On the oriented chromatic index of oriented graphs

Pascal Ochem, Alexandre Pinlou, Éric Sopena*

LaBRI, Université Bordeaux 1,
351, cours de la Libération
33405 Talence Cedex, France

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Abstract

A homomorphism from an oriented graph G to an oriented graph H is a mapping φ from the set of vertices of G to the set of vertices of H such that $\overrightarrow{\varphi(u)\varphi(v)}$ is an arc in H whenever $\overrightarrow{uv}$ is an arc in G. The oriented chromatic index of an oriented graph G is the minimum number of vertices in an oriented graph H such that there exists a homomorphism from the line digraph $LD(G)$ of G to H (the line digraph $LD(G)$ of G is given by $V(LD(G)) = A(G)$ and $\overrightarrow{ab} \in A(LD(G))$ whenever $a = \overrightarrow{uv}$ and $b = \overrightarrow{vw}$).

We give upper bounds for the oriented chromatic index of graphs with bounded acyclic chromatic number, of planar graphs and of graphs with bounded degree. We also prove that the problem of deciding whether an oriented graph has oriented chromatic index at most k is polynomial time if $k \leq 3$ and is NP-complete if $k \geq 4$.

Keywords: Oriented graph, oriented coloring, arc coloring.

1 Introduction

We consider finite simple *oriented graphs*, that is digraphs with no opposite arcs. For an oriented graph G, we denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs. For two adjacent vertices u and v, we denote by $\overrightarrow{uv}$ the arc from u to v or simply uv whenever its orientation is not relevant (therefore, $uv = \overrightarrow{uv}$ or $uv = \overrightarrow{vu}$).

The notion of oriented vertex-coloring was introduced by Courcelle [6] as follows: an *oriented k-vertex-coloring* of an oriented graph G is a mapping φ from $V(G)$ to a set of

*E-mail: {Pascal.Ochem,Alexandre.Pinlou,Eric.Sopena}@labri.fr

k colors such that (i) $\varphi(u) \neq \varphi(v)$ whenever $\overrightarrow{uv} \in A(G)$ and (ii) $\varphi(v) \neq \varphi(x)$ whenever $\overrightarrow{uv}, \overrightarrow{xy} \in A(G)$ and $\varphi(u) = \varphi(y)$. The *oriented chromatic number* of G, denoted by $\chi_o(G)$, is defined as the smallest k such that G admits an oriented k-vertex-coloring. Observe that conditions (i) and (ii) above insure that two vertices linked by a directed path of length one or two must get distinct colors in any oriented vertex-coloring.

Let G and H be two oriented graphs. A *homomorphism* from G to H is a mapping φ from $V(G)$ to $V(H)$ that preserves the arcs: $\overrightarrow{\varphi(u)\varphi(v)} \in A(H)$ whenever $\overrightarrow{uv} \in A(G)$. An oriented k-vertex-coloring of an oriented graph G can be equivalently defined as a homomorphism φ from G to H, where H is an oriented graph of order k. The existence of such a homomorphism from G to H is denoted by $G \to H$. The vertices of H are called *colors*, and we say that G is H-colorable. The oriented chromatic number of G can then be equivalently defined as the smallest order of an oriented graph H such that $G \to H$.

Oriented vertex-colorings have been studied by several authors in the last decade and the problem of bounding the oriented chromatic number has been investigated for graphs with bounded acyclic chromatic number [14], graphs with bounded maximum average degree [5], graphs with bounded degree [9], graphs with bounded treewidth [15, 16] and graphs subdivisions [17].

