out-degree $d_G^+(x)$ for every vertex x of the graph.)

So we shall deal here with oriented trees only. For undirected trees everything what follows holds too but the whole subject becomes trivial. As we shall see this is far from beeing so in the directed case.

The above problem can be stated even better if we allow a bit more abstract setting. Towards this end let $\mathcal{T}$ denotes the class of all finite (oriented) trees. Define the relation $\leq$ on $\mathcal{T}$ by letting $T \leq T'$ if and only if there exists a homomorphism $T \to T'$. This relation is

(i) reflexive,

(ii) transitive,

however not necessarily weakly antitransitive: for distinct trees T, T' it may happen that T and T' are hom-equivalent. A relation satisfying (i) and (ii) is called a *quasiorder*. However this is a step from partial order: if we factorize the relation $\leq$ by the (homomorphism) equivalence $\sim$ we get a quasiorder. (Recall that $T \sim T'$ if both $T \to T'$ and $T' \to T$).

This is perhaps a bit quick and this calls for

Question 13
Let $\leq$ be a quasiorder on a set X. As above define the equivalence $\sim$. Define the factor relation $\leq_{|\sim}$ and prove that X endowed with $\leq_{|\sim}$ is always a partially ordered set.

The core graphs make the *representants* of the classes of equivalence $\sim$: if T and T' are trees then $T \sim T'$ if and only if the core of T is isomorphic to the core of T'. Thus if we denote $\mathcal{T}_c$ the set of all finite trees which are cores then $\mathcal{T}_c$ together with the quasiorder $\leq$ is a partially ordered set.

Question 14 [A remark and a Research Problem]
It is NP-hard to decide whether a given graph is a core or not. On the other side there is a polynomial algorithm to decide whether a given tree is a core or not. However for both of these questions the known proofs are involved and elaborate (and they are particular cases of a more general theorem). Can you devise a short proof?

Question 15
(i) Give examples of tree cores.
(ii) Prove that the number of tree cores grows exponentially (in the number of vertices).
(iii) Is it true that any asymmetric undirected tree possesses an orientation which is a core?

Using all this we can pose our tree problem as follows:

Problem B3 (Third version)

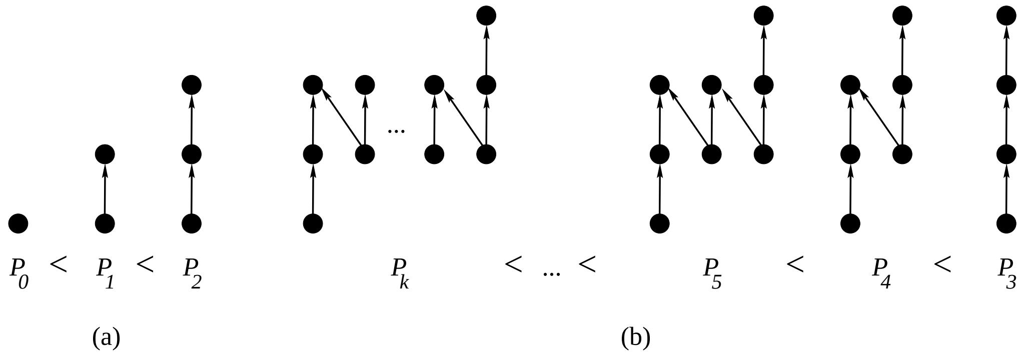


Figure 5: Tree pairs having no tree "between"

Characterize pairs of tree cores T_1 and T_2, $T_1 < T_2$, for which there exists a tree T satisfying $T_1 < T < T_2$.

That of course sounds already familiar and justifies if we call the above problem (any of the three equivalent versions) the *density problem for trees*. Examples of similar statements and/or problems you know from analysis and topology: the set of all rational numbers is dense in the set of all real numbers and a topological space is called separable if it has a countable dense set.

The density problem for trees is presently open (it was posed first in [42]) and we consider it a nice problem per se. But the problem is also related to several other problems and results and we are going to introduce these and discuss their relevance. This will be done in the following subsections 2.1 to 2.5.

2.1 Gaps

Observe that we have changed the statement of the problem – somehow the third version is more carefully worded. We better do so as the answer to Problems **C1** and **C2** as they stand is very simple and negative: one can easily find examples of tree pairs T_1 and T_2, $T_1 < T_2$, for which one cannot find a tree T with $T_1 < T < T_2$. Indeed examples given on Figure 5(a) are such examples.

And this is not all, one can find some other examples: for instance (K_0, K_1) (however this is a bit cheating and this depends on the definition of the tree). But even this is not all and it does not need much efforts (as we are on a good track) to find some other examples. They can be defined as follows: denote by P_0, P_1, P_2, P_3 the directed paths of respective lengths $0, 1, 2, 3$ and by P_k, $k > 3$, the oriented path of length $2k-3$ (with $2k-2$ vertices), as depicted in Figure 5(b).

More precisely, the graphs P_k are defined by:

$$V(P_k) = \{0, x_1, x_2, \ldots, x_{k-2}, y_1, y_2, \ldots, y_{k-2}, 1\}$$

with arcs

$$E(P_k) = \{(0, x_1), (x_1, y_1), (x_2, y_1), (x_2, y_2), \ldots, (x_{k-2}, y_{k-3}), (x_{k-2}, y_{k-2}), (y_{k-2}, 1)\}.$$

One can check that $P_{k+1} \to P_k$ while $P_k \not\to P_{k+1}$ and not much harder is to prove that there is no tree T satisfying $P_{k+1} < T < P_k$ (for every $k \geq 3$).

Question 16
Prove that there is no oriented graph G which satisfies $P_{k+1} < G < P_k$ for $k \geq 4$.

Question 17
Prove that all oriented path cores of height at most 4 are depicted on Figure 5 (i.e. prove that these are directed paths of length 0,1,2,3 and paths P_k for $k > 3$).

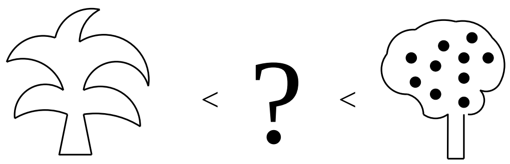
Can we find some more examples? Try to experiment with path orientations which have height 5 (our paths P_k are of the height 4)!

Well , it seems difficult to find further examples and so far no other pairs of trees were found! So we have infinitely many such pairs but all of a very simple structure (depicted in Figure 5). This leads to yet another formulation of the tree density problem.

To facilitate a more concise formulation let us introduce the following notion. Let $\mathcal{K}$ be a class of graphs. We say that the pair (G_1, G_2) , $G_i \in \mathcal{K}$, is a *gap* in $\mathcal{K}$ if there is no $G \in \mathcal{K}$ such that $G_1 < G < G_2$. Thus (K_0, K_1) and graphs depicted above in Figure 5 are all gaps in the class $\mathcal{T}$. This leads to the following

Problem B4 (Tree density problem - final version)

Prove that the pair (K_0, K_1) and all pairs depicted in Figure 5 are the only gaps of the class $\mathcal{T}$.



2.2 Undirected graph density

Why do we consider trees? The questions make sense for other graphs too! Of course this was investigated and several things are known. This will be reviewed in this and following subsections.

For undirected graphs we have the following

Theorem 11 (Density Theorem for Undirected Graphs)
The pairs (K_0, K_1) and (K_1, K_2) are the only gaps for the class of all undirected graphs. Explicitely, given undirected graphs G_1, G_2, $G_1 < G_2$, $G_1 \neq K_0$ and $G_1 \neq K_1$ there is a graph G satisfying $G_1 < G < G_2$.

Question 18
Convince yourself that the following is an equivalent formulation: given undirected graphs G_1, G_2, $G_1 < G_2$, G_2 not bipartite, there is a graph G satisfying $G_1 < G < G_2$.

This theorem was proved in 1982 by Welzl [52] by an elaborate ad hoc argument. The problem originated (a bit surprisingly) in Computer Science (theory of formal languages) where it was used to prove that certain class of languages (defined by so-called "graph grammars") forms a dense hierarchy (see [31]). Later the result had been better understood and the density theorem was put in another context. Also several proofs emerged and we shall show it thus displaying conceptual richness of the area.

We shall include three (very much different) proofs later on.

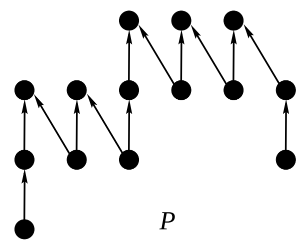


Figure 6: Coding of an oriented path

2.3 On Paths, Words and WQO

Here we consider the problem motivated by the existence of homomorphisms between perhaps the simplest type of graphs: oriented paths.

An *oriented path* P is any oriented graph where $V(P) = \{v_0, v_1, \dots, v_n\}$ and for every $i = 1, 2, \dots, n$ either $v_{i-1}v_i \in E(P)$ or $v_iv_{i-1} \in E(P)$ (but not both), and there are no other edges. Thus an oriented path is any orientation of an undirected path.

One can also express a given path P by a code $c(P) = a_1a_2 \dots a_n$, where n is the number of edges in P, and $a_i = 0$ or 1 depending on whether the i-th edge is a forward or a backward edge. For example the path P_4 on Figure 5(b) has code $c(P) = 00100$ and the path P_5 on Figure 5(b) has code $c(P') = 0010100$ (another example is depicted in Figure 6). Given a code $c = (a_1a_2 \dots a_n)$ the code c^{-1} is the code $(a'_1a'_2 \cdots a'_n)$ given by $a'_i + a_{n+1-i} = 1$. The code $c(P)^{-1}$ corresponds to the 'flipping' of the path P.

For two oriented paths P_1 and P_2 with codes $c(P_1)$ and $c(P_2)$ respectively, the concatenation P_1P_2 of P_1 and P_2 is the path with code $c(P_1P_2) = c(P_1)c(P_2)$. In other words, P_1P_2 is the path obtained from the disjoint union of P_1 and P_2 by identifying the terminal vertex of P_1 with the initial vertex of P_2. Considering the codes of oriented paths, an oriented path corresponds to a word on the alphabet $\{0,1\}$. Homomorphism of oriented paths can be interpreted as a kind of operation on words, the elementary operation being called *folding*. One can easily see that the following holds:

Theorem 12 (Word Homomorphism)
For oriented paths P and P' the following two statements are equivalent:

1. *P is homomorphic to P';*
2. *There exist words c, c' on alphabet $\{0,1\}$ such that $c(P)$ is a (consecutive) segment of c, c' is a (consecutive) segment of $c(P')$, and c' can be obtained from c by means of a sequence of substitutions (foldings) of the form*
$$ww^{-1}w \to w$$
where w is any word on the alphabet $\{0,1\}$ (see Figure 7).

