

Graph minor operations for the marking game

Paul Dorbec ¹, Gabriela Paris ², Éric Sopena ¹

¹ LaBRI-CNRS, Univ. Bordeaux, France

² LIRIS, University of Lyon 1, France

EXTENDED ABSTRACT

The marking game is a 2-player game played on a graph, that was introduced in 1999 by Zhu [3]. The players, called Alice and Bob, alternate turns marking a yet unmarked vertex. When a vertex is marked, we define its score as one plus the number of its marked neighbors. The score of the whole game is the maximum of the scores of the vertices of the graph, independently of who marked the vertex. We say Alice *has a strategy with score k* if Alice has a strategy that ensures all vertices get score at most k . We say Bob has a strategy with score k if Bob has a strategy to ensure at least one vertex gets score more than k .

In the following, we call A -marking game the game where Alice starts, and B -marking game when Bob starts. In order to be able to consider games under progress, we denote $G|M$ the game played on G where the set of vertices M is considered already marked (and we ignore their score). The A -marking number (resp. B -marking number) denoted $col_A(G|M)$ (resp. $col_B(G|M)$) is the minimum k such that Alice has a strategy with score k on the graph $G|M$ in the A -marking game (resp. the B -marking game).

In the following, we study how the A - and B -marking numbers change when canonical operations are applied to the graph. We first prove that the A - and B -marking numbers of a graph differ by at most one. We then give some bounds on the marking game numbers of a graph under three operations: vertex removal, edge removal, and edge contraction.

1 Marking numbers

In this section we study and compare the A -marking number and the B one. The results on this section also involve some work by Elżbieta Sidorowicz [4].

For a graph $G = (V, E)$ and an integer s , we let $A_s(G) = \{v \in V | d(v) \geq s\}$ and $B_s(G) = V \setminus A_s(G)$.

Proposition 1 ([4]) *Let $G = (V, E)$ be a graph, s an integer, and $M \subseteq V$ a set of marked vertices in G . We then have:*

- if $|A_s \setminus M| > |B_s \setminus M|$, $col_A(G|M) > s$
- if $|A_s \setminus M| \geq |B_s \setminus M|$, $col_B(G|M) > s$.

A strategy for Bob to ensure the previous proposition is to play only vertices in $|B_s \setminus M|$. Then the last vertex to be marked is necessarily in A_s and thus the score is at least $s + 1$.

We improve this result by showing that in general, Alice has no advantage playing on B_s if she wants to ensure a score s . This is similar to saying it is never useful for Alice to let Bob play first.

Lemma 2 *Let $G(V, E)$ be a graph and M a set of marked vertices. We have :*

$$col_A(G|M) \leq col_B(G|M) \leq col_A(G|M) + 1.$$

Proof: We first prove that $col_A(G|M) \leq col_B(G|M)$ by using the imagination strategy argument (see [1]). Consider a strategy for Alice to ensure a score at most s in the B -marking game. Alice adapts that same strategy to play the A -marking game. Before her first move, she imagines Bob played on any vertex $x \in B_s \setminus M$, then she plays the vertex y she would have played as answer in the B -marking game. As the game goes on, she plays as if x

was marked, and she follows her strategy step by step to the answers of Bob. If Bob happens to mark the vertex x , then she imagines Bob played any other unplayed vertex $x' \in B_s \setminus M$ and continues as in the imagined game.

Each time a vertex is marked in the A -marking game, it has no more marked neighbors than in the imagined B -marking game, since Alice uses the same strategy. Moreover, the imagined vertex that is played at the end of the game belongs to B_s so it has less than s marked neighbors. Hence the maximum score is at most s . (This statement requires some induction that is omitted here.)

We now prove that $col_B(G|M) \leq col_A(G|M) + 1$. Assume Alice has a strategy in the A -marking game with score s . Playing the B -marking game, Alice uses the same imagination strategy. Bob starts playing some vertex x and Alice plays as if Bob did not play that move. If at some point she is supposed to mark the vertex x marked by Bob, Alice marks any vertex in A_s and imagines she played the vertex x (since that vertex would be played later on, it has no more than $s - 1$ marked neighbors in the imagined game. If there is no vertex left in A_s , there are only vertices of B_s unmarked, and we can immediately conclude the proof). At each step there is one more vertex marked in the A -marking game than in the imagined B -marking game. Thus when a vertex is marked, it has at most s marked neighbors. So the maximum score is at most $s + 1$.