One can define *oriented arc-colorings* of oriented graphs in a natural way by saying that, as in the undirected case, an oriented arc-coloring of an oriented graph G is an oriented vertex-coloring of its line digraph $LD(G)$ (recall that $LD(G)$ is given by $V(LD(G)) = A(G)$ and $\overrightarrow{ab} \in A(LD(G))$ whenever $a = \overrightarrow{uv}$ and $b = \overrightarrow{vw}$). We say that an oriented graph G is *H-arc-colorable* if there exists a homomorphism φ from $LD(G)$ to H and φ is then an *H-arc-coloring* or simply an *arc-coloring* of G. Therefore, an oriented arc-coloring φ of G must satisfy (i) $\varphi(\overrightarrow{uv}) \neq \varphi(\overrightarrow{vw})$ whenever $\overrightarrow{uv}$ and $\overrightarrow{vw}$ are two consecutive arcs in G, and (ii) $\varphi(\overrightarrow{vw}) \neq \varphi(\overrightarrow{xy})$ whenever $\overrightarrow{uv}, \overrightarrow{vw}, \overrightarrow{xy}, \overrightarrow{yz} \in A(G)$ with $\varphi(\overrightarrow{uv}) = \varphi(\overrightarrow{yz})$. Note that these two conditions insure that two arcs belonging to a direted path of length two or three must get distinct colors in any oriented arc-coloring. Also note that two incident but non-consecutive arcs (i.e. two arcs incoming into a same vertex or two arcs outgoing from a same vertex) can get the same color since the two corresponding vertices in $LD(G)$ are not adjacent and does not belong to a directed 2-path. The *oriented chromatic index* of G, denoted by $\chi'_o(G)$, is defined as the smallest order of an oriented graph H such that $LD(G) \to H$.

The first easy result concerning oriented arc-coloring relates the oriented chromatic index to the oriented chromatic number:

Theorem 1 *Let G be an oriented graph. Then $\chi'_o(G) \leq \chi_o(G)$.*

To see that, consider an oriented graph G with $\chi_o(G) = k$ and an oriented k-vertex-coloring f of G. The mapping g defined by $g(\overrightarrow{uv}) = f(u)$ for every arc $\overrightarrow{uv} \in A(G)$ is clearly an oriented arc-coloring of G.

Therefore, all upper bounds for the oriented chromatic number are also valid for the oriented chromatic index. In this paper, we provide better upper bounds for the oriented chromatic index of several classes of graphs and consider the complexity of the oriented arc-coloring problem.

This paper is organised as follows. The link between oriented chromatic index and acyclic chromatic number is discussed in Section 2. The oriented chromatic index of planar graphs and of graphs with bounded degree are respectively considered in Sections 3 and 4. Finally, the complexity of determining the oriented chromatic index of a graph is studied in Section 5.

In the rest of the paper, we will use the following notation. A vertex of degree k will be called a k-vertex. If $\overrightarrow{uv}$ is an arc, u is a *predecessor* of v and v is a *successor* of u. A vertex will be called a *source* if it has no predecessors and a *sink* if it has no successors.

For a graph G and a vertex v of $V(G)$, we denote by $G \setminus v$ the graph obtained from G by removing v together with the set of its incident arcs; similarly, for an arc a of $A(G)$, $G \setminus a$ denotes the graph obtained from G by removing a. These two notions are extended to sets of vertices or arcs in a standard way.

Let G be an oriented graph and f be an oriented arc-coloring of G. For a given vertex v of G, we denote by $C_f^+(v)$ and $C_f^-(v)$ the *outgoing color set* of v (i.e. the set of colors of the arcs outgoing from v) and the *incoming color set* of v (i.e. the set of colors of the arcs incoming to v), respectively.

2 Oriented chromatic index and acyclic chromatic number

A proper vertex-coloring is *acyclic* if every subgraph induced by any two color classes is a forest (in other words, the graph has no bichromatic cycle). The *acyclic chromatic number* of a graph G, denoted by $\chi_a(G)$, is the smallest k such that G admits an acyclic k-vertex-coloring.

One of the first problems considered for oriented vertex-colorings was to characterize the families of graphs having bounded oriented chromatic number. It was shown that these families are exactly the ones having bounded acyclic chromatic number [9, 14].