Question 19
Prove this theorem!

Thus homomorphisms between paths give rise to a quasiorder of finite $\{0,1\}$- words (*folding ordering*) , denoted by $\leq_f$, defined by $w \leq_f w'$ if $P \to P'$ where $w = c(P)$ and $w' = c(P')$.

One can consider further natural orderings for 0-1 sequences:

We write $w \leq_s w'$ (*segment ordering*) if w' contains a (consecutive) segment w'' such that $w = w''$.

We write $w \leq w'$ (*subword ordering*) if w' contains a subword (i.e. a subsequence, not necessarily a consecutive segment) which is equal to w.

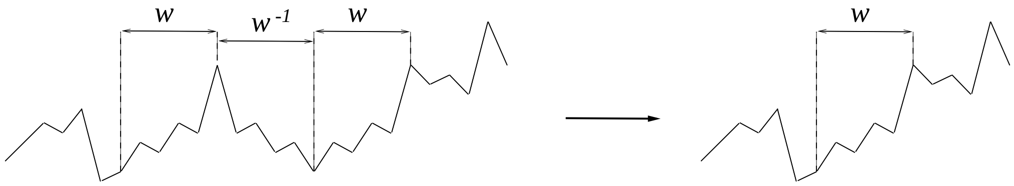


Figure 7: Folding of an oriented path

All these orderings are used in various context. One of them is related to the following concept: a set $\mathcal{K}$ endowed with a quasiordering $\leq$ is said to be WQO (*Well Quasi Ordered*) if

(i) $\mathcal{K}$ does not contain infinitely many incomparable elements (i.e. an infinite antichain),

(ii) $\mathcal{K}$ does not contain an infinite descending chain (i.e. a chain $x_1 > x_2 > x_3 > \ldots$).

In a combinatorial setting the condition (ii) is usually valid (such are all our quasiorderings which we are considering in this paper) but the condition (i) is far more subtle.

Question 20
Prove that $\leq_s$ is not WQO of finite sequences.

For ordering $\leq$ we have the following important

Theorem 13 (Higman Theorem)
The class of all finite 0-1 sequences is WQO with the subword ordering $\leq$.

Higman theorem [20] is a corner stone of the whole theory WQO. As some of you know no assumption is necessary in its statement: we may consider not only 0-1 sequences but also sequences with values in another WQO set; we may also consider infinite sequences; and we may also consider (finite, or infinite) trees with values in another WQO set. In all these instances the result remains valid: all these classes are WQO. In this direction some of the strongest results were obtained by Igor Kříž, see [25, 26].

Now our folding ordering $\leq_f$ is intermediate to $\leq$ and $\leq_s$. In the next subsection we show that $\leq_f$ is not WQO in a very strong sense. This is related to the density problem.

2.4 Density of Finite Paths

Let $\mathcal{P}$ denotes the set of all finite oriented paths. We have then the following, proved by Nešetřil and Zhu in [42]:

Theorem 14 (Path Density)
Suppose that P and P' are paths such that $P \to P'$ and $P' \not\to P$. Then there exists a path P'' with $P \to P'' \to P'$ but $P' \not\to P''$ and $P'' \not\to P$, provided that P' has height at least 4.

If you worked out the Question 17 then you can see immediately that this has the following

Corollary 15
The only gaps of $\mathcal{P}$ are of the form depicted on Figure 5.

The proof of Theorem 14 is given in [42].

Denote by $\mathcal{W}$ the set of all finite $\{0,1\}$-words endowed with the quasiordering. By the Subsection 2.1 we also have:

Theorem 16 (Folding Density)
All the gaps of $(\mathcal{W}, \leq_f)$ are formed by pairs containing the following words $()$, (0), (01), $(0\{01\}^k1)$ for every $k > 0$ and words where '0' and '1' are exchanged.

We do not give either Path Density or Folding Density as an exercise. Presently these results have a complicated and non-trivial proof which involves other results. Particularly, it involves Path Duality for graphs (proved in [19]).

At the end of this part let us relate Path density to WQO. As it is suggested by its name WQO is a generalization of well ordering (represented by ordinals which you know from the set theory) to quasiorders (and posets). Well ordering is of course to sharp contrast to density and we leave this as

Question 21
(i) Prove that if $(X, \leq)$ is WQO then for any pair $x < y$ in X there exists a gap $x' < y'$ satisfying $x < x' < y' < y$.
(ii) Assuming Theorem 14 (Path Density), construct a pair of paths which does not contain a gap.

2.5 Oriented Graphs Density - All Oriented Gaps

Thus oriented graphs present us with a much more complicated situation even in the simple case of paths. In this sense the Tree Problem which motivated this section is natural and next problem to solve. Not beeing able to solve this one is discouraged to proceed further and to try to solve the problem for other classes, not to speak about all oriented graphs. These problems seemed beyond the reach of the existing methods. However this is exactly what happened. The problem of Tree Density is still open while the solution of the general problem was achieved recently by Nešetřil and Tardif in [40]:

Theorem 17 (Chacterization of Gaps for Oriented Graphs)
For every oriented tree T there exists a unique oriented graph H_T such that (H_T, T) is a gap in the class $\mathcal{G}$ of all finite oriented graphs. Moreover, up to a homomorphism equivalence, all gaps (G_1, G_2) with G_2 connected are of this form.

We do not describe disconnected gaps. An example we leave as an exercise:

Question 22
Consider the unbalanced cycle C_{2k+3} formed by the alternating path of length $2k+1$ together with the terminal vertex joined with the first and the last vertex of this path. Then the disconnected graphs G_1 with components C_{2k+3} and P_{k+1} and G_2 with components C_{2k+3} and P_k form a gap in the class of all oriented graphs $\mathcal{G}$. Can you prove this? (There are infinitely many such examples even for a fixed pair of paths.)

The above Oriented Graph Density also shows that the density problem for trees (i.e. our original problem) is in a sense extremal: below every tree is a gap in $\mathcal{G}$ however it is not known whether one can get gaps in $\mathcal{T}$ in this way. And perhaps $\mathcal{T}$ has some gaps which are of completely different form.

This ends the expository part of Tree Density Problem and its surrounding. Now we include proofs of some of the above theorems.

2.6 Probabilistic proof of undirected graph density

The proof is based on the following *Sparse Incomparability Lemma* first isolated in [39]:

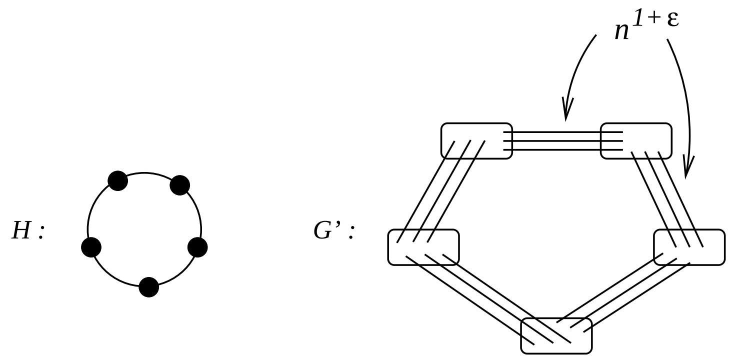


Figure 8: Construction for the proof of Sparse Incomparability Lemma

Lemma 18
Let G, H be fixed graphs, H non-bipartite, ℓ a positive integer. Assume $G \to H$ and $H \not\to G$. Then there exists a graph G' with the following properties:
(i) $G' \to H$,
(ii) $G' \not\to G \not\to G'$, $H \not\to G'$,
(iii) G' has girth $> \ell$.

$$\begin{tikzcd} G' \arrow[r, "+" marking] \arrow[dr] & G \arrow[d] \\ & H \arrow[u, bend right, "+" marking] \end{tikzcd}$$

This of course strengthens the classical Erdős result [11] on high chromatic graphs with given girth (say, for $H = K_k$ and $G = K_{k-1}$). Sparse Incomparability Lemma seems to be an useful tool and [39] originally applied this result to graphs without given symmetries and endomorphisms (the so-called *rigid graphs*).

First, we show that Density of Undirected graphs follows easily from the Sparse Incomparability Lemma:

Proof.
Let $G_1 < G_2$ be given. Apply Sparse Incomparability Lemma for $\ell = |V(G_2)|$, $H = G_2$, $G = G_1$ to get a graph G' with stated properties. Put $G = G' \cup G_1$. Then G has all the desired properties: $G_1 \to G$ obviously and $G \to G_2$ by (i) of Sparse Incomparability Lemma. On the other hand $G_2 \not\to G$ by girth and $G \not\to G_1$ by (ii). $\square$

An interested reader observed that we did not use condition (ii) of Sparse Incomparability Lemma in full. We used only the fact that H was non bipartite while G' contained no short *odd* cycles. This is much easier to guarantee and we shall return to this in the next section.

Proof of Sparse Incomparability Lemma.
Let graphs G, H be given, let ℓ be a given positive integer. Let t denotes the number of vertices of the graph G and, without loss of generality, let the set of vertices of H be $\{1, 2, ..., k\}$. For a (large) positive integer n consider pairwise disjoint sets $V_1, V_2, ..., V_k$, each of size n.

Let $\mathbf{G}$ be a random graph with vertex set $V = \cup_{i=1}^{k} A_i$ where the edges are chosen independently from the family $\{\{x,y\};\ x \in V_i,\ y \in V_j,\ \{i,j\} \in E(H)\}$, each with the probability $p = n^{\delta-1}$, where $0 < \delta < 1/\ell$ (compare schematic Figure 8).