Both bounds are tight. We consider the graphs $K_n \vee S_m$, where K_n denotes the complete graph on n vertices, S_m the edgeless graph on m vertices, and $G \vee H$ the joint of two graphs (see Fig. 1 for examples). For any integers $n > m$, we have that $col_A(K_n \vee S_m) = col_B(K_n \vee S_m) = n + m$. Indeed, playing only the vertices of S_m , Bob enforces that the last vertex of K_n played has all its $n + m - 1$ neighbors marked. This tightens the lower bound. On the other hand, for $n = m$, $col_A(K_n \vee S_m) = n + m - 1$ while $col_B(K_n \vee S_m) = n + m$. Optimal strategies for Alice and Bob are respectively to mark vertices from K_n and from S_m . \square

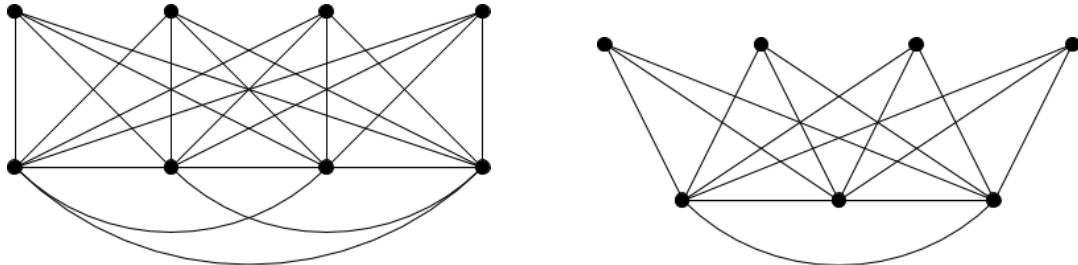


Figure 1: Graphs $K_4 \vee S_4$ and $K_3 \vee S_4$ that show tightness

2 Basic operations

In this section, we describe the possible effect of deleting a vertex, deleting an edge or contracting an edge on the marking number of a graph. We first propose the following result:

Theorem 3 (Vertex deletion) *Let v be a vertex of $V \setminus M$ (and $V \setminus M \neq \{v\}$). Then we have:*

$$\begin{aligned} col_A(G|M) - 2 &\leq col_A(G - \{v\}|M) \leq col_A(G|M), \\ col_B(G|M) - 2 &\leq col_B(G - \{v\}|M) \leq col_B(G|M), \end{aligned}$$

and these bounds are tight.

The proof is omitted here, but it is also based on some imagination argument. We observed earlier that $col_A(K_n \vee S_n) = 2n - 1$ while $col_A(K_{n-1} \vee S_n) = 2n - 3$, tightening the lower bound. For the B -marking game, we have $col_B(K_n \vee S_n) = 2n$ while $col_A(K_{n-1} \vee S_n) = 2n - 2$. Removing a vertex of S_m in $K_n \vee S_m$ where $m \geq n + 2$ does not change any of its marking numbers, tightening the upper bound.

Theorem 4 (Edge deletion) *Let e be an edge of G . Then we have:*

$$\begin{aligned} \text{col}_A(G|M) - 1 &\leq \text{col}_A(G \setminus \{e\}|M) \leq \text{col}_A(G|M), \\ \text{col}_B(G|M) - 1 &\leq \text{col}_B(G \setminus \{e\}|M) \leq \text{col}_B(G|M), \end{aligned}$$

and these bounds are tight.

The tightness is obtained by taking $G = (S_n \vee K_n)$ minus a perfect matching. We have then : $\text{col}_A(G) = 2n - 1$ and $\text{col}_B(G) = 2n - 1$. When an edge in K_n is removed we obtain : $\text{col}_A(G \setminus \{e\}) = \text{col}_B(G \setminus \{e\}) = 2n - 2$.

As a corollary of Theorems 3 and 4, we get:

Theorem 5 *For H a subgraph of G , we have:*

$$\begin{aligned} \text{col}_A(H) &\leq \text{col}_A(G), \\ \text{col}_B(H) &\leq \text{col}_B(G). \end{aligned}$$

This is a nice behavior of the marking game for subgraphs. We could expect that the parameter behave less nicely with the edge contraction. Indeed, we prove the following result.

Theorem 6 (Edge contraction) *Denote by G/e the graph G where the edge e has been contracted. Then we have:*

$$\text{col}_A(G|M) - 2 \leq \text{col}_A(G/e|M) \leq \text{col}_A(G|M) + 2.$$

Both bounds are tight, as show the following examples. First consider the family of graphs G_n obtained as follows. Remove from $K_n \vee S_n$ a perfect matching. Duplicate one of the vertices of K_n , changing a vertex w into two adjacent twins u and v . Then add a vertex adjacent to all the vertices of K_n (including u and v). We can check that $\text{col}_A(G_n) = 2n + 1$, but if we contract the edge uv , then $\text{col}_A(G_n/uv) = 2n - 1$. This tighten the lower bound. Similarly, for G obtained from $C_n \vee S_n$ minus a matching, we have $\text{col}_A(G) = n + 2$ but if we split a vertex of the cycle in two (distributing evenly the neighbors), we get a A-marking number of n .

References

- [1] B. Brešar, S. Klavžar, D. F. Rall, Domination game and an imagination strategy, SIAM J. Discrete Math. 24 (2010):979–991.
- [2] U. Faigle, U. Kern, H. Kierstead, W.T. Trotter, On the Game Chromatic Number of some Classes of Graphs, Ars Combin. 35 (1993) (17):143–150
- [3] X. Zhu. The Game Coloring Number of Planar Graphs, J. Combin. Theory, Series B 75 (1999) (2):245–258
- [4] E. Sidorowicz, private communication.