In particular, Raspaud and Sopena [14] proved that every oriented graph whose underlying undirected graph has acyclic chromatic number at most k has oriented chromatic number at most $k \cdot 2^{k-1}$. Recently, Ochem [12] proved that this bound is tight by constructing, for every $k \geq 3$, an oriented graph G such that $\chi_a(G) = k$ and $\chi_o(G) = k \cdot 2^{k-1}$.

By Theorem 1, every oriented graph with acyclic chromatic number k has oriented chromatic index at most $k \cdot 2^{k-1}$. By adapting the proof of the above-mentionned result of Raspaud and Sopena, we get a new bound which is quadratic in terms of the acyclic chromatic number:

Theorem 2 *Every oriented graph whose underlying undirected graph has acyclic chromatic number at most k has oriented chromatic index at most $2k(k-1) - \left\lfloor \frac{k}{2} \right\rfloor$.*

To show that, we need the two following easy lemmas :

Lemma 3 *Let F be an oriented forest. Then F admits a C_3-arc-coloring where C_3 is the directed cycle on three vertices.*

Proof. Since the line digraph of an oriented forest is an oriented forest, the result follows from the fact that every oriented forest has an oriented C_3-vertex-coloring [15]. $\square$

Lemma 4 *Let F be a forest, c be a 2-vertex-coloring of F using i and j $(i < j)$ and $\overrightarrow{F}$ be any orientation of F. There exists an oriented 4-arc-coloring f of $\overrightarrow{F}$ using $\{i, j\} \times \{0, 1\}$ such that for every vertex u, $c(u)$ and the first component of $f(u)$ coincide.*

Proof. The coloring f can easily be obtained from c by using the following rule: if $\overrightarrow{uv}$ and $\overrightarrow{vw}$ are two consecutive arcs in $\overrightarrow{F}$, then let $f(\overrightarrow{uv}) = (c(u), \alpha)$ and $f(\overrightarrow{vw}) = (c(v), \beta)$ with $\alpha = \beta$ if and only if $c(u) < c(v)$. $\square$

Proof of Theorem 2 : Let G be an oriented graph, $V_1, \ldots, V_k$ be the k color classes of $V(G)$ induced by an acyclic k-coloring of G, and $A_1, \ldots, A_k$ be the k arc sets of $A(G)$ defined as $A_i = \{\overrightarrow{uv} \in A(G), u \in V_i\}$. By definition, $F_{i,j} = G[V_i, V_j]$ is a forest for $i, j \in [1, k]$, $i < j$ and there are $\frac{k(k-1)}{2}$ such forests; moreover, consider the $l = \lfloor \frac{k}{2} \rfloor$ forests $F_{1,2}, F_{3,4}, \ldots, F_{2l-1,2l}$ (these forests do not share any vertex). We say that these l forests are of *type 1* while the remaining ones are of *type 2*. Then, we define an oriented arc-coloring $f_{i,j}$ for each forest $F_{i,j}$ as follows. If $F_{i,j}$ is of type 1, let $f_{i,j}$ be any oriented 3-arc-coloring given by Lemma 3. If $F_{i,j}$ is of type 2, let $f_{i,j}$ be any oriented 4-arc-coloring given by Lemma 4.

Recall that each $\overrightarrow{uv} \in A(G)$ belongs to a unique forest $F_{i,j}$, $i, j \in [1, k], i < j$. We now define the following mapping f on $A(G)$:

$$\forall\ \overrightarrow{uv} \in F_{i,j},\ f(\overrightarrow{uv}) = (x, i, j) \qquad \text{where } x = f_{i,j}(\overrightarrow{uv}).$$

