

On the list incidence chromatic number of graphs

*Brahim BENMEDJDOUB*¹, *Éric SOPENA*^{2,3} and *Isma BOUCHEMAKH*¹

¹ L'IFORCE, University of Sciences and Technology Houari Boumediene (USTHB)

² Univ. Bordeaux, LaBRI, UMR5800, F-33400 Talence, France

³ CNRS, LaBRI, UMR5800, F-33400 Talence, France

EXTENDED ABSTRACT

In this paper, we introduce and study the list version of incidence colourings. All the graphs we consider in this paper are simple and loopless undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G , respectively, and by $\Delta(G)$ the maximum degree of G .

Recall that a graph G is *k-list colourable* (or *k-choosable*) if, whenever each vertex v of G is given a list $L(v)$ of k colours, G admits a proper colouring in which each vertex receives a colour from its own list. The *list chromatic number* of G is then defined as the smallest integer k such that G is k -choosable. This type of colouring was independently introduced by Vizing [9] and Erdős, Rubin and Taylor [3] (see the surveys by Alon [1] and Tuza [8]).

An *incidence* of a graph G is a pair (v, e) where v is a vertex of G and e is an edge of G incident with v . We denote by $Inc(G)$ the set of incidences of G . Two incidences (v, e) and (w, f) of $Inc(G)$ are *adjacent* whenever (i) $v = w$, or (ii) $e = f$, or (iii) $vw = e$ or f . An *incidence k-colouring* of G is a mapping from $Inc(G)$ to the set of colours $\{1, \dots, k\}$ such that every two adjacent incidences receive distinct colours. The smallest k for which G admits an incidence k -colouring is the *incidence chromatic number* of G , denoted $\chi_i(G)$. Incidence colourings were first introduced and studied by Brualdi and Massey [2]. This problem has attracted much interest in recent years, see for instance [6, 5, 10, 11].

The list version of incidence colouring is defined in a way similar to the case of ordinary proper vertex colourings. We thus say that a graph G is *k-list incidence colourable* if, whenever each incidence (v, e) of G is given a list $L(v, e)$ of k colours, G admits an incidence colouring in which each incidence receives a colour from its own list. The *list incidence chromatic number* of G , denoted $\chi_i^\ell(G)$, is then the smallest integer k such that G is k -list incidence colourable.

We clearly have $\chi_i^\ell(G) \geq \chi_i(G)$ and $\chi_i(G) \geq \Delta(G) + 1$ for every graph G . Moreover, since every incidence (v, e) in a graph G has at most $3\Delta(G) - 2$ adjacent incidences, we get:

Proposition 1 *For any graph G , $\Delta(G) + 1 \leq \chi_i(G) \leq \chi_i^\ell(G) \leq 3\Delta(G) - 1$.*

Observe also that an incidence k -colouring of a graph G can be viewed as a proper colouring of the square of its line graph. Since the line graph of an n -cycle is an n -cycle, by a result of Erdős, Rubin and Taylor [3], we get:

Proposition 2 *For every cycle G , $\chi_i^\ell(G) = \chi_i(G)$.*

Recall that a graph G is *d-degenerated* if every subgraph of G contains a vertex of degree at most d . In [5], Hosseini Dolama and Sopena proved the following:

Theorem 3 (Hosseini Dolama and Sopena [5]) *For every d -degenerated graph G , $\chi_i(G) \leq \Delta(G) + 2d - 1$.*

We prove that the same result holds for the list incidence chromatic number:

Theorem 4 *For every d -degenerated graph G , $\chi_i^\ell(G) \leq \Delta(G) + 2d - 1$.*

Since every tree is 1-degenerated, every K_4 -minor free graph (and thus every outerplanar graph) is 2-degenerated, and every planar graph is 5-degenerated, Theorem 4 gives the following:

Corollary 5 For every graph G ,

1. if G is a tree then $\chi_i^\ell(G) = \Delta(G) + 1$,
2. if G is a K_4 -minor free graphs (and thus, if G is an outerplanar graph) then $\chi_i^\ell(G) \leq \Delta(G) + 3$,
3. if G is a planar graph then $\chi_i^\ell(G) \leq \Delta(G) + 9$.

The square grid $G(m, n)$ is the graph defined as the Cartesian product of two paths, that is, $G(m, n) = P_m \square P_n$, where P_n denotes the path on n vertices. Since every square grid is 2-degenerated, we get $\chi_i^\ell(G(m, n)) \leq \Delta(G(m, n)) + 3 \leq 7$ for every square grid $G(m, n)$ by Theorem 4. We can decrease this bound to 6:

Theorem 6 For every integers $m, n \geq 1$, $\chi_i^\ell(G(m, n)) \leq 6$.

