LaBRI Research Report No. Compiled on April 11, 2001
The final version of this paper has been published in *Comment. Math. Univ. Carolinae* **38-1** (1997), 125–136.

T-PRESERVING HOMOMORPHISMS OF ORIENTED GRAPHS [1]

Jaroslav NEŠETŘIL[2]
Dept of Appl. Math., Charles University, Prague, Czech Republic.

Éric SOPENA and Laurence VIGNAL
LaBRI, Univ. Bordeaux I, 33405 Talence, France.

Abstract. A homomorphism of an oriented graph $G = (V, A)$ to an oriented graph $G' = (V', A')$ is a mapping φ from V to V' such that $\varphi(u)\varphi(v)$ is an arc in G' whenever uv is an arc in G. A homomorphism of G to G' is said to be T-preserving for some oriented graph T if for every connected subgraph H of G isomorphic to a subgraph of T, H is isomorphic to its homomorphic image in G'. The T-preserving oriented chromatic number $\vec{\chi}_T(G)$ of an oriented graph G is the minimum number of vertices in an oriented graph G' such that there exists a T-preserving homomorphism of G to G'. This paper discusses the existence of T-preserving homomorphisms of oriented graphs. We observe that only families of graphs with bounded degree can have bounded T-preserving oriented chromatic number when T has both in-degree and out-degree at least two. We then provide some sufficient conditions for families of oriented graphs for having bounded T-preserving oriented chromatic number when T is a directed path or a directed tree.

Keywords. Graph homomorphisms, Oriented Graphs.

1 Introduction

The vertex set of a graph G is denoted by $V(G)$. The edge set of an undirected graph is denoted by $E(G)$ and $A(G)$ stands for the set of arcs of a digraph. A homomorphism of a digraph G to a digraph G' is a mapping φ from $V(G)$ to $V(G')$ such that $\varphi(u)\varphi(v)$ is an arc in G' whenever uv is an arc in G. Homomorphisms of undirected graphs are defined in a similar way. Homomorphisms of digraphs and undirected graphs have been studied as a generalization of graph coloring [7, 8, 9, 13, 17]. It is easy to see that an undirected graph U is k-colorable if and only if U admits a homomorphism to the complete graph K_k. The chromatic number of U can then equivalently be defined as the minimum number of vertices in an undirected graph U' such that there exists a homomorphism of U to U'. Therefore, if a graph G has a homomorphism to a graph G' we will say that G is *G'-colorable* and the vertices of G' will be called *colors*.

An *orientation* of an undirected graph U is a digraph obtained from U by giving to every edge one of its two possible orientations. A digraph is an *oriented graph* if it is an orientation of some undirected graph. Thus, oriented graphs are digraphs which contain no opposite arcs. Homomorphisms of oriented graphs have been studied in [12, 16, 18, 19]. The *oriented chromatic number* of an oriented graph G is defined as the minimum number of vertices in an oriented graph H such that there exists a homomorphism of G to H. The oriented chromatic number of an undirected graph U is then defined as the maximum of the oriented chromatic numbers of its orientations. In particular, we know that classes of graphs with bounded genus, bounded degree or bounded treewidth have bounded oriented

[1]The results of this paper were partially obtained while the first author visited LABRI and the second author visited KAM Charles University.

[2]Partially supported by GAUK and GACR grants.

chromatic number [12, 19]. However, having bounded chromatic number is not a sufficient condition for an undirected graph for having bounded oriented chromatic number. For instance, the family of bipartite graphs has unbounded oriented chromatic number [12, 19].

The oriented chromatic number of an oriented graph can also be defined via the existence of *oriented colorings*: an oriented k-coloring of an oriented graph G is a mapping c from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that *(i)* $c(x) \neq c(y)$ if $xy \in A(G)$ (c is therefore a k-coloring of the underlying undirected graph of G) and *(ii)* if $xy, zt \in A(G)$ then $c(x) = c(t) \Rightarrow c(y) \neq c(z)$. The oriented chromatic number of G then corresponds to the minimum k such that G has an oriented k-coloring.