We shall prove that f is an oriented arc-coloring of G. We first have to check that any pair of consecutive arcs $\overrightarrow{uv}$ and $\overrightarrow{vw}$ get distinct colors. If $\overrightarrow{uv}$ and $\overrightarrow{vw}$ belong to two distinct forests, say $\overrightarrow{uv} \in F_{i,j}$ and $\overrightarrow{vw} \in F_{j,k}$, then $f(\overrightarrow{uv}) = (x, i, j) \neq (y, j, k) = f(\overrightarrow{vw})$ for any x, y since $i \neq k$. Now if $\overrightarrow{uv}, \overrightarrow{vw} \in F_{i,j}$, then $f(\overrightarrow{uv}) = (f_{i,j}(\overrightarrow{uv}), i, j) \neq (f_{i,j}(\overrightarrow{vw}), i, j) = f(\overrightarrow{vw})$ since $f_{i,j}$ is an oriented arc-coloring. Therefore, if f is not an oriented arc-coloring of G, there are four arcs $\overrightarrow{uv}, \overrightarrow{vw}, \overrightarrow{xy}, \overrightarrow{yz}$ of G with $f(\overrightarrow{uv}) = f(\overrightarrow{yz})$ and $f(\overrightarrow{vw}) = f(\overrightarrow{xy})$. Since any $f_{i,j}$ is an oriented arc-coloring, the arcs $\overrightarrow{uv}, \overrightarrow{vw}, \overrightarrow{xy}, \overrightarrow{yz}$ does not belong to the same forest and then $\overrightarrow{uv}, \overrightarrow{yz} \in F_{i,j}$ and $\overrightarrow{vw}, \overrightarrow{xy} \in F_{j,k}$, $i \neq k$. Since two forests of type 1 does not share any vertex, we assume w.l.o.g. that $F_{i,j}$ is of type 2. Suppose that $f(\overrightarrow{uv}) = f(\overrightarrow{yz}) = ((i, p), i, j)$ (resp. $((j, p), i, j,))$ for some $p \in \{0, 1\}$. This implies that $y \in V_i$ (resp. $v \in V_i$). Then, since $f(\overrightarrow{vw}) = f(\overrightarrow{xy}) = (r, j, k)$ for some r, the vertex y (resp. v) belongs either to V_j or to V_k. This is a contradiction since $i \neq j \neq k \neq i$. It is easy to see that this coloring uses at most $2k(k-1) - \lfloor \frac{k}{2} \rfloor$ colors. That completes the proof. $\square$

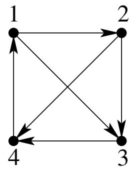

Figure 1: The tournament T_4

3 Planar graphs

A celebrated result of Borodin [2] states that every planar graph has acyclic chromatic number at most five. Thus, from their previously mentionned result, Raspaud and Sopena [14] obtained that every oriented planar graph has oriented chromatic number at most 80, which is the best known upper bound for planar graphs up to now.

Sopena [15] constructed an oriented planar graph with oriented chromatic number 16. More recently, Marshall [10] showed that an oriented planar graph with oriented chromatic number at least 17 exists. The gap between the lower and the upper bound is very large and seems to be very hard to reduce.

Concerning oriented arc-coloring of planar graphs, Theorem 2 and Borodin's result give the following upper bound:

Corollary 5 *Let G be a planar graph. Then $\chi'_o(G) \leq 38$.*

We can easily construct oriented planar graphs with oriented chromatic index 7. The gap between lower and upper bound is thus also large for the oriented chromatic index of planar graphs.

The *girth* of a planar graph is the size of a smallest cycle. Oriented chromatic number of planar graphs with large girth was widely studied [3, 4, 5, 11]. In particular, Nešetřil *et. al* [11] proved that for every $g \geq 3$, there exists a planar graph G with girth g such that $\chi_o(G) \geq 5$.

We now prove that in case of oriented arc-coloring, this bound can be decreased to 4 for some high girth. Let T_4 be the tournament on four vertices depicted in Figure 1. We say that a T_4-arc-coloring f of an oriented graph G is *good* if

1. $\forall\, u \in V(G)$, $C_f^+(u) \in \{\{1\}, \{2\}, \{3\}, \{4\}, \{2,3\}, \{3,4\}\}$,
2. $\forall\, u \in V(G)$, $C_f^-(u) \in \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}\}$.

