

A Study of k -dipath Colourings of Oriented Graphs

*Christopher Duffy*¹, *Gary MacGillivray*² and *Éric Sopena*³

¹ Department of Mathematics and Statistics, Dalhousie University, Canada

² Department of Mathematics and Statistics, University of Victoria, Canada

³ Univ. Bordeaux, LaBRI, UMR5800, F-33400 Talence, France.

CNRS, LaBRI, UMR5800, F-33400 Talence, France

EXTENDED ABSTRACT

Recall that an *oriented graph* is a digraph obtained from a simple, undirected graph by giving each edge one of its two possible orientations. The *directed girth* of an directed graph H is defined to be the minimum length of a directed cycle in H , or infinity if H has no directed cycle. For a pair of vertices $u, v \in V(G)$, the *weak distance from u to v* , denoted $d_{weak}(u, v)$, is the least k such that there is a directed path of length k from u to v or from v to u , or infinity if no such path exists.

Recall, also, that if G and H are oriented graphs, then a *homomorphism of G to H* is a function ϕ from the vertices of G to the vertices of H such that $\phi(x)\phi(y) \in E(H)$ whenever $xy \in E(G)$. If G and H are oriented graphs such that there is a homomorphism ϕ of G to H , then we write $\phi : G \rightarrow H$, or $G \rightarrow H$ if the name of the function ϕ is not important.

Let k and t be a positive integers, and let G be an oriented graph. Chen and Wang [2] defined a *k -dipath t -colouring* of G to be an assignment of t colours to the vertices of G so that any two vertices joined by a directed path of length at most k are assigned different colours. A 1-dipath t -colouring of an oriented graph G is a colouring of the underlying undirected graph of G . Figure 1 gives of a 3-dipath 4-colouring of an oriented graph.

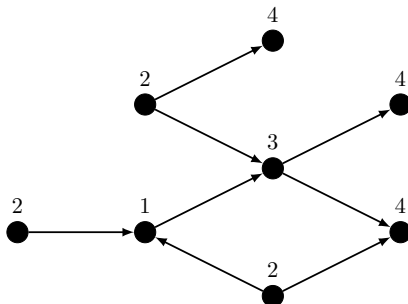


Figure 1: A 3-dipath 4-colouring.

The *k -dipath chromatic number of G* , denoted by $\chi_{kd}(G)$, is the smallest positive integer t such that there exists a k -dipath t -colouring of G . Chen and Wang showed that any orientation of a Halin graph has 2-dipath chromatic number at most 7, and there are infinitely many such graphs G with $\chi_{2d}(G) = 7$ [2].

For a positive integer t , an *oriented t -colouring* of an oriented graph G is a homomorphism of G to some oriented graph on t vertices. Oriented colourings were first introduced in 1994 [3], and have been a topic of considerable interest in the literature since then; see the recent survey by Sopena [8], and also [1], [4] for related topics. Since oriented graphs have no directed cycles of length two, the definition implies that any two vertices of G joined by a directed path of length at most two are assigned different colours (i.e. they have different images) in an oriented colouring of G . The 2-dipath chromatic number is of interest, in part, because it

gives a lower bound for the *oriented chromatic number* $\chi_o(G)$ – the smallest positive integer t such that G admits an oriented t -colouring. It follows from the definition that any oriented colouring of G is a 2-dipath colouring of G ; hence $\chi_{2d}(G) \leq \chi_o(G)$.

In her Master’s thesis [9] (more recently published as [6]), Sherk (née Young) gives a *homomorphism model* for 2-dipath t -colouring. That is, for each positive integer t , she describes an oriented graph \mathcal{G}_t with the property that an oriented graph G has a 2-dipath t -colouring if and only if there is a homomorphism of G to \mathcal{G}_t . As is common with such theorems, it is possible to use the homomorphism to \mathcal{G}_t to find a 2-dipath t -colouring of G . The existence of this model implies an upper bound for the oriented chromatic number as a function of the 2-dipath chromatic number. It also leads to a proof that deciding whether a given oriented graph has a 2-dipath t -colouring is Polynomial if the fixed integer $t \leq 2$, and NP-complete if $t \geq 3$.

A natural question is whether Sherk’s results can be generalized to k -dipath t -colouring. We seek a model similar to hers, where the homomorphism to the target oriented graph can be used to find a k -dipath t -colouring of the given oriented graph. However, we suggest that such model only exists for colouring oriented graphs with no directed cycles of length k or less. Consider the case of 3-dipath t -colouring, where $t \geq 3$. Suppose there exists a digraph $H_{3,t}$ with the property that an oriented graph G has a 3-dipath t -colouring if and only if there is a homomorphism of G to $H_{3,t}$. The digraph $H_{3,t}$ has no loops, otherwise all vertices of G can be assigned the same colour.

By definition, the directed 3-cycle has a 3-dipath 3-colouring: assign each vertex a different colour. Thus, there is a homomorphism of the directed 3-cycle to $H_{3,t}$. Consequently, $H_{3,t}$ has a directed 3-cycle. But, now there is a homomorphism of a directed path of length three to $H_{3,t}$ in which the two end vertices have the same image. Since the ends of a directed path of length three must be assigned different colours in a 3-dipath t -colouring, this model will not have the desired property. Similar considerations apply to k -dipath t -colouring for all pairs of positive integers k and t with $t \geq k$. Hence, a homomorphism model of the type we seek will not exist if the oriented graphs being coloured can have directed cycles of length k or less. This is consistent with Sherk’s results for 2-dipath colouring of oriented graphs as these graphs have no directed cycles of length two or less.