Let $\vec{P}_n$ denote the directed n-path, that is the oriented graph defined by $V(\vec{P}_n) = \{x_0, x_1, \ldots, x_n\}$ and $A(\vec{P}_n) = \{x_i x_{i+1},\ 0 \leq i < n\}$. It is easy to observe that if φ is a homomorphism of an oriented graph G to an oriented graph G' then for every directed 2-path uvw in G, $\varphi(u)\varphi(v)\varphi(w)$ is a directed 2-path in G'. We say that φ is $\vec{P}_2$-preserving. It is then natural to generalize this notion as follows: let T be any oriented graph; a homomorphism of an oriented graph G to an oriented graph G' is said to be *T-preserving* if every connected subgraph H of G which is isomorphic to a subgraph of T is isomorphic to its image under φ in G' (in other words, any two vertices of H must be assigned distinct colors). A *T-preserving oriented coloring* of an oriented graph G will thus be an oriented coloring such that any two vertices belonging to some subgraph isomorphic to a subgraph of T are assigned distinct colors. We then define the T-preserving oriented chromatic number $\vec{\chi}_T(G)$ of an oriented graph G as the minimum number of vertices in an oriented graph H such that there exists a T-preserving homomorphism of G to H. As before, the T-preserving oriented chromatic number of an undirected graph is defined as the maximum of the T-preserving oriented chromatic numbers of its orientations.

This paper is devoted to the study of T-preserving homomorphisms of oriented graphs. This notion of preservation could be considered in the undirected case. However, families of undirected graphs with bounded T-preserving chromatic number (for non trivial T) must have bounded degree. To see that, let P_2 denote the undirected 2-path and S_n the star with n branches (that is $V(S_n) = \{x, y_1, y_2, \ldots, y_n\}$ and $E(S_n) = \{xy_i,\ 1 \leq i \leq n\}$). It is easy to observe that if S_n admits a P_2-preserving homomorphism to an undirected graph U then U cannot have fewer than $n+1$ vertices. In other words, the P_2-preserving chromatic number of stars is unbounded. On the contrary, every oriented star has a $\vec{P}_2$-preserving homomorphism to $\vec{P}_2$ itself and has $\vec{P}_2$-preserving oriented chromatic number 3. Therefore, T-preserving homomorphisms can be studied in the oriented case in a wider class of graphs. Note that P_2-preserving homomorphisms of undirected graphs were already considered in [15] and were called *local homomorphisms*. Nešetřil proved in [15] that for every connected undirected graph G, automorphisms are the only P_2-preserving homomorphisms of G to itself. Obviously, the same holds for T-preserving homomorphisms for every connected graph T with at least two edges. For directed graphs the situation is more complicated. For instance, one can easily see that for every directed graph T there exists a directed graph G_T with a non-injective T-preserving homomorphism of G_T to itself.

Let S_n^+ (resp. S_n^-) denote the oriented star with n branches and all arcs directed outwards (resp. inwards). It is easy to check that S_n^+ (resp. S_n^-) has S_2^+-preserving (resp. S_2^--preserving) oriented chromatic number $n+1$. Therefore, whenever an oriented graph T has maximal in-degree and maximal out-degree at least 2, only families of graphs with bounded degree can have bounded T-preserving oriented chromatic number. But it is not difficult to prove that for every oriented graph T, the family of graphs with degree at most k has bounded T-preserving oriented chromatic number for every k. In the rest of the paper, we will thus restrict ourselves to oriented graphs T having in-degree at most 1, that is directed paths and directed trees (observe that it is not necessary to consider directed cycles $\vec{C}_n$ since $\vec{P}_n$-preserving homomorphisms are indeed $\vec{C}_n$-preserving homomorphisms). All the results we shall prove can obviously be restated for oriented graphs with out-degree at most 1.

In Section 2 we introduce a new notion of acyclic coloring: a k-coloring of an undirected graph is said to be p-acyclic if every p colors induce an acyclic subgraph. We prove in particular that for every p, planar graphs with sufficiently large girth can be p-acyclically colored (see Corollary 5). This notion of p-acyclic coloring will be useful in the study of T-preserving homomorphisms.

In Section 3 we consider the case when T is a directed path. Our main results here state that the family of undirected graphs with bounded p-acyclic chromatic number have bounded $\vec{P}_p$-preserving oriented chromatic number (see Theorem 7) and that the family of oriented graphs with bounded out-degree have bounded $\vec{P}_p$-preserving oriented chromatic number for every p (see Theorem 11).

In Section 4 we study T-preserving homomorphisms when T is a directed tree (that is a rooted tree whose all arcs are directed from the root towards the leaves). We prove that the family of oriented graphs with bounded p-acyclic chromatic number and bounded out-degree have bounded T-preserving oriented chromatic number for every directed tree T of depth at most $\lfloor \frac{p-1}{2} \rfloor$ (see Theorem 14) and show that both conditions are necessary.

We finally discuss some problems related to our work in Section 5.