We first prove the following:

Lemma 6 *Let $P = v_0v_1 \dots v_9v_{10}$ be an oriented 10-path of 2-vertices ($d(v_i) = 2$ for $1 \leq i \leq 9$). Then, any good T_4-arc-coloring φ of $P' = P \setminus \{v_2, \dots, v_8\}$ can be extended to a good T_4-arc-coloring of P.*

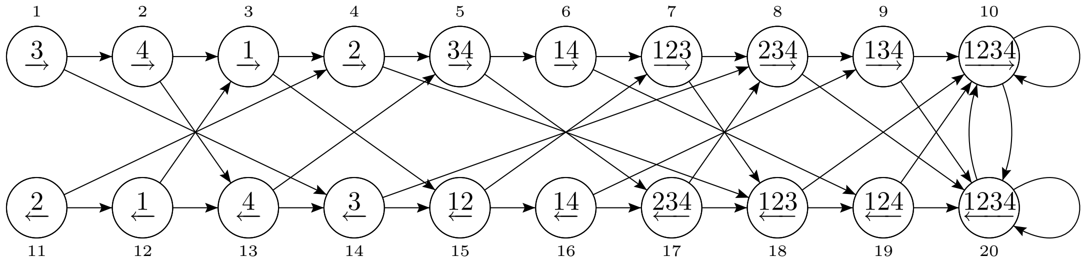

Figure 2: Automaton $\mathcal{A}$ of Lemma 6

Proof. Let f' be a good T_4-arc-coloring of P'. Let $f'(v_9v_{10}) = c$. We uncolor $f'(v_9v_{10})$ and prove in the following that there exist a good T_4-arc-coloring f of P extending f' such that $f(v_9v_{10}) = c$. We shall extend f' to P from v_1v_2 to v_9v_{10} and we shall reason in term of reachable colors (i.e. for each i, we shall consider the set of colors we are able to obtain on v_iv_{i+1}). For instance, if the reachable color set of v_iv_{i+1} is $S = \{c_1, \dots, c_n\}$, that means there exist n good T_4-arc-colorings $f_1, \dots, f_n$ of $P \setminus \{v_{i+1}v_{i+2}, \dots, v_9v_{10}\}$ extending f' such that $f_j(v_iv_{i+1}) = c_j$ for $1 \leq j \leq n$. Therefore, showing that the reachable color set of v_9v_{10} is $\{1, 2, 3, 4\}$ means that there exists a good T_4-arc-coloring f of P with $f(v_9v_{10}) = c$, that will complete the proof.

We associate to each arc of P a *state*, which is its orientation (from left to right if $\overrightarrow{v_iv_{i+1}} \in A(G)$ and from right to left otherwise) and its reachable color set. A case study of T_4 allows us to determine twenty possible states of an arc of P. Let us consider the automaton of Figure 2. Its twenty states are the twenty possible states of an arc of P. The first row stands for the arc of P oriented from left to right while the second one stands for the ones oriented from right to left. Each state of the first row (resp. second row) will be called a *1-state* (resp. a *2-state*). Two transitions leave each state, one towards a 1-state and one towards a 2-state; these transitions are respectively called *1-transition* and *2-transition*. The four first states of each row (i.e. the states $1, 2, 3, 4, 11, 12, 13, 14$) correspond to the eight possible states of v_0v_1 (i.e. one color and one orientation). Then, knowing the state of an arc v_iv_{i+1} and the orientation of the next arc $v_{i+1}v_{i+2}$, we are able to determine the state of $v_{i+1}v_{i+2}$ by following either the 1-transition if $\overrightarrow{v_{i+1}v_{i+2}} \in A(P)$ or the 2-transition if $\overrightarrow{v_{i+2}v_{i+1}} \in A(P)$. Clearly, the state of the v_9v_{10} will be either the state 10 or the state 20, that completes the proof. $\square$

Theorem 7 *Let G be a planar graph with girth $g \geq 46$. Then G admits a good T_4-arc-coloring.*

Proof. Consider a minimal counter-example H to Theorem 7. We prove that H contains neither a 1-vertex nor an oriented 10-path of 2-vertices.