A set $A \subset V$ is said to be *large* if there are i, j, $1 \le i < j \le k$, such that $|A \cap V_i| \ge n/t$ and also $|A \cap V_j| \ge n/t$. For evey large set A we consider all such pairs $\{i,j\}$ and we call them *good pairs* of A. For a large set A denote by $|\mathbf{G}/A|$ the minimum number of edges of $\mathbf{G}$ which lie in the set $\{\{x,y\};\ x \in V_i,\ y \in V_j\}$ for a good pair of A.

We first estimate probability

$$\alpha = Prob[A \text{ large implies } |\mathbf{G}/A| \geq n].$$

We have

$$1-\alpha \leq \sum_{A \text{ large}} Prob[|\mathbf{G}/A|] < n] \leq 2^{kn} \cdot \binom{\binom{kn}{2}}{n} \cdot (1-p)^{\frac{n^2}{r^2}}.$$

Now bounding very roughly

$$\binom{\binom{kn}{2}}{n} \leq \binom{k^2n^2}{n} \leq k^{2n}n^{2n} < e^{cn\log_2 n}$$

and

$$(1-p)^{\frac{n^2}{t^2}} \leq e^{-p\frac{n^2}{t^2}}$$

we obtain

$$1-\alpha < e^{cn\log_2 n - c'n^{1+\delta}}$$

for some positive constants c and c' which are independent on n.

Thus we get $Prob[A \text{ large implies } |\mathbf{G}/A| \geq n] = 1 - o(1)$.

On the other hand if we denote by $c(\mathbf{G})$ the number of edges contained in all cycles of length $3, 4, \ldots, \ell$ in $\mathbf{G}$ then by the linearity of expected value we have

$$E(c(\mathbf{G})) \leq 3!\binom{kn}{3}p^3 + 4!\binom{kn}{4}p^4 + \ldots + \ell!\binom{kn}{\ell}p^\ell = \ell \cdot \frac{k^l n^\ell n^{\delta\ell}}{n^\ell} < n^{\delta\ell} = o(n).$$

Thus there exists a graph G'' (an instance of the random graph $\mathbf{G}$) such that
(i) If i,j is a good pair of a large set A then G'' has at least n edges in the set $\{\{x,y\};\ x \in V_i,\ y \in V_j\}$,
(ii) There exists $n-1$ edges $e_1, e_2, \ldots, e_{n-1}$ such that the graph G' which we obtain from G'' by deleting edges $e_1, e_2, ..., e_{n-1}$ has girth $> \ell$.

We prove that the graph G' satisfies the above conditions of Sparse Incomparability Lemma. Properties (i) and (iii) are evident from the construction of G'. To prove (ii) let us suppose that f is a homomorphism $G' \to G$. Define a mapping $g : V(H) \to V(G)$ by $g(i) = y$ if $|f^{-1}(y) \cap V_i| \geq n/t$ (we could call g a *majority mapping*). Clearly for every i one can choose $g(i)$ (if there are more possibilities we choose one arbitrarily). It follows from the properties (i) and (ii) of graph G' that the majority mapping g is a homomorphism $H \to G$ which is a final contradiction.

This is the only non-arrow which one has to prove for (ii) (the remaining non-arrows follow from the girth of graph G'). □

2.7 Constructive proof of undirected graph density via products

This proof is due to M. Perles and J. Nešetřil (see e.g. [35]) and it is particularly simple. It uses product and the construction of high chromatic graphs without short odd cycles (construction for existence), see Question 4 (see how dilligent you are).

Proof of Theorem 11.
Let G_1 and G_2 be given undirected graphs, let $f : G_1 \to G_2$ be a homomorphism, and suppose there is no homomorphism $G_2 \to G_1$. As this pair is not equivalent to the gap (K_1, K_2), at least one component of the graph G_2 has chromatic number greater than 2. Also, at least one component of G_2 fails to be homomorphic to G_1, and this component may be assumed to be non-bipartite; let it contain an odd cycle of length k. Now choose a graph H with the following properties: H contains no odd cycle of length k or less, and the chromatic number of H is greater than $n_1^{n_2}$, where n_1 and

n_2 denote the number of vertices of the graphs G_1 and G_2 respectively. Such a graph exists by the celebrated theorem of Erdős [11]. But the existence follows much more easily from Question 4 and there we gave also a construction of such graphs.

Now let $G = G_1 \cup (H \times G_2)$. Here $\times$ denotes the direct product (see Definition 5) of two graphs and $\cup$ means the disjoint union. We shall prove that G has the desired properties. Obviously $G_1 \to G$ and $G \to G_2$ follows as the second projection of $H \times G_2$ is a homomorphism into G_2. On the other hand there is no homomorphism from G_2 into G, as homomorphisms preserve odd cycles and they cannot increase the length of the shortest of them. Thus it suffices to prove that there is no homomorphism $G \to G_1$. Let us suppose for the contradiction that there is a homomorphism $f : H \times G_2 \to G_1$. Thus for any vertex x of H we have an induced mapping $f_x : V(G_2) \to V(G_1)$ defined by $f_x(y) = f(x, y)$. (This mapping need not be a homomorphism.) As there are at most $n_1^{n_2}$ such mappings there are vertices x and x' forming an edge of H such that the mappings f_x and f_x' are identically equal, say to g. However in this case g is a homomorphism of G_2 into G_1, contrary to our assumption. $\square$

Note that this construction of graph G given in the proof can be used to prove that Sparse Incomparability Lemma holds for large *odd girth*.

This is a good place to review another construction which we introduced in the introduction (see Subsection 1.1 on chromatic number):

Given two graphs G and H one can define G *power of* H, denoted by H^G, as the following graph: $V(H^G) = \{f : V(G) \to V(H)\}$ and a pair (f, g) forms an edge if $(f(x), g(y)) \in E(H)$ for every edge $(x, y) \in E(G)$. (We define the G power of H by the same formula for both undirected or directed graphs.)

This construction was isolated in the graph theoretic concept in [27], however as we have seen in Subsection 1.1 this is also a standard category theory construction (where it is called “map object” or exponentiation). This led to the notion of *cartesian closed category* as a category where one can define the power for every pair of graphs. The following is the crucial property which we use (and, as we have seen in Subsection 1.1, which in fact defines the power construction):

$$\text{for every graph } K,\ K \to H^G \text{ holds if and only if } K \times G \to H.$$

This is easy to see: given $f : K \to H^G$, define $g : K \times G \to H$ by $g(x, y) = f(x)(y)$. Conversely, given g we may define f by the same formula. One can easily check that f is a homomorphism $K \to H^G$ if and only if g is a homomorphism $K \times G \to H$.

Thus in the above proof we have $H \times G_2 \to G$ if and only if $H \to {G_1}^{G_2}$. Thus in the above proof we may assume that the chromatic number of H is greater than the chromatic number of ${G_1}^{G_2}$. As $G_2 \not\to G_1$ there are no loops in ${G_1}^{G_2}$ and it is also clear that the chromatic number of ${G_1}^{G_2}$ is at most the number of vertices of ${G_1}^{G_2}$. (However we do not try to optimize at this point.)

2.8 Density via Amalgamation

Besides giving a new proof of undirected density theorem we prove the following result, which generalizes and solves a longstanding problem in this area. This proof was given by Nešetřil and Tardif and it appeared in [41].

Theorem 19
Let G, H be directed graphs which are cores. Let H be connected and assume that H fails to be an orientation of a tree. Further assume that $G \to H \not\to G$ holds. Then there exists a directed graph K with $G \to K \to H$ and $H \not\to K \not\to G$.

The proof is based on the following construction which goes back to [13] and [14], see also [34]:

Let G be a directed graph, I a graph with two distinguished vertices $a, b \in V(I)$. Denote by $G \star (I, a, b)$ the graph S defined as follows (see Figure 9):

$$V(S) = (E(G) \times V(I))_{|\sim}$$

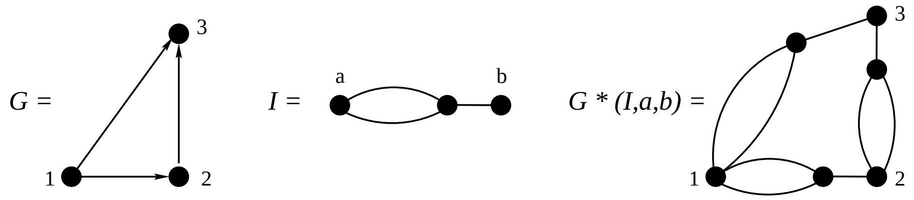


Figure 9: The indicator construction

where the equivalence $\sim$ is generated by the following rules: $((x,y),a) \sim ((x,y'),a)$, $((x,y),b) \sim ((x',y),b)$, $((x,y),b) \sim ((y,z),a)$. The equivalence class containing $((x,y),z)$ will be denoted by $[((x,y),z)]$. We put

$$E(S) = \{ \ ([((x,y),z)],[((x,y),z')]); (x,y) \in E(G), (z,z') \in E(I) \ \}.$$

Clearly $G \star (I,a,b)$ arises by replacing every arrow of G by a copy of I (thus we also call I *indicator*).

This construction is known as *arrow construction* (in some of the earlier papers also called *šíp construction*; šíp = arrow in czech). Note that if (I,a,b) is an undirected graph (considered as a symmetric relation), then $G \star (I,a,b)$ is also an undirected graph.

We shall make use of the following obvious (but key) property of the arrow construction:

Lemma 20
Let G and H be directed graphs with $\chi(G) > |V(H)|$ and let every homomorphism $f : I \to H$ satisfies $f(a) \neq f(b)$. Then $G \star (I,a,b) \not\to H$.

Third proof of Density Theorem for Undirected Graphs.
Let G,H be undirected graphs, H non bipartite, with $G \to H \not\to G$. Without loss of generality, assume that H is connected. Clearly we may assume that G and H are cores. Let $e = \{a,a'\} \in E(H)$ belong to a circuit in H. Put $I = H - e + \{a',b\}$ where $b \notin V(H)$. (Thus I arises from H by deleting the edge e, adding a new vertex $b \notin V(H)$ together with the edge $\{a',b\}$, see Figure 10.)