2 p-Acyclic colorings of undirected graphs

A coloring of an undirected graph U is said to be acyclic if every two colors induce a forest (ie. the subgraph of U induced by the vertices with any two given colors has no cycle). The acyclic chromatic number $\chi_a(U)$ is then defined as the minimum number k such that U has an acyclic k-coloring. In [18] Raspaud and Sopena proved the following:

Theorem 1 **[18]** *If an undirected graph U has acyclic chromatic number at most k then every orientation of U has oriented chromatic number at most $k \cdot 2^{k-1}$.*

Conversely, Kostochka *et al.* proved in [12] that if all the orientations of an undirected graph U have bounded oriented chromatic number then U has bounded acyclic chromatic number. The links between the oriented chromatic number and other parameters of a graph (eg. maximum degree, arboricity) have also been studied in [12]. Thomas [20] observed that every family of graphs defined by a non-empty set of forbidden minors has bounded acyclic chromatic number and thus bounded oriented chromatic number.

Borodin proved in [3] that every planar graph has acyclic chromatic number at most 5. From Theorem 1 we get that every oriented planar graph has oriented chromatic number at most $5 \cdot 2^4 = 80$ [18]. The study of oriented planar graphs is thus particularly challenging in this context. Despite many efforts, no better upper bound is known up to now (we know that there exist oriented planar graphs with oriented chromatic number at least 15). This upper bound can be significantly decreased under some large girth assumptions [16].

We now introduce a generalization of the notion of acyclic coloring. We shall see that this new notion is of interest in the study of T-preserving homomorphisms. Recall that the girth $g(U)$ of an undirected graph U is the length of a shortest cycle in U. The girth of an oriented graph is then defined as the girth of its underlying undirected graph.

Definition 2 Let U be an undirected graph; a coloring of U is said to be *p-acyclic* if every cycle C in U uses at least $p+1$ colors. The *p-acyclic chromatic number* $\chi_a^p(U)$ of U is the minimum number k such that U has a p-acyclic k-coloring.

Note that any p colors in a p-acyclic coloring induce a forest. Thus a graph cannot have a p-acyclic coloring if its girth is less than $p+1$. Observe that the notion of 2-acyclic coloring corresponds to the usual notion of acyclic coloring.

It is easy to see that every undirected graph U with maximum degree k and girth $g(U) > p$ has bounded p-acyclic chromatic number for every p: for every vertex x in U there are at most $t = k((k-1)^{\lfloor p/2 \rfloor} - 1)/(k-2)$ distinct vertices at distance at most $\lfloor p/2 \rfloor$ from x. We can thus color the graph U with $t+1$ colors in such a way that every two vertices at distance at most $\lfloor p/2 \rfloor$ get distinct colors. Such a coloring is obviously p-acyclic.

Clearly, every graph with bounded p-acyclic chromatic number has bounded q-acyclic chromatic number for every $q < p$. However, this hierarchy of p-acyclic chromatic numbers is strict as we have the following:

Observation 3 *For every p, there exist families of graphs with bounded p-acyclic chromatic number and unbounded $(p+1)$-acyclic chromatic number.*

To see that, we generalize a construction proposed by Kostochka and Mel'nikov in [11]. Let $G_{n,p}$ $(n, p \geq 2)$ denote the graph obtained by linking two vertices s and t by n distinct paths of length p (observe that every cycle in $G_{n,p}$ has length $2p$). Let now $\mathcal{G}_p = \{G_{n,p}; n \geq 2\}$. We define a coloring c of $G_{n,p}$ as follows: $c(s) = 0$, $c(t) = p$ and for every path $sx_1x_2 \ldots x_{p-1}t$, let $c(x_i) = i$. The coloring c is p-acyclic since every cycle in $G_{n,p}$ uses exactly $p+1$ colors. Thus the family $\mathcal{G}_p$ has p-acyclic chromatic number at most $p+1$. Suppose now that the $(p+1)$-acyclic chromatic number of $\mathcal{G}_p$ is at most K. If $n > \binom{K}{p-1}$ then there exist two paths in $G_{n,p}$ whose internal vertices uses the same subset of colors. Thus $G_{n,p}$ contains a cycle using at most $p+1$ colors, a contradiction.

We know that the families of planar graphs and of graphs with bounded treewidth have bounded 2-acyclic chromatic number [12]. The graphs $G_{n,2}$ from the previous observation are obviously planar and have treewidth 2. This shows that there exist planar graphs and graphs of treewidth 2 having arbitrarily large 3-acyclic chromatic number. In particular, if we want to get families of planar graphs with bounded p-acyclic chromatic number we have to make some additional assumption, for instance on the maximal degree or on the girth. In [16] Nešetřil *et al.* proved the following:

Theorem 4 [16] *Let G be a planar graph with girth $g \geq 5d+1$ having no vertex of degree one. Then G contains an induced $(d+1)$-path.*

Using that result, we easily get the following:

Corollary 5 *For every $d > 1$, if U is an undirected planar graph with girth $g \geq 5d+1$ then the d-acyclic chromatic number of U is at most $d+1$.*