- Suppose that H contains a 1-vertex u, let v be its neighbor and suppose that $\overrightarrow{uv} \in A(H)$. Let $H' \setminus u$. Due to the minimality of H, H' admits a good T_4-arc-coloring f.

Therefore, $C_f^+(v) \in \{\{1\},\{2\},\{3\},\{4\},\{2,3\},\{3,4\}\}$. For each possible case, there clearly exists a predecessor in T_4 we can use to extend f to good T_4-arc-coloring of H. The proof of the case $\overrightarrow{vu} \in A(H)$ is similar.

- Suppose that H contains a 10-path $uv_1v_2\dots v_9w$ of 2-vertices (therefore $d(v_i) = 2$ for all $i \in [1,9]$) and let $H' = H \setminus \{v_2, v_3, \dots, v_8\}$. Due to the minimality of H, H' admits a good T_4-arc-coloring f. Lemma 6 insures that f can be extended to a good T_4-arc-coloring of H.

In [11], the authors proved that every planar graph G of girth $g(G) \geq 5d+1$ contains either a 1-vertex or a $(d+1)$-path of 2-vertices. Therefore, since $g(H) \geq 46$, a counter-example to Theorem 7 does not exist. That completes the proof. □

Observe that for a directed cycle of length $l \not\equiv 0 \pmod 3$, any oriented arc-coloring needs at least 4 colors. Then, for any $g \geq 3$, consider the connected oriented graph G_g obtained by taking two directed cycles sharing one arc, the first one of size g and the second one of size $k \geq g$, $k \not\equiv 0 \pmod 3$. Clearly, the bound of Theorem 7 is tight since G_g has girth g and oriented chromatic index 4.

4 Graphs with bounded degree

Every oriented graph with maximum degree three has oriented chromatic number at most 11 [16]. In [15], Sopena conjectured that the oriented chromatic number of *connected* oriented graphs with maximum degree three is at most 7. In case of oriented arc-coloring, Pinlou [13] recently proved that every oriented graph with maximum degree three has oriented chromatic index at most 7.

For the general case, Kostochka *et al.* [9] proved that every oriented graph with maximum degree Δ has oriented chromatic number at most $2\Delta^2 2^\Delta$ using a probabilistic argument. Therefore, for such a graph G we also have $\chi'_o(G) \leq 2\Delta^2 2^\Delta$. In [1], Alon *et al.* proved that every graph with maximum degree Δ has acyclic chromatic number at most $O(\Delta^{4/3})$. Using Theorem 2, we thus get the better upper bound of $O(\Delta^{8/3})$ for the oriented chromatic index of oriented graphs with maximum degree Δ.

We improve this latter bound and show the following:

Theorem 8 *Let G be an oriented graph with maximum degree Δ. Then, $\chi'_o(G) \leq 2\Delta^2$.*

Proof. Let G^* be the undirected graph defined by $V(G^*) = V(G)$ and $uv \in E(G^*)$ if and only if $\overrightarrow{uv} \in A(G)$ or $\overrightarrow{vu} \in A(G)$ or there exists a vertex t such that $\overrightarrow{ut}, \overrightarrow{tv} \in A(G)$. We clearly have $\Delta(G^*) \leq \Delta + \Delta \times (\Delta - 1) = \Delta^2$. Let p be an optimal proper vertex-coloring of G^* using at most $\Delta^2 + 1$ colors from $\{0, \dots, \Delta^2\}$ by Brook's Theorem.

