

Packing coloring of circular ladders and generalised H –graphs

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EXTENDED ABSTRACT

All the graphs we consider are simple and loopless undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G , respectively. The *distance* $d_G(u, v)$ between vertices u and v in G is the length (number of edges) of a shortest path joining u and v . We denote by P_n , $n \geq 1$, the path of order n and by C_n , $n \geq 3$, the cycle of order n .

A *packing k -coloring* of G is a mapping $\pi : V(G) \rightarrow \{1, \dots, k\}$ such that, for every two distinct vertices u and v , $\pi(u) = \pi(v) = i$ implies $d_G(u, v) > i$. The *packing chromatic number* $\chi_\rho(G)$ of G is then the smallest k such that G admits a packing k -coloring. In other words, $\chi_\rho(G)$ is the smallest integer k such that $V(G)$ can be partitioned into k disjoint subsets V_1, \dots, V_k , in such a way that every two vertices in V_i are at distance greater than i in G for every i , $1 \leq i \leq k$. This type of coloring has been introduced by Goddard, Hedetniemi, Hedetniemi, Harris and Rall [6, 7] under the name *broadcast coloring* and has been studied by several authors in recent years (see for instance [1, 2, 3, 4, 5, 9, 10, 11]).

In this work, we determine some exact values, or upper and lower bounds, of the packing chromatic number of some families of graphs, namely circular ladder graphs, H-graphs and generalized H-graphs.

We first recall a few known results. The following proposition, which states that having packing chromatic number at most k is a hereditary property, will be useful in the sequel:

Proposition 1 (Goddard, Hedetniemi, Hedetniemi, Harris and Rall [7]) *If H is a subgraph of G , then $\chi_\rho(H) \leq \chi_\rho(G)$.*

The packing chromatic numbers of paths and cycles have been determined by Goddard *et al.* [7]:

Theorem 2 (Goddard, Hedetniemi, Hedetniemi, Harris and Rall [7])

- $\chi_\rho(P_n) = 2$ if $n \in \{2, 3\}$,
- $\chi_\rho(P_n) = 3$ if $n \geq 4$,
- $\chi_\rho(C_n) = 3$ if $n = 3$ or $n \equiv 0 \pmod{4}$,
- $\chi_\rho(C_n) = 4$ if $n \geq 5$ and $n \equiv 1, 2, 3 \pmod{4}$.

The *corona* $G \odot K_1$ of a graph G is the graph obtained from G by adding a degree-one neighbor to every vertex of G . We call such a degree-one neighbor a *pendant vertex* or a *pendant neighbor*. In [8], we have determined in particular the packing chromatic number of coronae of cycles:

Theorem 3 (Laïche, Bouchemakh, Sopena [8]) *The packing chromatic number of the corona graph $C_n \odot K_1$ is given by:*

$$\chi_\rho(C_n \odot K_1) = \begin{cases} 4 & \text{if } n \in \{3, 4\}, \\ 5 & \text{if } n \geq 5. \end{cases}$$

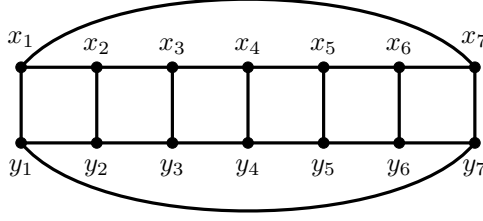


Figure 1: The circular ladder CL_7

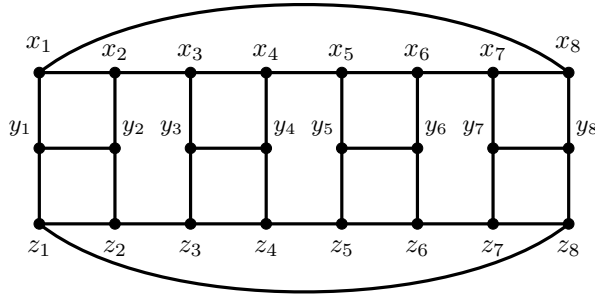


Figure 2: The H-graph $H(4)$

Recall that the Cartesian product $G \square H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, two vertices (u, u') and (v, v') being adjacent if and only if either $u = v$ and $u'v' \in E(H)$ or $u' = v'$ and $uv \in E(G)$.

The *circular ladder* CL_n of length $n \geq 3$ is the Cartesian product $CL_n = C_n \square K_2$. Figure 1 depicts the circular ladder CL_7 . Note that the circular ladder CL_n contains the corona graph $C_n \odot K_1$. Therefore, by Proposition 1, Theorem 3 provides a lower bound for the packing chromatic number of circular ladders.