Proof. W.l.o.g. we may suppose that U has no vertex of degree one since no such vertex belongs to a cycle. We proceed by induction on the number n of vertices in U. This is certainly true if U has at most $d+1$ vertices. Suppose now that $n > d+1$. By Theorem 4 we know that U contains an induced $(d+1)$-path $x_0x_1 \ldots x_{d+1}$. Let f be any d-acyclic coloring of $G \setminus \{x_1, \ldots, x_d\}$. If $f(x_0) = f(x_{d+1})$ we color the vertices $x_1, \ldots, x_d$ with the d remaining colors. If $f(x_0) \neq f(x_{d+1})$ we color the vertices $x_1, \ldots, x_d$ with any d distinct colors. The $(d+1)$-coloring thus obtained is clearly d-acyclic. □

3 Path-preserving homomorphisms

In this section, we study T-preserving homomorphisms of oriented graphs when the graph T is a directed path. We first consider T-preserving homomorphisms of oriented forests:

Proposition 6 *Let U be an undirected forest. Then for every p, every orientation of U has $\vec{P}_p$-preserving oriented chromatic number at most $p+1$.*

Proof. Clearly every oriented tree has a homomorphism to $\vec{C}_{p+1}$, the directed cycle on $p+1$ vertices. Obviously, every such homomorphism is $\vec{P}_p$-preserving. □

Using that, we get the following result which generalizes Theorem 1 [18] introduced in Section 2:

Theorem 7 *Let U be an undirected graph with p-acyclic chromatic number k. Then the $\vec{P}_p$-preserving oriented chromatic number of every orientation of U is at most $k \cdot (p+1)^{\binom{k-1}{p-1}}$.*

Proof. Suppose that U has p-acyclic chromatic number k and let c_0 denote a p-acyclic coloring of U which uses k colors. Let $\vec{U}$ be any orientation of U. Recall that any p colors induce a forest in U. Moreover, every color i belongs to $t = \binom{k-1}{p-1}$ subsets of p colors. We assume that for every i those subsets containing i are ordered as $\sigma_1^i, \sigma_2^i, \ldots, \sigma_t^i$. We now define a coloring c of $\vec{U}$ as follows: for every $x \in V(\vec{U})$ let $c(x) = (c_0(x), c_1(x), \ldots, c_t(x))$ where for every i, $1 \leq i \leq t$, $c_i(x)$ corresponds to the $\vec{P}_p$-preserving coloring by $p+1$ colors of the forest induced by the p-subset $\sigma_i^{c_0(x)}$ (such a coloring exists by Proposition 6). We claim that the coloring c thus defined is $\vec{P}_p$-preserving. To see that, consider a directed path $x_0 x_1 \ldots x_q$ of length $q \leq p$ in $\vec{U}$. If all these vertices get distinct colors, there is nothing to prove. Suppose that two vertices x_i and x_j get the same color. The number of colors used for coloring $x_0, \ldots, x_q$ is then at most p, and there exists at least one p-subset σ such that the whole path $x_0 x_1 \ldots x_q$ is contained in the forest induced by σ. But then the colors assigned to x_i and x_j differ on the component associated with σ, a contradiction. Thus c is indeed a $\vec{P}_p$-preserving oriented coloring of $\vec{U}$ which uses at most $k \cdot (p+1)^{\binom{k-1}{p-1}}$ colors. □

In case of oriented planar graphs, using Corollary 5 and Theorem 7 we get:

Corollary 8 *If U is an undirected planar graph with girth $g \geq 5d+1$ then the $\vec{P}_d$-preserving oriented chromatic number of every orientation of U is at most $(d+1)^{d+1}$.*

Unlike the case of acyclic chromatic numbers and oriented chromatic numbers [12], an undirected graph with bounded $\vec{P}_p$-preserving oriented chromatic number may have unbounded p-acyclic chromatic number. For instance, the family $\mathcal{G}_2$ from Observation 3 has unbounded 3-acyclic chromatic number although every orientation of a graph $G_{n,2}$ in $\mathcal{G}_2$ has $\vec{P}_p$-preserving oriented chromatic number at most 6 for every $p \geq 1$. To see that, observe that such an orientation contains no subgraph isomorphic to $\vec{P}_p$ for every $p \geq 4$. Hence we only need to consider $\vec{P}_3$-preserving oriented colorings. The coloring f defined by $f(s) = 0$, $f(t) = 5$ and for every internal vertex x_i, $1 \leq i \leq n$, $f(x_i) = 1$ (resp. 2,3,4) if x_i is linked to s, t by the arcs sx_i and tx_i (resp. sx_i and x_it, x_is and tx_i, x_is and x_it) is obviously $\vec{P}_3$-preserving and uses at most 6 colors.