Let now c be the mapping from $A(G)$ to $\{1, \dots, \Delta^2\} \times \{0,1\}$ defined by $c(\overrightarrow{uv}) = (p(v), 0)$ if $p(u) < p(v)$ and $c(\overrightarrow{uv}) = (p(u), 1)$ if $p(u) > p(v)$.

We will show that c is an arc-coloring of G.

Suppose first that $\overrightarrow{uv}$ and $\overrightarrow{vw}$ are two consecutive arcs of G such that $c(\overrightarrow{uv}) = c(\overrightarrow{vw}) = (\alpha, i)$. If $i = 0$ (resp. $i = 1$) then $p(v) = p(w)$ (resp. $p(u) = p(v)$), a contradiction since $vw \in E(G^*)$ (resp. $uv \in E(G^*)$).

Suppose now that there are four arcs $\overrightarrow{uv}, \overrightarrow{vw}, \overrightarrow{xy}$, and $\overrightarrow{yz}$ of G such that $c(\overrightarrow{uv}) = c(\overrightarrow{yz}) = (\alpha, i)$ and $c(\overrightarrow{vw}) = c(\overrightarrow{xy}) = (\beta, j)$. If $i = j = 0$, we get $\alpha = p(v) = p(z)$ and $p(v) < p(w)$ on one hand, $\beta = p(y) = p(w)$ and $p(w) = p(y) < p(z) = p(v)$ on the other hand, a contradiction. The case $i = j = 1$ leads to a contradiction in a similar way. Assume now that $i \neq j$ and w.l.o.g. that $i = 0$ and $j = 1$. Then we have $\alpha = p(v) = p(z)$ on one hand and $\beta = p(v) = p(x)$ on the other hand. Thus $p(z) = p(x)$, a contradiction since $xz \in E(G^*)$.

Therefore, the mapping c is an arc-coloring of G which uses at most $2\Delta^2$ colors. □

Concerning the lower bound, we are only able to construct oriented graphs with maximum degree Δ and oriented chromatic index $2\Delta - 1$. For that, let $n = 2\Delta - 1$ and consider the oriented bipartite graph $B_{n,n}$ defined by $V(B_{n,n}) = \{x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}\}$, $\overrightarrow{x_iy_i} \in A(B_{n,n})$ for all $0 \leq i < n$ and $\overrightarrow{y_ix_k} \in A(B_{n,n})$ for all $0 \leq i < n$, $1 \leq j < \Delta$ and $k = i+j \pmod n$. Clearly, $B_{n,n}$ is a Δ-regular graph and any pair of arcs of $\{\overrightarrow{x_iy_i}, 0 \leq i < n\}$ belongs to a directed 3-path and thus need distinct colors.

5 NP-completeness

Complexity results for the oriented chromatic number were established recently. Klostermeyer and MacGillivray [8] have shown that given an oriented graph G, deciding whether $\chi_o(G) \leq k$ is polynomial time if $k \leq 3$ and is NP-complete if $k \geq 4$. Culus and Demange [7] extended the above result to the case of bipartite oriented graphs and circuit-free oriented graphs.

In this section, we determine the complexity of deciding whether the oriented chromatic index of a given oriented graph is at most a fixed positive integer. Since the oriented chromatic index of an oriented graph G is the oriented chromatic number of its line digraph $LD(G)$, the result we provide below is then an extension of Klostermeyer and MacGillivray's result to the case of line digraphs.

Theorem 9 *Given an oriented graph G, deciding whether $\chi'_o(G) \leq k$ is polynomial time if $k \leq 3$ and NP-complete if $k \geq 4$.*

Proof. The case $k \leq 3$ directly follows from Klostermeyer and MacGillivray's result since $\chi'_o(G) = \chi_o(LD(G))$ and $LD(G)$ can be constructed from G in polynomial time.