It is clear that $I \to H$ (identifying vertices a and b) but any homomorphism $f : I \to G$ satisfies $f(a) \neq f(b)$ (for otherwise we get a contradiction with $H \not\to G$). Now let F be any graph satisfying $\chi(F) > |V(G)|$ and let F' be any orientation of F. Consider the arrow construction $F' \star (I,a,b)$ and define the graph K by $K = (F' \star (I,a,b)) \cup G$.

We prove that K has properties claimed by Theorem 19: Clearly $G \to K$. We also have $K \to H$ as the mapping f defined by $f([e,x]) = x$ for $x \in V(H)$ and $e \in E(F')$ and $f([e,b]) = a$ is a homomorphism $K \to H$ (we preserve the above notation concerning the arrow construction $F' \star (I,a,b)$). Further, by the above Lemma 20, $K \not\to G$ (as $\chi(F) > |V(G)|$). Thus it remains to be shown that $H \not\to K$. Suppose the contrary and let $g : H \to F \star (I,a,b)$ be a homomorphism. Then $f \circ g : H \to H$ where f is the above defined homomorphism $F \star (I,a,b) \to H$. As H is a core $f \circ g$ is a homomorphism. Put $h = (f \circ g)^{-1}$. Then $f \circ g \circ h(x) = x$ for every $x \in V(H)$. Put $g \circ h(a) = [(e,a)]$ with $e = (u,v)$. Then the image $g \circ h(H)$ of H is a connected subgraph of $F \star (I,a,b)$ which is (by the injectivity of the mapping $f \circ g \circ h$) contained in the set of all $[(e',x)]$ where e' is incident with u and $x \in V(I)$ (this set is the "star" induced by those edges of F' which are incident with the vertex u). But then the edge $\{[g \circ h(a)],[g \circ h(a')]\}$ is a cut edge in the graph $g \circ h(G)$ which is the final contradiction as a,a' was contained in a cycle of H. $\square$

Proof of Theorem 19.
Let G,H satisfy the assumption of the theorem. Let H be a connected core (again without loss of generality) and let $(a,a') \in E(H)$ belong to a cycle in H. Put $I = H - (a,a') + (b,a')$ where

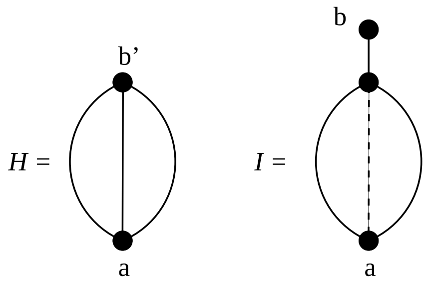


Figure 10: Construction of the graph I

$b \notin V(H)$ (i.e. we first delete arc (a, a') and then add a new vertex b together with the arc (b, a')). Let F be an oriented graph with $\chi(F) > |V(G)|$ and consider the arrow construction $F \star (I, a, b)$. Put $K = G \cup (F \star (I, a, b))$. Then we have:

- $G \to K$ (by the inclusion map);
- $K \to H$ (by the same mapping as in the above proof);
- $K \not\to G$ (by the chromatic number assumption);
- $H \not\to K$ (as above in the proof for undirected graphs).

Thus the graph K has the desired properties. □

This proof can be generalized to finite models of general relational systems and thus it constitutes the essential part of [40].

We do not have place to include another proof which uses some non trivial machinary and which is related to good characterizations (see [40] or the next Spring School!).

3 A planar graph problem

As observed before (see (4) in Subsection 1.3) homomorphisms and colorings are very closely related: an undirected graph G is k-colorable if and only if $G \to K_k$ and, therefore, $\chi(G) = k$ if and only if $G \to K_k$ and $G \not\to K_{k-1}$. In these last two sections we shall consider the case of *(simple) oriented graphs*, that is directed graphs with no multiple arcs and no directed cycle of length less than 3. Hence oriented graphs are simply *orientations* of undirected graphs, that is directed graphs obtained by giving to each edge one of its two possible orientations.

As in the undirected case, the notion of homomorphisms of oriented graphs naturally corresponds to the (new) notion of oriented k-colorings of oriented graphs:

Definition 21 (Oriented k-colorings)
An *oriented k-coloring* of an oriented graph G is a partition of $V(G)$ into k color-classes $V_1, V_2, \ldots, V_k$ such that
(i) each color-class V_i is an independent set,
(ii) for every two color-classes V_i and V_j, all the arcs linking V_i and V_j have the same direction.

Or, equivalently,

Definition 5' (Oriented k-colorings - Second version)
An oriented k-coloring of an oriented graph G is a mapping $f : V(G) \to \{1, 2, \ldots, k\}$ such that
(i) $(x, y) \in E(G) \implies f(x) \neq f(y)$,
(ii) $(x, y), (z, t) \in E(G),\ f(x) = f(t) \implies f(y) \neq f(z)$.

Observe that the conditions (ii) above implies in particular that if xyz is a directed path in G (that is $(x, y), (y, z) \in E(G)$) then $f(x) \neq f(y) \neq f(z) \neq f(x)$.

Question 23
Find an integer function φ such that for every oriented graph G, if $g : V(G) \to \{1, 2, \ldots, \alpha\}$ is a mapping satisfying

$$(x, y), (y, z) \in E(G) \implies f(x) \neq f(y) \neq f(z) \neq f(x)$$

for every x, y, z, then there exists an oriented $\varphi(\alpha)$-coloring of G.

3.1 Oriented chromatic number

As everyone guess, the *oriented chromatic number* of an oriented graph G, denoted by $\vec{\chi}(G)$, is defined as the smallest k such that G admits an oriented k-coloring or, equivalently, as the minimal order of an oriented graph H such that $G \to H$.

This notion can be extended to undirected graphs by setting

$$\vec{\chi}(U) = max\{\ \vec{\chi}(\vec{U});\ \vec{U} \text{ orientation of } U\ \}$$

for every undirected graph U, and to families of graphs by setting

$$\vec{\chi}(\mathcal{F}) = max\{\ \vec{\chi}(\vec{G});\ G \in \mathcal{F}\ \}$$

for every family $\mathcal{F}$ of graphs.

Question 24
If U is an undirected graph, what is the value of $min\{\ \vec{\chi}(\vec{U});\ \vec{U}$ orientation of $U\ \}$?

Question 25
For every $n > 0$ give an orientation $\vec{B}_n$ of a bipartite graph such that $\vec{\chi}(\vec{B}_n) > n$. What is the minimal order of such a bipartite graph?

Question 26 *(Difficult)*
The oriented chromatic number of any tournament on n vertices is obviously n. But it is easy to convince oneself that all the arcs are not necessary. For instance the oriented chromatic number of the directed cycle on five vertices is 5.
What is the minimum number of arcs in an oriented graph G with $\vec{\chi}(G) = |V(G)|$?

Determining the oriented chromatic number of some specific graph families then arises as a natural question. One possible way for getting upper bounds on the oriented chromatic number of a family $\mathcal{F}$ is to exhibit some oriented graph $H_{\mathcal{F}}$ which is universal for $\mathcal{F}$:

Definition 22 (Universal graphs)
An oriented graph $H_{\mathcal{F}}$ is *universal* for a family $\mathcal{F}$ of oriented graphs if every graph F in $\mathcal{F}$ has a homomorphism to $H_{\mathcal{F}}$.

The oriented chromatic number of graphs has been studied in this way in several papers (see e.g. [7, 38, 48, 49]). It appears that some circulant graphs can be proved to be universal for some families of graphs. Recall that the *circulant graph* $G = G(n; c_1, c_2, \ldots, c_\ell)$ is the directed graph defined by $V(G) = \{0, 1, \ldots, n-1\}$ and $(x, y) \in E(G)$ whenever $y \equiv x + c_i \pmod{n}$ for some i, $1 \leq i \leq \ell$. (Under which obvious conditions on the c_i's the graph G is an oriented graph? This is easy, it is not a question.) This is in particular the case when n is a prime with $n \equiv 3 \pmod 4$, and the $c_i^\prime s$ are all the non-zero quadratic residues[5] of n (see Questions 27(i) and 28 below).

[5] Recall that q, $0 \leq q < n$, is a quadratic residue of n if $a^2 \equiv q \pmod{n}$ for some a.

Question 27
Determine the oriented chromatic number of
(i) trees,
(ii) graphs with maximum degree two,
(iii) graphs with maximum degree three.

Question 28
Prove that every oriented outerplanar graph has a homomorphism to $T_7 = G(7;1,2,4)$.

Question 29 [Research question]
Determine the oriented chromatic number of the n-dimensional (undirected) hypercube H_n.
(Recall that the hypercube H_n is the graph whose vertices are all binary words of length n and whose edges link any two such words that differ on only one digit. Equivalently, $V(H_n) = \{0, 1, \ldots, 2^n - 1\}$ and $(i, j) \in E(H_n)$ if and only if $|i - j| = 2^p$ for some $p \in \{0, 1, \ldots, n - 1\}$. For instance, H_2 is the 4-cycle and H_3 the usual 3-dimensional cube on 8 vertices.)

In the next two subsections we shall deal with the oriented chromatic number of some families of planar graphs.

3.2 Planar graphs

Our problem is the following:

Problem C1 (Oriented chromatic number of planar graphs)

Determine the best possible upper bound for the oriented chromatic number of planar graphs.

Up to now, the best known upper bound for this problem is 80. This bound is deduced from a more general result linking the oriented chromatic number and the acyclic chromatic number of a graph. On the other hand, we know that there exist oriented planar graphs with oriented chromatic number at least 16. These two bounds will be discussed in the following.

What is the best way to attack this problem? Unfortunately, we are not able to answer this question. That's certainly why this problem is particularly challenging...

Maybe one useful hint can be the following

Lemma 23
If every oriented planar graph has oriented chromatic number at most k then there exists an oriented graph H of order k such that every oriented planar graph has a homomorphism to H.

Question 30
Prove Lemma 23.

Therefore, in order to decrease the upper bound of 80, it "sufficies" to exhibit some *magic enough* oriented graph M on $k < 80$ vertices such that every oriented planar graph has a homomorphism to M. The main difficulty here is to find a target graph M having "sufficiently good" properties, that is such that one can effectively prove that every oriented planar graph homomorphically maps to M.