Recall that an undirected graph U is *d-degenerate* if every subgraph of U contains a vertex with degree at most d. It is floklore to prove that every d-degenerate graph has chromatic number at most $d+1$.

One other way to obtain families of graphs with bounded $\vec{P}_p$-preserving oriented chromatic numbers is to consider families of oriented graphs with bounded out-degree (or, similarly, with bounded in-degree). For undirected graphs, the existence of orientations with bounded out-degree ensures that the chromatic number is bounded as shown by the following easy observation:

Observation 9 *Let U be an undirected graph; if U can be oriented in such a way that every vertex x in U has out-degree at most d then $\chi(G) \leq 2d+1$.*

To see that, observe that if every vertex has out-degree at most d in the corresponding orientation $\vec{U}$ of U, then there exists a vertex x with in-degree at most d. Thus, the total degree of x is at most $2d$. Since this property is obviously true for every subgraph of U, we get that U is $2d$-degenerate and thus $(2d+1)$-colorable.

However, families of oriented graphs with bounded out-degree have not necessarily bounded p-acyclic chromatic number. For instance, every graph $G_{n,p}$ from Observation 3 can be oriented in such a way that every vertex has out-degree at most 2, although the family $\mathcal{G}_p$ has unbounded $(p+1)$-acyclic chromatic number.

We will now prove that Observation 9 can be generalized to the case of oriented chromatic numbers:

Theorem 10 *Let G be an oriented graph with out-degree at most d. Then G has oriented chromatic number at most $2^{2d(d+1)+1} - 1$.*

Proof. Let U_G be the undirected graph defined by $V(U_G) = V(G)$ and xy is an edge in U_G if and only if x and y are joined in G by a directed path of length 1 or 2. Clearly U_G can be oriented in such a way that every vertex has out-degree at most $d(d+1)$. By Observation 9, U_G has chromatic number at most $t = 2d(d+1)+1$. Let now H_t be the oriented graph defined as follows: the vertices of H_t are all the tuples of the form $(i; a_1, \dots, a_{i-1})$ with $1 \le i \le t$ (i is called the *identity* of the vertex) and $a_j \in \{0,1\}$ for every j, $1 \le j < i$. Let $a = (i; a_1, \dots, a_{i-1})$ and $b = (j; b_1, \dots, b_{j-1})$ be two vertices in H_t with $i < j$. There is arc from a to b if $b_i = 0$ and an arc from b to a otherwise. Observe that the graph H_t has $2^t - 1$ vertices and that all the arcs in H_t link vertices with distinct identities.
Let now c be any given t-coloring of U_G. We denote by V_i the set of vertices x such that $c(x) = i$ and by $G_{i,j}$ the (oriented) subgraph of G induced by $V_i \cup V_j$. From the definition of U_G we get that $G_{i,j}$ contains no directed 2-path. In other words, either all the arcs are directed from V_i to V_j or from V_j to V_i. We then define a mapping φ_c from $V(G)$ to $V(H_t)$ as follows: for every $x \in V_i$, $1 \le i \le t$, let $\varphi_c(x) = (i; a_1, \dots, a_{i-1})$ where for every j, $1 \le j < i$, $a_j = 0$ if there is an arc from V_j to x in G and $a_j = 1$ otherwise. From above, φ_c is clearly a homomorphism from G to H_t. □

This result can be extended to the $\vec{P}_p$-preserving oriented chromatic number:

Theorem 11 *Let G be an oriented graph with out-degree at most d. For every $p \ge 3$, the $\vec{P}_p$-preserving oriented chromatic number of G is at most*

$$(2^{2d(d+1)+1} - 1) \cdot \left(1 + 2d^3 \cdot \frac{d^{p-2} - 1}{d - 1}\right)$$

.

Proof. Let c_1 be an oriented coloring of G using at most $2^{2d(d+1)+1} - 1$ colors (such a coloring exists by Theorem 10). We define the undirected graph U_G as follows: $V(U_G) = V(G)$ and $xy \in E(U_G)$ if and only if x and y are joined in G by a directed q-path, $3 \le q \le p$. Every vertex in U_G has degree at most

$$q = d^3 + d^4 + \ldots + d^p = d^3 \cdot \frac{d^{p-2} - 1}{d - 1}$$

and thus U_G has chromatic number at most $2q + 1$. Let c_2 be a coloring of U_G using at most $2q + 1$ colors. The oriented coloring of G defined by $c(x) = (c_1(x), c_2(x))$ for every vertex x is then clearly $\vec{P}_p$-preserving and uses at most $(2^{2d(d+1)+1} - 1) \cdot (2q + 1)$ colors. □

Since every planar graph has oriented chromatic number at most 80 [18] we get:

Theorem 12 *Every oriented planar graph G with out-degree at most d has $\vec{P}_p$-preserving oriented chromatic number at most*

$$80 \cdot \left(1 + 2d^3 \cdot \frac{d^{p-2} - 1}{d - 1}\right).$$

Proof. The proof is similar to that of Theorem 11 by using an oriented 80-coloring c_1. □

By a result of Nash-Williams [14] we know that every planar graph can be oriented in such a way that every vertex has out-degree at most 3. Therefore, by bounding the out-degree of oriented planar graphs we still get a large class of oriented graphs.