Let G be an oriented graph. Define G^k to be the simple graph formed from G as follows:

- $V(G^k) = V(G)$, and
- $E(G^k) = \{uv \mid 0 < d_{weak}(u, v) \leq k\}$.

Observe that if G is an oriented graph with directed girth at least $k + 1$, then there is a one-to-one correspondence between k -dipath colourings of G and proper colourings of G^k .

The main result of this work is the construction of a homomorphism model for k -dipath t -colouring of oriented graphs with no directed cycles of length k or less. That is, for all positive integers k and t we describe an oriented graph $\mathcal{G}_{k,t}$ with the property that an oriented graph G , with no directed cycle of length at most k , has a k -dipath t -colouring if and only if G admits a homomorphism to $\mathcal{G}_{k,t}$. Further, such a homomorphism is sufficient to construct an explicit k -dipath t -colouring of G . The oriented graph $\mathcal{G}_{k,t}$ is constructed by considering a set of matrices which encode the possible colours of vertices (in a k -dipath t -colouring) in an closed $(k - 1)$ -neighbourhood of a fixed vertex $.$ The set of such matrices is taken as the vertex set of $\mathcal{G}_{k,t}$. Adjacency between a pair of matrices encodes that the respective fixed vertices in those matrices have consistent $(k - 1)$ -neighbourhoods. This construction generalises Sherk’s homomorphism model for 2-dipath colouring using vectors to encode the colours of neighbours of a vertex with a fixed colour.

This homomorphism model for k -dipath t -colouring captures the following facts.

- If $t < k$, then no oriented graph with directed girth at least $k + 1$ and a k -dipath t -colouring can have a directed path of length greater than t . Any such digraph is k -dipath t -colourable.

- If $t \geq k$, then any k -dipath t -colouring of an oriented graph is also a k -dipath $(t + 1)$ -colouring, and there exist oriented graphs with k -dipath chromatic number $t + 1$.
- Every $(k + 1)$ -dipath t -colouring of an oriented graph is a k -dipath t -colouring

In addition to considering homomorphisms and k -dipath coloring, we determine the complexity of deciding the existence of a k -dipath t -colouring for all pairs of fixed positive integers k and t . When instances are restricted to oriented graphs with no directed cycles of length k or less, it is shown that that this problem is NP-complete whenever $t > k \geq 3$, and Polynomial if $t = k$ or $k \leq 2$. When there are no restrictions, it is shown that this problem is NP-complete whenever $k \geq 3$ and $t \geq 3$, and Polynomial whenever $k \leq 2$ or $t \leq 2$. These results generalise the result of Sherk for 2-dipath colouring; however, the here reduction is directly from the t -colouring problem, rather than following from the homomorphism model.

The problem of deciding whether the k -th power of a graph G is t -colourable is known to be NP-complete [7, 5]. Our results imply that if $t > k$, the problem of deciding whether the underlying graph of the k -th power of an oriented graph G (i.e. two vertices are adjacent if and only if they are at weak distance at most k) is t -colourable is NP-complete, even when restricted to powers of oriented graphs with directed girth at least $k + 1$, and also that, if $t = k \geq 3$, the problem of deciding whether the k -th power of an oriented graph G is t -colourable is NP-complete.

References

- [1] O.V. Borodin, A.V. Kostochka, J. Nešetřil, A. Raspaud and É. Sopena, On the Maximum Average Degree and the Oriented Chromatic Number of a Graph. In **Discrete Mathematics**, 206:77–89, 1999.
- [2] M. Chen and W. Wang, The 2–dipath Chromatic Number of Halin Graphs. In **Information Processing Letters**, 99(2):47–53, 2006.
- [3] B. Courcelle, The Monadic Second Order Logic of Graphs VI: On several representations of graphs by relational structures. In **Discrete Applied Mathematics**, 54(2):117–149, 1994.
- [4] M. Hosseini Dolama and É. Sopena, On the Oriented Chromatic number of Halin Graphs. In **Information Processing Letters**, 98(6):247–252, 1998.
- [5] Y. Lin and S. Skiena, Algorithms for Square Roots of Graphs. In **SIAM Journal on Discrete Mathematics**, 8(1):99–118, 1995.
- [6] G. MacGillivray and K. Sherk, A Theory of 2–dipath Colourings. In **Australasian Journal of Combinatorics**, 60(1):11–26, 2014.
- [7] S T. McCormick, Optimal Approximation of Sparse Hessians and its Equivalence to a Graph Coloring Problem. In **Mathematical Programming**, 26(2):153–171, 1983.
- [8] É. Sopena, Homomorphisms and Colourings of Oriented Graphs: An updated survey. In **Discrete Mathematics**, 26(2):1993–2005, 2015.
- [9] K. Young, 2-dipath and Proper 2-dipath Colouring. Master’s Thesis, University of Victoria, 2009.