In the rest of this subsection we shall prove that every oriented planar graph has oriented chromatic number at most 80 and exhibit an oriented planar graph with oriented chromatic number 15. In the next subsection, we shall prove that every oriented planar graph with no cycle (not considering the orientation of the arcs) of length less than 14 has oriented chromatic number at most 5 (which is best possible: you cannot get 4 by the girth assumption, so no 4CC here!).

A k-coloring of a graph G is *acyclic* if every cycle in G uses at least three colors or, equivalently, if the union of every two color-classes induces a forest. The *acyclic chromatic number* of G, denoted by $a(G)$, is then defined as the smallest k for which there exists an acyclic k-coloring of G.

Question 31
For every n, construct a bipartite graph B_n with $a(B_n) > n$.

Question 32
Determine the acyclic chromatic number of outerplanar graphs.
(Hint: recall that every maximal outerplanar graph contains a vertex with degree two...)

Then Raspaud and Sopena [45] proved the following

Theorem 24
Every graph with acyclic chromatic number at most k has oriented chromatic number at most $k \times 2^{k-1}$.

Proof.
Consider a graph G with $a(G) \leq k$ and let c be an acyclic k-coloring of G. Consider any two colors i and j and assume $i < j$. We know that the set

$$V_{i,j} = \{\ x \in V(G) :\ c(x) = i \ \text{ or } \ c(x) = j\ \}$$

induces a forest in G, that we shall denote by $F_{i,j}$. Consider now any orientation $\vec{G}$ of G and denote by $\vec{F}_{i,j}$ the corresponding orientation of $F_{i,j}$. Recall that the forest $F_{i,j}$ is already 2-colored by i and j.

We then define a mapping $c_{i,j} : V_{i,j} \longrightarrow \{0,1\}$ in such a way that the mapping $\varphi_{i,j}$ given by $\varphi_{i,j}(x) = (c(x), c_{i,j}(x))$ for every $x \in V_{i,j}$ is an oriented 4-coloring of $\vec{F}_{i,j}$. The mapping $c_{i,j}$ is constructed inductively as follows:

- In every connected component T of $F_{i,j}$ we choose a vertex v_T and set $c_{i,j}(v_T) = 0$.
- Now let y be any vertex in $V_{i,j}$ such that xy is an edge in $F_{i,j}$ and $c_{i,j}(x)$ has already been defined. If the corresponding arc in $\vec{F}_{i,j}$ is directed from the i-colored vertex to the j-colored vertex then we set $c_{i,j}(y) = c_{i,j}(x)$ otherwise we set $c_{i,j}(y) = 1 - c_{i,j}(x)$.

In other words, two adjacent vertices in $\vec{F}_{i,j}$ are assigned the same value by $c_{i,j}$ if and only if the arc linking them goes from the vertex with the smallest color to the vertex with the largest color in the acyclic coloring c. Since $F_{i,j}$ has no cycle the mapping $c_{i,j}$ is well-defined.

It is then not difficult to check that the mapping $\varphi_{i,j}$ is indeed a homomorphism of $\vec{F}_{i,j}$ to the directed 4-cycle $(i,0) \to (j,0) \to (i,1) \to (j,1) \to (i,0)$, that is an oriented 4-coloring of $\vec{F}_{i,j}$. Let us check it:

1. Adjacent vertices get distinct colors, thanks to component c.

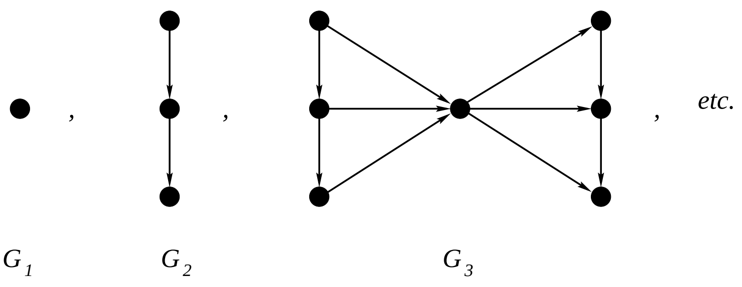


Figure 11: Oriented graphs G_i with $\vec{\chi}(G_i) = |V(G_i)|$

2. Suppose now that (x, y) and (z, t) are two arcs with $\varphi_{i,j}(x) = \varphi_{i,j}(t)$, that is $c(x) = c(t)$ and $c_{i,j}(x) = c_{i,j}(t)$. By the definition of $c_{i,j}$ we then necessarily have $c_{i,j}(y) \neq c_{i,j}(z)$ and therefore $\varphi_{i,j}(y) \neq \varphi_{i,j}(z)$.

Let us now return to the whole oriented graph $\vec{G}$. Every vertex v, with color $c(v)$, belongs to exactly $k-1$ bicolored forests, namely

$$\vec{F}_{1,c(v)},\ \vec{F}_{2,c(v)},\ \ldots,\ \vec{F}_{c(v)-1,c(v)},\ \vec{F}_{c(v),c(v)+1},\ \ldots,\ \vec{F}_{c(v),k}.$$

Suppose that we have defined as above all the $\frac{k(k-1)}{2}$ mappings $c_{i,j}$, $1 \leq i < j \leq k$. We can then define an oriented coloring φ of $\vec{G}$ by setting

$$\varphi(v) \;=\; (\ c(v)\ ;\ c_{1,c(v)}(v),\ c_{2,c(v)}(v),\ \ldots,\ c_{c(v)-1,c(v)}(v),\ c_{c(v),c(v)+1}(v),\ \ldots,\ c_{c(v),k}(v)\ )$$

for every vertex v. As before, one can check that the mapping φ is an oriented coloring of $\vec{G}$:

1. Adjacent vertices get distinct colors thanks to component c.

2. If (x, y) and (z, t) are two arcs with $\varphi(x) = \varphi(t)$ (which implies in particular that $c(x) = c(t)$) then either $c(y) \neq c(z)$, and thus $\varphi(y) \neq \varphi(z)$, or $c(y) = c(z)$ (which means that x, y, z and t all belong to the bicolored forest $\vec{F}_{c(x),c(y)}$) and thus $\varphi(y) \neq \varphi(z)$ thanks to component $c_{c(x),c(y)}$.

Since the mapping φ above defined uses exactly $k \cdot 2^{k-1}$ colors, we are done. □

Using a celebrated result of Borodin [6] which states that every planar graph can be acyclically 5-colored[6] (and this is best possible), we get from Theorem 24 that every planar graph has oriented chromatic number at most $5 \cdot 2^4 = 80$.

Question 33
Construct a planar graph which is not acyclically 4-colorable.

Conversely to Theorem 24, Kostochka, Sopena and Zhu proved in [24] that every family of graphs with bounded oriented chromatic number has bounded acyclic chromatic number.

On the other hand we know that there exist oriented planar graphs with oriented chromatic number at least 16. Consider the infinite sequence of oriented graphs $G_1, G_2, \ldots$, inductively defined as follows (see Figure 11): G_1 is the graph with one vertex and no arc and for every $i > 1$ the graph G_{i+1} is obtained by taking two disjoint copies of G_i, a new vertex v_{i+1}, and adding all possible arcs from the vertices of the first copy to v_{i+1} and all possible arcs from v_{i+1} to the second copy.

[6] Borodin's proof is reminiscent of the four-color proof of Appel and Haken and uses an unavoidable set of some 450 reducible configurations (see Subsection 1.2), but no computer!

Question 34
Prove that $\vec{\chi}(G_i) = |V(G_i)| = 2^i - 1$ holds for every i.

Question 35
Compare the number of arcs in G_i with the answer you (maybe) gave to Question 26...

The graph G_3 above defined is outerplanar (see again Figure 11) and the graph G_4 is therefore a planar graph. Moreover, it can be checked that $\vec{\chi}(G_3) = 7$ and $\vec{\chi}(G_4) = 15$. (If you worked out the Question 28 this proves that the oriented chromatic number of the family of outerplanar graphs is 7).

In order to push up this number to 16 we have to do more work... but we shall not do it here[7], due to the lack of space.

Viewing this you possibly may prefer the following version of our problem:

Problem C2 (Oriented chromatic number of planar graphs)

Determine the best possible lower bound for the oriented chromatic number of planar graphs.

3.3 Planar graphs with large girth

Another celebrated result on planar graph coloring is due to H. Grötzsch who proved in 1959 that every triangle-free planar graph is 3-colorable[8]. The shortest length of a cycle in a graph G is called the *girth* of G. Grötzsch's theorem thus says that every planar graph with girth at least 4 is 3-colorable. By considering the case of large odd cycles, it is clear that we cannot go on playing with this girth parameter to decrease again by one the chromatic number of planar graphs.

The situation is quite different for oriented planar graphs. Since the upper bound given by Theorem 24 is 80 we can try to progressively decrease this bound to 5 (here again because there exist planar graphs with arbitrarily large girth and oriented chromatic number 5).

Question 36
(i) Prove that every oriented cycle, except the directed cycle on 5 vertices, has oriented chromatic number at most 4.
(ii) Prove that there exist planar graphs with arbitrarily large girth and oriented chromatic number at least 5.

Upper bounds on the oriented chromatic number of planar graphs with large girth has been obtained by exhibiting some so-called nice graphs which are universal for these families:

Definition 25 (Nice graphs)
An oriented graph G is *k-nice* for some $k > 0$ if for every two vertices $x, y \in V(G)$ (not necessarily distinct) and every oriented path P of length k (with $V(P) = \{u_0, u_1, \ldots, u_k\}$ and $(u_i, u_j) \in E(P)$ if and only if either $i = j + 1$ or $j = i + 1$) there exists a homomorphism $f_{x,y} : P \to G$ with $f_{x,y}(u_0) = x$ and $f_{x,y}(u_k) = y$.

[7] The smallest oriented planar graph with oriented chromatic number 16 that we know at this time has 1565 vertices. The basic idea is to exhibit an oriented planar graph G such that every oriented graph H with $G \to H$ has to satisfy some properties and to prove that there exists no graph on less than 16 vertices satisfying these properties. The proof is quite complicated and has not been published yet.

[8] A short proof of Grötzsch's theorem was given by C. Thomassen [51] in 1993. In 1974, Grünbaum made the assertion that every planar graph with *at most three* triangles is 3-colorable. This extension of Grötzsch's theorem was proved by V.A. Aksionov: On continuation of 3-coloring of planar graph (in Russian), *Metody Diskret. Analyz.* **26**, 3–19 (1974).