A Halin graph is a planar graph obtained from a tree with no vertex of degree 2 by adding a cycle connecting all its leaves. Wang, Chen and Pang proved that $\chi_i(G) = \Delta(G) + 1$ for every Halin graph G with $\Delta(G) \geq 5$ [10] and Meng, Guo and Su that $\chi_i(G) \leq \Delta(G) + 2$ for every Halin graph G with $\Delta(G) = 4$ [7] (it is known, by a result of Maydanskiy [6], that if G is subcubic, then $\chi_i(G) \leq \Delta(G) + 2$).

For this class of graph, we prove the following results:

Lemma 7 If G is a Halin graph then $\chi_i^\ell(G) \leq \max\{\Delta(G) + 1, 7\}$.

Lemma 8 If G is a Halin graph then $\chi_i^\ell(G) \leq \max\{\Delta(G) + 2, 6\}$.

By Lemmas 7 and 8, we thus get:

Theorem 9 If G is a Halin graph then

$$\begin{cases} \chi_i^\ell(G) \leq 6, & \text{if } \Delta(G) \in \{3, 4\}, \\ \chi_i^\ell(G) \leq 7, & \text{if } \Delta(G) = 5, \\ \chi_i^\ell(G) = \Delta(G) + 1, & \text{otherwise.} \end{cases}$$

A cactus is a (planar) graph such that every vertex belongs to at most one cycle. We prove the following:

Theorem 10 If G is a cactus then $\chi_i^\ell(G) \leq \max\{\Delta(G) + 1, 8\}$. Moreover, if $\Delta(G) \in \{3, 4\}$ then $\chi_i^\ell(G) \leq \Delta(G) + 2$.

Apart from improving, if possible, some of the above given bounds, we propose the following open questions:

1. What is the best upper bound on the list incidence chromatic number of subcubic graphs? (By Theorem 4, we know that this bound is at most 8.)
2. What is the value of $\chi_i^\ell(K_n)$? (By Proposition 1, we know that this value is at most $3n - 4$.)
3. Which classes of graphs satisfy the incidence version of the list colouring conjecture, that is, for which graphs G do we have $\chi_i^\ell(G) = \chi_i(G)$? (By Proposition 1 and Theorem 4, we know that this equality holds for every tree.)

References

- [1] N. Alon. Restricted colorings of graphs. In “Surveys in combinatorics”, Proc. 14th British Combinatorial Conference, **London Math. Soc. Lecture Notes Ser.** 187:1–33, 1993.
- [2] R.A. Brualdi and J.Q. Massey. Incidence and strong edge colorings of graphs. In **Discrete Math.** 122:51–58, 1993.
- [3] P. Erdős, A. L. Rubin and H. Taylor. Choosability in graphs. In Proc. West Coast Conference on Combinatorics, Graph Theory and Computing, Humboldt State Univ., Arcata, **Congress. Numer.** 26:125–157, 1979.
- [4] R. Halin. Studies on minimally n -connected graphs. In **Proc. Combinatorial Mathematics and its Applications**, Oxford, Academic Press, London, 1969.
- [5] M. Hosseini Dolama, É. Sopena and X. Zhu. Incidence coloring of k -denegerated graphs. In **Discrete Math.** 283:121–128, 2004.
- [6] M. Maydanskyi. The incidence coloring conjecture for graphs of maximum degree 3. In **Discrete Math.** 292:131–141, 2005.
- [7] X. Meng, J. Guo and B. Su. Incidence coloring of pseudo-Halin graphs. In **Discrete Math.** 312:3276–3282, 2012.
- [8] Z. Tuza. Graph colorings with local constraints – A survey. In **Discuss. Math. Graph Theory** 17:161–228, 1997.
- [9] V. G. Vizing. Coloring the vertices of a graph in prescribed colors. In **Metody Diskret. Anal. Teorii Kodov Shem** 101:3–10 (in Russian), 1976.
- [10] S.-D. Wang, D.-L. Chen and S.-C. Pang. The Incidence colorings number of Halin graphs and outerplanar graphs. In **Discrete Math.** 256(1-2):397–405, 2002.
- [11] J. Wu. Some results on the incidence coloring number of graphs. In **Discrete Math.** 309:3866–3870, 2009.