We show that the case $k = 4$ is NP-complete using a reduction from 3-COLORABILITY. We construct the oriented graph G' from an undirected graph G as follows. For every vertex v of G, we put an arc v' in G'. For every edge xy in G, we add a directed 4-path of 2-vertices joining the head of x' to the tail of y', and another 4-path of 2-vertices joining the head of y' to the tail of x'. Hence, G' contains 10-circuits (i.e. a directed cycles on ten vertices) induced by the edges of G: such a 10-circuit induced by the edge xy is denoted by $C_{x,y}$. Thus, any oriented arc-coloring needs at least four colors. Therefore, we have $\chi'_o(G') \leq 4$ if and only if G' has an homomorphism to the tournament T_4 depicted in Figure 1 (T_4 is the only tournament on four vertices containing a 4-circuit). Notice that, for any edge xy of G, the arcs x' and y' are opposite arcs on $C_{x,y}$. We easily check by a case study that any T_4-arc-coloring h of $C_{x,y}$ is such that $h(x') \neq h(y')$ and that every couple of distinct colors can be obtained for $(h(x'), h(y'))$ except $(2, 3)$ and $(3, 2)$. If c is a proper 3-vertex-coloring of G, then G' admits a T_4-arc-coloring h such that $h(v') = 1$ if $c(v) = 1$, $h(v') = 2$ if $c(v) = 2$, and $h(v') = 4$ if $c(v) = 3$. Conversely, if G' admits a T_4-arc-coloring h, then the coloring c of G such that $c(v) = 1$ if $h(v') = 1$, $c(v) = 2$ if $h(v') = 2$ or $h(v') = 3$, and $c(v) = 3$ if $h(v') = 4$, is a proper 3-vertex-coloring.

We now consider the case $k \geq 4$, k even. We consider the problem whether $\chi'_o(G) \leq k$ restricted to oriented graphs G containing neither sources nor sinks. This case is done by induction on k. Notice that the oriented graphs in the proof of the case $k = 4$ contain neither sources nor sinks, so $k = 4$ is our base case. We construct an oriented graph G' without sources nor sinks from an oriented graph G without sources nor sinks, such that $\chi'_o(G') = \chi'_o(G) + 2$. The graph G' is obtained from G by adding three vertices v_1, v_2, v_3, the arcs $\overrightarrow{v_1v_2}$, $\overrightarrow{v_2v_3}$, and the arcs $\overrightarrow{vv_1}$, $\overrightarrow{v_3v}$, for every vertex v of G. Any oriented k-arc-coloring f of G can be extended to an oriented $(k+2)$-coloring of G' as follows. The arcs $\overrightarrow{vv_1}$ (resp. $\overrightarrow{v_3v}$) get the same color as one of $C^+_f(v)$ (resp. $C^-_f(v)$) since $C^+_f(v) \neq \emptyset$ and $C^-_f(v) \neq \emptyset$. The arcs $\overrightarrow{v_1v_2}$ and $\overrightarrow{v_2v_3}$ get additional colors. Conversely, any oriented k-arc-coloring of G' induces an oriented $(k-2)$-arc-coloring of G. To see this, notice that every arc $\overrightarrow{xy}$ of G is contained in the 5-circuit $\overrightarrow{xy}, \overrightarrow{yv_1}, \overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}, \overrightarrow{v_3x}$, which implies that the color of $\overrightarrow{xy}$ is distinct from those of $\overrightarrow{v_1v_2}$ and $\overrightarrow{v_2v_3}$.

We finally consider the case $k \geq 4$, k odd. We construct an oriented graph G' from an oriented graph G without sources nor sinks, such that $\chi'_o(G') = \chi'_o(G) + 1$. The graph G' is obtained from G by adding two vertices v_1, v_2, the arc $\overrightarrow{v_1v_2}$, and the arcs $\overrightarrow{vv_1}$, for every vertex v of G. As above, we check that any oriented k-arc-coloring of G can be extended to an oriented $(k+1)$-coloring of G', any oriented k-arc-coloring of G' induces an oriented $(k-1)$-arc-coloring of G. □

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