Recall that every such oriented path can be encoded by a $\{0,1\}$-word $c(P) = c_0c_1 \dots c_{k-1}$ defined by $c_i = 0$ if $(u_i, u_{i+1}) \in E(P)$ and $c_i = 1$ otherwise for every i, $0 \leq i < k$ (see subsection 2.3). Now, for every oriented graph G and every vertex $x \in G$ let us denote by $N_0(x)$ and $N_1(x)$ the sets

$$N_0(x) = \{y;\ (x,y) \in E(G)\} \qquad N_1(x) = \{y;\ (y,x) \in E(G)\}.$$

This notion can be inductively extended to any non-empty word $w = w_1w_2 \dots w_q$ on the alphabet $\{0,1\}$ by letting $N_{w_1w_2\dots w_q}(x) = N_{w_q}(N_{w_1w_2\dots w_{q-1}}(x))$.

An oriented graph G is then k-nice if for every word w of length k on the alphabet $\{0,1\}$ and every vertex $x \in V(G)$ we have $N_w(x) = V(G)$.

Question 37
Prove that every k-nice oriented graph is k'-nice for every $k' > k$.

Question 38
(i) Prove that the circulant oriented graph $G(5;1,2)$ is 4-nice.
(ii) Prove that the circulant oriented graph $G(7;1,2,4)$ is 3-nice.
(iii) Prove that every circulant oriented graph $G_{n,d} = G(n;1,2,\dots,d)$, $2d < n$, is $(\frac{n-1}{d-1})$-nice.

Question 39
Can you exhibit some nice oriented planar graphs? (We say that an oriented graph is nice if it is k-nice for some k).

Hell, Kostochka, Raspaud and Sopena proved in [15] that if an oriented graph is universal for a family of outerplanar or planar oriented graphs with girth at least g (and minimal with respect to this property) then it is necessarily nice.

Let $\mathcal{P}_g$ denote the family of oriented planar graphs with girth at least g. Obviously, if we want to prove that some oriented graph H_g is universal for the family $\mathcal{P}_g$ it is enough to prove that every graph G which is "maximal in $\mathcal{P}_g$" has a homomorphism to H_g (we mean here that if we add any arc to G the obtained graph does no longer belong to $\mathcal{P}_g$). For these maximal graphs we have the following (see Nešetřil, Raspaud and Sopena [38])

Lemma 26
For every $k > 0$, if G is maximal in $\mathcal{P}_{5k+1}$ and has no vertex with degree 1 then G contains an ear *of length $k+1$ (that is an induced path of length $k+1$).*

From this lemma we then get

Theorem 27 (Universality of Nice Graphs)
If H_g is k-nice then it is universal for the family $\mathcal{P}_{5k-4}$.

Question 40
Prove Lemma 26 and Theorem 27.

Question 41
Adapt Lemma 26 and Theorem 27 to the case of outerplanar graphs.

Using the fact that the oriented tournaments $T_5 = G(5;1,2)$ and $T_7 = G(7;1,2,4)$ are respectively 4-nice and 3-nice we get

Corollary 28
Every oriented planar graph with girth at least 16 has oriented chromatic number at most 5.
Every oriented planar graph with girth at least 11 has oriented chromatic number at most 7.

Question 42
(i) What is the maximal girth of an oriented planar graph G such that $G \not\to T_5$? such that $G \not\to T_7$?
(ii) Can you construct an oriented graph of any given girth with $\vec{\chi} > 4$?

Using more complicated techniques, Borodin, Kostochka, Nešetřil, Raspaud and Sopena proved in [7] that every oriented planar graph with girth at least 14 (resp. 8, 6, 5) has oriented chromatic number at most 5 (resp. 7, 11, 19).

More precisely, the result of [7] links the oriented chromatic number of a graph G with the so-called maximum average degree of G. For a graph G, the *average degree* of G, denoted by $ad(G)$, is nothing else than the average of the degrees of its vertices:

$$ad(G) = \frac{\sum_{v \in V(G)} deg(v)}{|V(G)|} = \frac{2 \cdot |E(G)|}{|V(G)|}.$$

The *maximum average degree* of G, denoted by $mad(G)$, is then the maximum of the average degrees of its subgraphs:

$$mad(G) = max\ \{\ ad(H) :\ H \text{ subgraph of } G\ \}.$$

In case of planar graphs, this mad parameter is related to the girth parameter as follows:

Observation 29
For every planar graph G with girth at least g, $mad(G) < \frac{2g}{g-2}$.

Question 43
Prove Observation 29 (Euler's formula may help...).

In order to illustrate the technique of *reducible configurations* (see Section 1.2) we shall prove the following

Theorem 30
Every graph with maximum average degree strictly less than 7/3 has oriented chromatic number at most 5.

Proof.
We shall prove that every graph G with $mad(G) < 7/3$ admits a homomorphism to the tournament $T_5 = G(5;1,2)$ on five vertices depicted by Figure 12(a). We shall proceed in two steps, proving:

(1) If G is a minimum counter-example to Theorem 30 then G does not contain any of the two configurations[9] A and B, depicted respectively in Figure 12(b) and 2(c).

(2) If G is a graph which does not contain any of the two configurations A and B, then $mad(G) \geq 7/3$.

[9] In these configurations, white vertices are vertices with *prescribed* degrees. In other words, a graph contains the configuration A if and only if it contains an induced path of length 4 (with three internal vertices of degree 2).

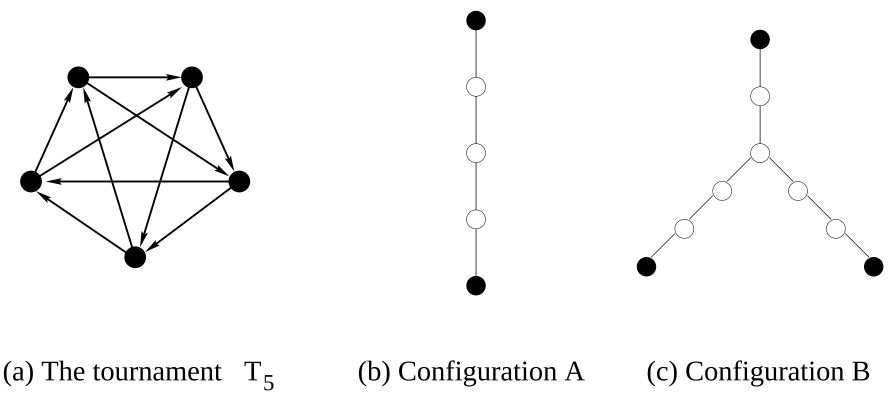


Figure 12: Proof of Theorem 14

From (1) and (2) we shall get that there exists no counter-example to Theorem 30 and we shall be done.

Let us start by the proof of (1). It suffices to observe that if G contains the configuration A (resp. B) then any T_5-coloring of the graph G_A (resp. G_B), obtained from G by deleting the vertices of degree two in the configuration A (resp. B), can be extended to a T_5-coloring of G. This can be done since the tournament T_5 is 4-nice (see Question 38).

Question 44
Unsing the fact that T_5 is 4-nice (and maybe more) prove that one can extend any T_5-coloring of G_A or G_B to a T_5-coloring of G.

The proof of (2) will make use of the "discharging" technique. The idea is as follows. Let G be any graph not containing any of the configurations A and B. To every vertex v of G we assign an *initial charge* $ch(v) = deg(v)$. Observe that we have

$$\sum_{v \in V(G)} ch(v) = \sum_{v \in V(G)} deg(v) = 2 \cdot |E(G)|.$$

We now define a *discharging procedure*, that is way of redistributing the initial charges among neighboring vertices, by the following rule:

If v is a vertex of degree *at least three* and w a neighbor of v *with degree two* then:
1. v gives $1/3$ to w if the other neighbor of w is of degree two,
2. v gives $1/6$ to w otherwise.

After having applied this rule on the whole graph (that is at every vertex of degree at least three) each vertex v has a *final charge*, denoted by $ch^*(v)$. Clearly, the total sum of charges is preserved:

$$\sum_{v \in V(G)} ch^*(v) = \sum_{v \in V(G)} ch(v).$$

The next step consists in proving that for every vertex v, $ch^*(v) \geq 7/3$. We proceed by case analysis, according to the degree of v, using the fact that G does not contains any of the configurations A and B.

1. If v has degree 2 then at least one of its neighbors has degree at least three (since G does not contain A). If v has a neighbor of degree 2 then it receives $1/3$ from its other neighbor. Otherwise, it receives $1/6$ from each of its two neighbors. In both cases, $ch^*(v) = 2 + 1/3 = 7/3$.

2. If v has degree 3 then it may have:
   (a) no neighbor of degree two: in that case $ch^*(v) = 3 > 7/3$,
   (b) one neighbor of degree two: v gives at most $1/3$ and $ch^*(v) \geq 3 - 1/3 > 7/3$,
   (c) two neighbors of degree two: v gives at most $2/3$ and $ch^*(v) \geq 3 - 2/3 = 7/3$,
   (d) three neighbors of degree two: in that case, since G does not contain B, at most one of these neighbors has itself a neighbor of degree two. Therefore, v gives at most $1/3+2/6$ and $ch^*(v) \geq 3 - 1/3 - 2/6 = 7/3$.
3. Finally, if v has degree $k \geq 4$ then it gives at most $k/3$ and $ch^*(v) = k - k/3 = 2k/3 > 7/3$.

In every case we thus get that $ch^*(v) \geq 7/3$. Using the previous observations we have

$$\sum_{v \in V(G)} ch^*(v) = \sum_{v \in V(G)} ch(v) = 2 \cdot |E(G)| \geq \frac{7}{3} \cdot |V(G)|$$

from which we get

$$mad(G) \geq \frac{2 \cdot |E(G)|}{|V(G)|} \geq 7/3$$

which contradicts our initial assumption $mad(G) < 7/3$.

This concludes the proof of (1) and (2), that is the proof of Theorem 30. □

Using Observation 29 we get from Theorem 30 that every planar graph with girth at least 14 has oriented chromatic number at most 5.