4 Tree-preserving homomorphisms

In this section we consider the case when T is a *directed tree*, that is a tree whose arcs are all directed from the root towards the leaves (directed trees are sometimes called *branchings* in the literature). The maximal length of a directed path in T is called the *depth* of T. We will first study T-preserving homomorphisms of oriented forests and then T-preserving homomorphisms of oriented graphs in general. Our first result is the following:

Theorem 13 *Let T be a directed tree with depth q and G be an oriented forest with out-degree at most d. The T-preserving oriented chromatic number of G is then at most $(q+1)\cdot d^q$.*

Proof. W.l.o.g. we suppose that G is an oriented tree. We will proceed in two steps. We first associate with G an undirected graph U_G whose chromatic number is bounded. We then construct a T-preserving homomorphism of G by combining a coloring c_1 of U_G and a $\vec{P}_q$-preserving oriented coloring c_2 of G (such a coloring exists by Proposition 6).
Step 1. Let U_G be the undirected graph defined by $V(U_G)=V(G)$ and $xy\in E(U_G)$ if and only if there exist two directed paths $zx_1\dots x_{\ell-1}x$ and $zy_1\dots y_{\ell-1}y$ in G with the same length $\ell\le q$ and having no vertex in common except z (two such paths will be called *vertex-disjoint* in the following). We claim that the graph U_G thus obtained is (d^q-1)-degenerate. To see that, assume on the contrary that there exists a subgraph U' of U_G such that every vertex in U' has degree at least d^q. By a result of Dirac [5], U' contains a cycle of length at least d^q+1. Let us denote such a cycle by $C=(u_0,u_1,\dots,u_m,u_0)$, $m\ge d^q$. By definition of U_G, for every edge u_iu_{i+1} (taken modulo m) in C, there exists a vertex z_i and two vertex-disjoint directed paths $\vec{P}_{z_iu_i}$ and $\vec{P}_{z_iu_{i+1}}$ of the same length $\ell_i\le q$. Since every vertex in G has out-degree at most d, no vertex can be joined to more than d^q vertices by vertex-disjoint directed paths of length q in G (this bound can be achieved by a vertex x which is a leaf in a complete d-ary directed tree subgraph of G). Thus there are at least two distinct vertices among the z_i's. The (generally not simple) closed path $\vec{P}^{-1}_{z_0u_0}\vec{P}_{z_0u_1}\vec{P}^{-1}_{z_1u_1}\dots\vec{P}_{z_{m-1}u_m}\vec{P}^{-1}_{z_mu_m}\vec{P}_{z_mu_0}$ (where $\vec{P}^{-1}$ denotes the path $\vec{P}$ taken in opposite direction) then contains a cycle, a contradiction since G is a tree. Therefore, there exists a coloring c_1 of U_G using at most d^q colors.
Step 2. Let c_2 be the $\vec{P}_q$-preserving oriented coloring of G defined in the proof of Proposition 6. We claim that the oriented coloring c of G defined by $c(x)=(c_1(x),c_2(x))$ for every vertex x is T-preserving. To see that suppose that u and v are two vertices in G belonging to some connected subgraph G' isomorphic to a subgraph of T. Let r denote the unique vertex of G' with in-degree zero (the *root* of the corresponding subgraph of T). It is not difficult to check that $c_2(u)=c_2(v)$ if and only if u and v have the same "level" in G', that is u and v are linked to r by two directed paths of the same length. But in that case, u and v are joined by two vertex-disjoint directed paths $ru_1\dots u_{\ell-1}u$ and $rv_1\dots v_{\ell-1}v$ in G and thus $c_1(u)\ne c_1(v)$. The coloring c is therefore T-preserving and uses at most $(q+1)\cdot d^q$ colors. This concludes the proof. □

From Theorem 13 we then get:

Theorem 14 *Let T be a directed tree with depth q and U be an undirected graph with $(2q+1)$-acyclic chromatic number at most k. Then every orientation of U with out-degree at most d has T-preserving oriented chromatic number at most $k\cdot[(q+1)\cdot d^q]^{\binom{k-1}{2q}}$.*