William and Roy [12] proved that the packing chromatic number of a circular ladder of length $n = 6q$, $q \geq 1$, is at most 5. Our following result determines the packing chromatic number of any circular ladder:

Theorem 4 *For every integer $n \geq 3$,*

$$\chi_\rho(CL_n) = \begin{cases} 5 & \text{if } n \text{ is even and } n \geq 16, \text{ or } n \in \{6, 10, 12\}, \\ 6 & \text{otherwise.} \end{cases}$$

The *H-graph* $H(r)$, is the 3-regular graph of order $3n$, $n = 2r$, with vertex set

$$V(H(r)) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\},$$

and edge set

$$E(H(r)) = \{(x_i, x_{i+1}), (z_i, z_{i+1}) : 1 \leq i \leq n-1\} \cup \{(x_n, x_1), (z_n, z_1)\} \\ \cup \{(y_{2i-1}, y_{2i}) : 1 \leq i \leq r\} \cup \{(x_i, y_i), (y_i, z_i) : 1 \leq i \leq n\}.$$

Figure 2 depicts the H-graph $H(4)$.

William and Roy [12] proved that the packing chromatic number of any H-graph $H(r)$ with $r \geq 4$, r even, is at most 5. Again, we complete their result as follows:

Theorem 5 For every integer $r \geq 2$, $\chi_\rho(H(r)) = 5$ if r is even and $6 \leq \chi_\rho(H(r)) \leq 7$ if n is odd.

We know consider a natural extension of H-graphs. The *generalized H-graph* $H^\ell(r)$ with ℓ levels, $\ell \geq 1$, is the 3-regular graph of order $n(\ell + 2)$, $n = 2r$, with vertex set

$$V(H^\ell(r)) = \{x_j^i : 0 \leq i \leq \ell + 1, 1 \leq j \leq n\}$$

and edge set

$$\begin{aligned} E(H(r)) = & \{(x_j^0, x_{j+1}^0), (x_j^{\ell+1}, x_{j+1}^{\ell+1}) : 1 \leq j \leq n-1\} \cup \{(x_n^0, x_1^0), (x_n^{\ell+1}, x_1^{\ell+1})\} \\ & \cup \{(x_{2j-1}^i, x_{2j}^i) : 1 \leq i \leq \ell, 1 \leq j \leq r\} \\ & \cup \{(x_j^i, x_j^{i+1}) : 0 \leq i \leq \ell, 1 \leq j \leq n\}. \end{aligned}$$

Figure 3 depicts the generalized H-graph with three levels $H^3(4)$.

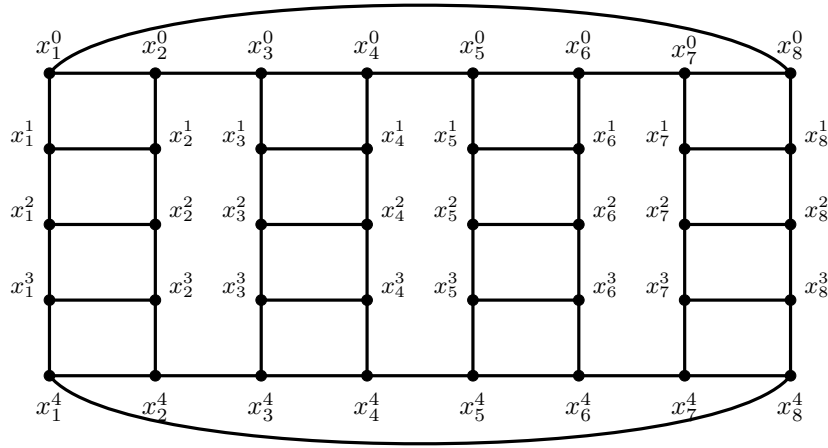


Figure 3: The generalized H-graph $H^3(4)$

Observe that any H-graph is a generalized H-graph with one level. For generalized H-graphs with two levels, we have the following:

Theorem 6 For every integer $r \geq 2$,

$$\chi_\rho(H^2(r)) = \begin{cases} 7 & \text{if } r \in \{2, 4, 7, 8, 11\}, \\ 6 & \text{otherwise.} \end{cases}$$

Finally, for generalized H-graphs with more than two levels, we have the following:

Theorem 7 For every integers $r \geq 2$ and $\ell \geq 3$,

$$\chi_\rho(H^\ell(r)) = \begin{cases} 5 & \text{if } r \text{ is even,} \\ 6 & \text{otherwise.} \end{cases}$$

Finally, we propose the following directions for future research:

1. In Theorem 5, we proved that $6 \leq \chi_\rho(H(r)) \leq 7$, for every odd r . What is the exact value of $\chi_\rho(H(r))$ in this case?
2. It would be interesting to study the packing coloring of some other classes of graphs such as, for instance, cactuses, triangulated graphs, outerplanar or planar graphs...

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