4 Games for planar graphs

Coloring the vertices of a graph may be a diverting game... but this game really becomes more interesting, and challenging, if you play it with a partner, especially with an uncooperative partner! It seems possible to play such a game on any kind of graph and maybe with any kind of coloring.

Consider for instance the following two-players game $GAME(G, k)$, where G is an undirected graph and k a fixed number of colors (we shall use $\{1, 2, \ldots, k\}$ as the set of colors). The two players are Alice and Bob and they will play alternatively, with Alice having the first move. Alice's goal is to provide a proper k-coloring of G and Bob's goal is to prevent her from doing that... A move consists in choosing an uncolored vertex v and assigning it a color from the set $\{1, 2, \ldots, k\}$ distinct from the colors previously assigned (by either player) to the neighbors of v. If after $|V(G)|$ moves the graph is colored then Alice wins. If an impass is reached before the whole graph is colored then Bob wins[10].

The notion of the game chromatic number of a graph can be defined as follows:

Definition 31 (Game chromatic number)
The *game chromatic number* of a graph G, denoted by $gcn(G)$, is the least integer k such that Alice has a winning strategy[11] when playing the game $GAME(G, k)$.

This parameter is well defined since Alice has always a strategy for the game $GAME(G, |V(G)|)$. Moreover, since every coloring produced by a winning game is a proper coloring we have

$$\chi(G) \leq gcn(G) \leq |V(G)|$$

. This game can be extended in a natural way to the case of oriented colorings of oriented graphs, leading to the notion of the *oriented game chromatic number* of a graph G, denoted by $ogcn(G)$.

[10]An impass is reached if after some move (not the last one) every uncolored vertex is such that all the colors are already used on its neighborhood.

[11]We say that Alice has a winning strategy if she can win the game whatever the strategy of Bob is.

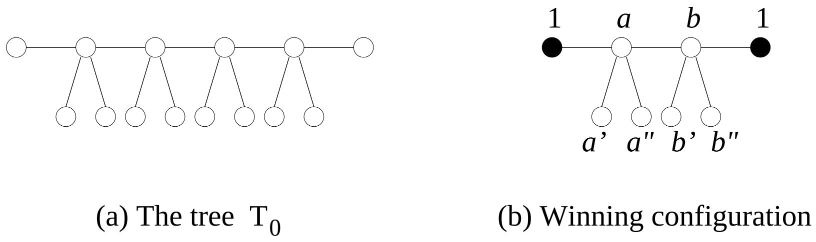


Figure 13: A tree with game chromatic number 4

Our last problem is then the following:

Problem D (Oriented game chromatic number of planar graphs)

Does there exists a constant C such that every oriented planar graph has oriented game chromatic number at most C?

The undirected version of this coloring game was first introduced by Bodlaender [3]. Bodlaender proved in particular that if T is a tree then $gcn(T) \leq 5$ and that there exist trees with game chromatic number at least 4. In [12], Faigle, Kern, Kierstead and Trotter improved Bodlaender's result and showed that $gcn(T) \leq 4$ for every tree T. They also proved that if G is a planar graph then $gcn(G) \leq O(\log |V(G)|)$. A constant (but large) upper bound, the existence of which was conjectured by Bodlaender, was later proved by Chen and Schelp [8]. Their bound was then improved by Kierstead and Trotter [22] who proved that $gcn(G) \leq 33$ for every planar graph G. This latter bound was recently decreased to 30 by Dinski and Zhu [9] who proved that if G has acyclic chromatic number[12] at most k then $gcn(G) \leq k(k+1)$. Dinski and Zhu proof is also easier and we reproduce it below.

In the next subsections we shall prove the results on the game chromatic number of trees and planar graphs and finally discuss the case of the oriented game chromatic number of oriented graphs.

4.1 The game chromatic number of trees

Concerning undirected trees we have the following

Theorem 32 (Game Chromatic Number of Trees)
If T is a tree then $gcn(T) \leq 4$ and this bound is best possible.

Proof.
We shall begin by proving that there exists a tree T_0 with $gcn(T_0) \geq 4$. This should allow the reader to become more familiar with our coloring game... Consider the tree T_0 depicted in Figure 13(a).

To prove our assumption we shall give a strategy for Bob to prevent Alice from producing a 3-coloring of T_0 in the game $GAME(T_0, 3)$. Suppose that on her first move Alice color some vertex $v \in V(T_0)$ with any color, say 1. Then, immediately, Bob has to choose a vertex w which is at distance 3 from v in T_0 (this is always possible) and to assign w the same color 1. After that, the configuration depicted in Figure 13(b) necessarily appears in T_0... Observe that there are exactly 6 vertices in T_0 "outside" the configuration and that it is now Alice's turn (in other words Bob can avoid playing inside the configuration unless Alice does). From now on, if Alice plays outside the configuration then so will do Bob. As soon as Alice plays inside the configuration Bob plays according to the following strategy:

---

[12] Recall that the acyclic chromatic number $a(G)$ of a graph G is the minimum k such that G has an acyclic k-coloring, that is a proper k-coloring such that no cycle in G uses less than three colors.

1. If in her last move Alice colored the vertex a (resp. b) with color $\alpha \neq 1$, Then Bob assigns the color $\beta \notin \{1, \alpha\}$ to vertex b' (resp. a') and wins the game since b (resp. a) can no longer be colored.

2. Otherwise, assume without loss of generality that in her last move Alice colored the vertex a'. We have two cases to consider:

   (a) a' is colored with $\alpha \in \{2, 3\}$: as before, Bob colors vertex a'' with color $\beta \notin \{1, \alpha\}$ and wins the game since a can no longer be colored.

   (b) a' is colored with 1: Bob then assigns color 2 to b'. After that, if Alice does not color vertex a then Bob colors a with 3 and wins the game since b or b'' can no longer be colored. Otherwise, Bob colors b'' with 3 and wins the game since b can no longer be colored.

Let us now prove that Alice has a winning strategy for the game $GAME(T, 4)$ for every tree T. The idea is as follows: Alice will play in such a way that after any of her moves the following property

(P) *every connected subgraph of T made of uncolored vertices has at most two colored neighbors*[13]

holds. Let us check that this invariant can effectively be preserved. Clearly, property (P) is satisfied after Alice's first move. Suppose now that in his last move Bob has colored the vertex v. Since property (P) was satisfied before Bob's last move we know that now at most one subtree of uncolored vertices may be adjacent to three colored vertices (just check it on a piece of paper if necessary). Alice then colors the *unique* vertex which is on each of the three paths linking pairs of these three colored vertices. Since we have four colors, this is always possible. After Alice's such move, property (P) clearly again holds.

Therefore, thanks to property (P), it is always possible for both players to color any uncolored vertex with a color in $\{1, 2, 3, 4\}$ and Alice will win the game. □

Question 45
Determine the game chromatic number of
(i) paths,
(ii) cycles.

4.2 The game chromatic number of planar graphs

Recall that Borodin's result [6] says that every planar graph G has acyclic chromatic number $a(G)$ at most five. The following result of Dinski and Zhu [9] thus states that the game chromatic number of every planar graph is at most 30:

Theorem 33
If G is a graph with $a(G) \leq k$ then $gcn(G) \leq k(k + 1)$.

Proof [sketch of].
For every $i \in \{1, 2, \ldots, k\}$ let $X_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,k+1}\}$ be a set of $k + 1$ colors and $X = \cup_{i=1}^{k} X_i$ be a set of $k(k + 1)$ colors.

Let c be any acyclic k-coloring of G. We denote by C_i, $1 \leq i \leq k$, the set of vertices with color i and by $F_{i,j}$ the forest subgraph of G induced by $C_i \cup C_j$.

We shall now describe a winning strategy for Alice in the game $GAME(G, k(k + 1))$, where X is used as the set of colors. In the process of the game, let us say that a vertex v has a *correct color* if

[13] That is at most two colored vertices are adjacent to some vertex of T.

$v \in C_i$ and v has been assigned (by either player) a color from X_i. We shall say that a color $x_{j,q}$ is *available* at an uncolored vertex w if no neighbor of w is colored with $x_{j,q}$.

The idea is as follows: Alice will always color a vertex with a correct color and will ensure that at every step, for every uncolored vertex $w \in C_i$, there exists at least one available color for w in X_i. To do that, Alice must prevent Bob from using all the colors of X_i on the neighbors of w...

In her first move, Alice chooses any vertex and assigns it any correct color. Suppose now that on his last move Bob has colored with $x_{j,q}$ a vertex $v \in C_i$. We then have two cases to consider:

1. If $i = j$ then v has been colored with a correct color and Alice chooses any uncolored vertex and assigns it an available correct color.

2. Suppose now that $i \neq j$. Let T be the connected subgraph of $F_{i,j}$ which contains v, and let S be the set of vertices of $C_i \cap T$ which have been colored by colors from X_j. We have the two following subcases:

   (a) If $S = \{v\}$ then Alice chooses any uncolored vertex and assigns it an available correct color.

   (b) Otherwise, for each $u \in S \setminus \{v\}$ let P_{uv} be the unique path in T connecting u and v. If for every such u the path P_{uv} contains an internal colored vertex (that is a colored vertex distinct from u and v) then Alice chooses any uncolored vertex and assigns it an available correct color. Otherwise, it can be proved that there is a unique vertex $u_0 \in S \setminus \{v\}$ such that all the internal vertices of P_{u_0v} are uncolored. Then Alice chooses an internal vertex of P_{u_0v} and assigns it an available correct color.

This finishes the description of Alice's strategy. Of course it remains to prove that the above discussed vertex u_0 is unique and that this strategy is a winning strategy. This is done in [9], but it is long and case analysis (those interested can look at the paper). □

If you worked out the Question 32 you know that the acyclic chromatic number of every outerplanar graph is at most 3. From Theorem 32 we then get that the game chromatic number of every outerplanar graph is at most 12.

Question 46 [Research question]
Can you improve this upper bound for the game chromatic number of outerplanar graphs?

4.3 The oriented game chromatic number

We can define in a similar way an oriented version of our coloring game, denoted by $OGAME(G, k)$. The only difference lies in the definition of a player's move: each player must color an uncolored vertex in such a way that the *partial* oriented k-coloring of G thus obtained can be extended to an oriented coloring of the whole graph G. Please note that we do not require to be able to extend it to an oriented k-coloring of G since in that case Alice would always win! (Provided that G is k-colorable).