Proof. The proof is similar to that of Theorem 7. Let c_0 denote a $(2q+1)$-acyclic coloring of U which uses k colors. Let $\vec{U}$ be any orientation of U with out-degree at most d. Recall that any $2q+1$ colors induce a forest in U. Moreover, every color i belongs to $t=\binom{k-1}{2q}$ subsets of $2q+1$ colors. We assume that for every i those subsets containing i are ordered as $\sigma^i_1,\sigma^i_2,\dots,\sigma^i_t$. We now define a coloring c of $\vec{U}$ has follows: for every $x\in V(\vec{U})$ let $c(x)=(c_0(x),c_1(x),\dots,c_t(x))$ where for every i, $1\le i\le t$, $c_i(x)$

corresponds to the T-preserving coloring of the forest induced by the $(2q+1)$-subset $\sigma_i^{c_0(x)}$ (such a coloring exists by Theorem 13). We claim that the coloring c thus defined is T-preserving. Therefore consider two vertices u and v belonging to some connected subgraph $\vec{U}'$ of $\vec{U}$ isomorphic to a subgraph of T. If u and v get distinct colors, there is nothing to prove. Suppose that u and v get the same color. There exist two vertex-disjoint paths in G of the form $zu_1 \dots u_\alpha u$ and $zv_1 \dots v_\beta v$ with $\alpha < q$ and $\beta < q$ (one of them may be empty if u and v are linked by a directed path). The number of colors used in these two paths is thus at most $2q+1$ and there exists at least one $(2q+1)$-subset such that these two paths are contained in the forest induced by σ. But then the colors assigned to u and v differ on the component associated with σ, a contradiction. Thus c is indeed a T-preserving oriented coloring of $\vec{U}$ which uses at most $k \cdot [(q+1) \cdot d^q]^{\binom{k-1}{2q}}$ colors. □

In case of planar graphs, using Corollary 5 and Theorem 14 we get:

Corollary 15 *Let T be a directed tree with depth q and U be an undirected planar graph with girth $g \geq 10q+6$. Then every orientation of U with out-degree at most d has T-preserving oriented chromatic number at most $2 \cdot (q+1)^{2q+2} \cdot d^{2q^2+q}$.*

We end this section by showing that having bounded out-degree and bounded p-acyclic chromatic number are both necessary conditions in Theorem 14. Recall that S_2^+ denotes the directed tree on three vertices $\{a, b, c\}$ with the arcs ab and ac. As observed in Section 1, having bounded out-degree is necessary (even for trees) since oriented stars with all arcs directed outwards have unbounded S_2^+-preserving oriented chromatic number. Let now H_n be the oriented graph defined as follows: $V(H_n) = \{u_1, u_2, \dots, u_n\} \cup \{v_{ij} \ ; \ 1 \leq i < j \leq n\}$ and $E(H_n) = \{v_{ij}u_i, \ v_{ij}u_j \ ; \ \ 1 \leq i < j \leq n\}$. Clearly H_n has out-degree at most 2 and S_2^+-preserving oriented chromatic number at least n since all the u_i's must be assigned distinct colors. It is not difficult to check that the 3-acyclic oriented chromatic number k of H_n is such that $k(k+1) \geq n$. To see that, suppose on the contrary that there exists a 3-acyclic coloring of H_n using k colors with $k(k+1) < n$. Then there are at least $k+2$ vertices among the u_i's, say $u_{i_1}, u_{i_2}, \dots, u_{i_{k+2}}$, which get the same color α. Moreover, at least two vertices $v_{i_1j_1}$ and $v_{i_1j_2}$ (with $j_1, j_2 \in \{i_2, i_3, \dots, i_{k+2}\}$) get the same color β and thus the 6-cycle $u_{i_1}v_{i_1j_1}u_{j_1}v_{j_1j_2}u_{j_2}v_{i_1j_2}u_{i_1}$ uses at most 3 colors, a contradiction.

5 Discussion

In this paper we were essentially interested in proving the existence of T-preserving homomorphisms for some special oriented graphs T. Thus, most of the bounds we have established can surely be improved by using more accurate arguments.

We end this paper by listing some open questions related to our work.

1. What is the minimum integer function $g(p)$ such that every planar graph with girth at least $g(p)$ has bounded p-acyclic chromatic number ? By Observation 3 we know that $g(p) > 2p-3$ for every p. However, by using techniques inspired from [4], we can prove for instance that every planar graph with girth $g \geq 12$ has 3-acyclic chromatic number at most 5, but we have no general result on this problem yet.