In other words, each player has to choose an uncolored vertex u and to assign it a color $\alpha \in \{1, 2, \ldots, k\}$ such that:

(i) no neighbour of u is already colored with α,

(ii) if uvw (resp. wvu) is a directed path of length 2 in G then w is not colored with α,

(iii) if (u, v) (resp. (v, u)) is an arc in G and v is colored with β then there exists no arc (x, y) (resp. yx) in G such that x is colored with β and y is colored with α.

Consider for instance the situation depicted in Figure 14: if we want to color the vertex u we cannot use the colors 1,3 and 55 (due to (i)), 4 and 8 (due to (ii)), 6,7 and 9 (due to (iii)); but the color 2 (for instance) is available.

The oriented game chromatic number of an oriented graph is then naturally defined as follows:

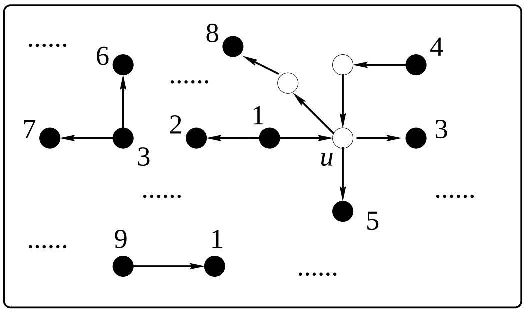


Figure 14: The oriented coloring game

Definition 34 (Oriented game chromatic number)
The *oriented game chromatic number* of an oriented graph G, denoted by $ogcn(G)$, is the least integer k such that Alice has a winning strategy when playing the game $OGAME(G,k)$.

This number is clearly well-defined since for every graph G the inequality $ogcn(G) \leq |V(G)|$ holds.
Actually we do not know the answer to the following questions so these are research questions, at least presently.

Question 47 [**Research question**]
Determine the best possible upper bound for the oriented game chromatic number of an oriented tree.

Question 48 [**Research question**]
Determine the best possible upper bound for the oriented game chromatic number of an oriented cycle.

Let us consider again our oriented version of the coloring game. In the game $OGAME(G,k)$ both players are simply using the set $\{1,2,\dots,k\}$ as the set of available colors... What happens if we try to play this game with a fixed target graph H of order k ?

Let us try to define more precisely this "new version" of the game that we shall denote by $OGAME_H(G)$, where H is any given oriented graph on k vertices. The "spirit" of the game remains the same but assume now that on his turn each player has to choose an uncolored vertex u and to assign it a color $\alpha \in V(H)$ such that:

(i) no neighbour of u is already colored with α,

(ii) if uvw (resp. wvu) is a directed path of length 2 in G then w is not colored with α,

(iii) if (u,v) (resp. (v,u)) is an arc in G and v is colored with β then $(\alpha,\beta) \in E(H)$ (resp. $(\beta,\alpha) \in E(H)$).

The game chromatic number corresponding to this modified version would then be defined as follows:

Definition 35 (Oriented H-game chromatic number)
The *oriented H-game chromatic number* of an oriented graph G, denoted by $ogcn_H(G)$, is the least integer k for which there exists an oriented graph H on k vertices such that Alice has a winning strategy when playing the game $OGAME_H(G)$.

Question 49
Is is clear that this new number is also well-defined?

In case of undirected graphs this modified version is obviously nonsense... Playing with a set of k colors or with the target graph K_k is clearly the same! But what about the oriented case? When Alice wins the oriented game we are left with an oriented k-coloring of the graph G, that is with a homomorphism of G to some tournament T on k vertices. (This homomorphism is not necessarily onto). But what happens if we try to impose this target T at the beginning of the game? Will Alice still win? Suppose that Bob is clever enough to assign some color, say α, to some vertex u such that (i) there is a 2-path uvw (with any orientation), (ii) w has previously been assigned some color β and (iii) v is uncolored and no color in T is now available for v. In that case, Bob wins the H-game and it would possibly not be the case if Alice and Bob were playing the first version of our game...

Question 50
Can you get some relation between the numbers $ogcn(G)$ and $ogcn_H(G)$?

One possible way for Alice to win the oriented coloring game (in any of its two versions) is to ensure that at any step every uncolored vertex is colorable! (we know that this is an evidence...). In other words, if u is an uncolored vertex we need for u a color α such that α is a predecessor of the set $S(u) = \{c(v);\ (u,v) \in E(G)\}$ and α is a successor of the set $P(u) = \{c(w);\ (w,u) \in E(G)\}$, where c stands for the oriented coloring in construction. Towards this end, let us introduce the following property of oriented graphs:

Definition 36 (k-good oriented graphs)
An oriented graph H is *k-good*, $k > 0$, if for every subset $\{x_1, x_2, \ldots, x_k\}$ of $V(H)$ and every word $w = w_1 w_2 \ldots w_k$ on the alphabet $\{0,1\}$ there exists a vertex y_w in $V(H)$ such that $(y_w, x_i) \in E(H)$ if $w_i = 0$ and $(x_i, y_w) \in E(H)$ otherwise.

Question 51
(i) Prove that if H is k-good then for every vertex $x \in V(H)$ the subgraphs $\Gamma^+(x)$ and $\Gamma^-(x)$ are both $(k-1)$-good.
(By $\Gamma^+(x)$ (resp. $\Gamma^-(x)$) we denote the subgraph of H induced by the successors (resp. predecessors) of x).
(ii) Is this condition also a sufficient condition?

Question 52
Prove that C_3 (the directed cycle on three vertices) is 1-good and that $T_7 = G(7;1,2,4)$ is 2-good.

Question 53
Try to construct a 3-good oriented graph (the smallest 3-good graph has 19 vertices).

Question 54
Can you prove that there exists k-good oriented graphs for every k ?

The idea here is the following: suppose that we play the H-version of the game on some graph G with a k-good target graph H... If Alice ensures that at any step the inequality $|P(u)| + |S(u)| \leq k$ holds for every uncolored vertex u, then Alice will win the game and, therefore, $ogcn_H(G) \leq k$!

Hmmm... it sounds well, but it is false! (As you surely observed). We simply forgot that u may also be linked to some colored vertices by a directed path of length 2! And these vertices may prevent us to assign the "good" color y_w...

Well, we just need to enforce the "good" condition:

Definition 37 ((k,p)-good oriented graphs)
An oriented graph H is *(k,p)-good*, $k \geq p > 0$, if for every subset $\{x_1, x_2, \ldots, x_k\}$ of $V(H)$ and every word $w = w_1 w_2 \ldots w_k$ on the alphabet $\{0,1\}$ there exist p distinct vertices $y_w^1, y_w^2, \ldots, y_w^p$ in $V(H)$ such that for every j, $1 \leq j \leq p$, $(y_w^j, x_i) \in E(H)$ if $w_i = 0$ and $(x_i, y_w^j) \in E(H)$ otherwise.

Now, if Alice can ensure that at any step and for every uncolored vertex u:

(i) the inequality $|P(u)| + |S(u)| \leq k$ holds,

(ii) there are at most $p - 1$ *dangerous* vertices for u (we say that w is dangerous for u if (a) w is colored, (b) w is linked to u by a directed path of length 2 whose internal vertex is uncolored and (c) there is no directed path of length 2 linking w to u whose internal vertex is colored),

then Alice will win the game $OGAME_H(G)$ whenever H is (k,p)-good. (Are we right this time?)

Question 55
Do you see the reason for condition (c) in the second item?

Question 56
Prove that Alice really wins if she ensures the above conditions.

Question 57
Can you construct (k,p)-good graphs for every $k, p > 0$?
(Maybe using a combination of k-good graphs?)

The problem now for Alice is to "control", for every uncolored vertex u, the number of its colored neighbours and the number of its "dangerous" quasi-neighbours as well...

Let us forget for a while the orientation of G and play a new game... We even forget the colors and assume that both players are just "marking" the vertices. The aim of Alice in this game is to avoid that an unmarked vertex has too much marked neighbours (at most k) − this is in some sense similar to the GO game... − and also too much dangerous quasi-neighbours (at most p)... Because of this second condition let us call our game the *extended-GO* game, and denote it by $eGO(G, k, p)$.

This game is certainly easier to play (it could even be sold all over the world, don't you think?) and we may hope that some results are easier to get!

For instance we have

Theorem 38
For every tree T, Alice as a winning strategy for the game $eGO(T, 3, 3)$.

Proof.
It suffices for Alice to use the strategy explained in the proof of Theorem 32: thanks to the property

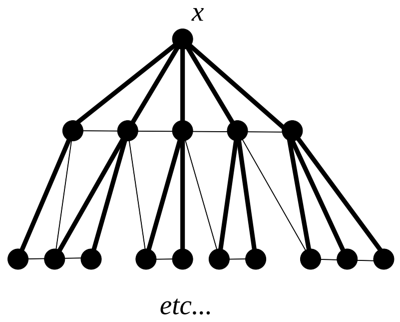


Figure 15: Special drawing of a maximal outerplanar graph

(P) (see the proof of Theorem 32) it is easy to see that all along the game every uncolored vertex has at most 3 colored neighbours and at most 3 dangerous quasi-neighbours. □

By looking more precisely to this proof, it appears that in fact the *total number* of colored neighbours and dangerous quasi-neighbours is at most 3. However we cannot directly derive from this fact that Alice has a winning strategy for the games $eGO(T,3,0)$ or $eGO(T,2,1)$ or etc. But there is maybe a way there to "weaken" the requirements on (k,p)-good graphs...

Question 58 [Research question]
Can you prove that Alice has also a winning strategy for the $eGO(G,k,p)$ game (for some k,p) when playing on an outerplanar graph G ?
(Hint: you may restrict yourself to the case of a *maximal* outerplanar graph (that is which is no longer outerplanar as soon as you add an edge). If you choose any vertex x you can then draw it as in Figure 15 (this drawing has several interesting properties...). Consider then any BFS spanning tree of the graph (such a tree is drawn with thick edges) and apply the strategy used in the proof of Theorem 32...).

All the games which fit to print! We better stop here.

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