2. Our definition of p-acyclic colorings forces to consider graphs with girth at least $p+1$. An alternate definition of p-acyclic coloring could be the following: let us say that a k-coloring of a graph G is *weakly* p-acyclic if every cycle C in G uses at least $min(|C|, p+1)$ colors (where $|C|$ denotes the length of C). This definition allows to consider p-acyclic colorings of graphs with any girth. Which families of graphs have bounded weak p-acyclic chromatic number ? In particular, what is the minimum integer function $g'(p)$ such that every planar graph with girth at least $g'(p)$ has bounded weak p-acyclic chromatic number ? By Observation 3 we still have $g'(p) > 2p-3$ for every p.

3. We proved in Section 3 that every planar graph with sufficiently large girth has bounded $\vec{P}_n$-preserving oriented chromatic number. What is the minimum integer function $h(n)$ such that every planar graph with girth at least $h(n)$ has bounded $\vec{P}_n$-preserving oriented chromatic number ? Christian Szegedy (Bonn) observed that $h(n) > n$ for every n. To see that, we construct an oriented planar graph $G_{n,p}$ for every $n \geq 3$, $p \geq 1$ with girth n and $\vec{P}_n$-preserving oriented chromatic number strictly greater than p. This graph is obtained as follows: let $y, x_1, x_2, \ldots, x_p$ be $p+1$ distinct vertices. For every i, $1 \leq i \leq p$, we add a directed $\lfloor \frac{n}{2} \rfloor$-path from y to x_i and a directed $\lfloor \frac{n}{2} \rfloor$-path (resp. $(\lfloor \frac{n}{2} \rfloor + 1)$-path) if n is even (resp. odd) from x_i to y. The graph $G_{n,p}$ thus obtained has clearly girth n and since any two vertices in $\{y, x_1, x_2, \ldots, x_p\}$ are joined by a directed path of length at most n we get that $G_{n,p}$ has $\vec{P}_n$-preserving oriented chromatic number at least $p+1$.

References

[1] Albertson M. and D. Berman. An acyclic analogue to Heawood's theorem. *Glasgow Math. J.* **19**, 163–166, 1978.

[2] Alon N., C. McDiarmid and B. Read. Acyclic colorings of graphs. *Random Structures and Algorithms* **2**, 277–289, 1991.

[3] Borodin O.V. On acyclic colorings of planar graphs. *Discrete Math.* **25**, 211–236, 1979.

[4] Borodin O.V., A.V. Kostochka, J. Nešetřil, A. Raspaud and E. Sopena. On the maximum average degree and the oriented chromatic number of a graph. Preprint, 1995.

[5] Dirac G.A. Some theorems on abstract graphs. *Proc. London. Math. Soc.* **2**, 69–81, 1952.

[6] Häggkvist R. and P. Hell. On A-mote universal graphs. *European J. Combin.* **13**, 23–27, 1993.

[7] Hell P. and J. Nešetřil, On the complexity of H-coloring. *J. Combin. Theory Series B* **48**, 92–110, 1990.

[8] Hell P., J. Nešetřil and X. Zhu. Duality theorems and polynomial tree-coloring. *Trans. AMS*, to appear.

[9] Hell P., J. Nešetřil and X. Zhu. Duality of graph homomorphisms. Combinatorics, Paul Erdös is eighty, Vol. 2, *Bolyai Society Mathematical Studies*, 1993.

[10] Jensen T.R. and B. Toft. *Graph coloring problems.* Wiley Interscience, 1995.

[11] Kostochka A.V. and L.S. Mel'nikov. Note to the paper of Grünbaum on acyclic colorings. *Discrete Math.* **14**, 403–406, 1976.

[12] Kostochka A.V., E. Sopena and X. Zhu. Acyclic and oriented chromatic numbers of graphs. Preprint 95-087, Univ. Bielefeld, 1995.

[13] Maurer H.A., A. Salomaa and D. Wood. Colorings and interpretations: a connection between graphs and grammar forms. *Discrete Applied Math.* **3**, 119-135, 1981.

[14] Nash-Williams C.St.J.A. Decomposition of finite graphs into forests. *J. London Math. Soc.* **39**, 12, 1964.

[15] Nešetřil J. Homomorphisms of derivative graphs. *Discrete Math.* **1**-3, 257–268, 1971.

[16] Nešetřil J., A. Raspaud and E. Sopena. Colorings and girth of oriented planar graphs. Research Report 1084-95, Univ. Bordeaux I, 1995.

[17] Nešetřil J. and X. Zhu, On bounded treewidth duality of graph homomorphisms. *J. Graph Theory*, to appear.

[18] Raspaud A. and E. Sopena. Good and semi-strong colorings of oriented planar graphs. *Inf. Processing Letters* **51**, 171–174, 1994.

[19] Sopena E. The chromatic number of oriented graphs. Research Report 1083-95, Univ. Bordeaux I, 1995.

[20] Thomas R. *Personal communication*, 1995.