

Strong Rainbow Connection in Digraphs

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EXTENDED ABSTRACT

In an edge-coloured graph G , a path is said to be rainbow if it does not use two edges with the same colour. Then the graph G is said to be rainbow-connected if any two vertices are connected by a rainbow path. This concept of rainbow connection in graphs was introduced by Chartrand et al. in [1]. Since then the rainbow connection number of various graph classes has been determined. Also, different other parameters similar to rainbow connection were introduced, such as strong rainbow connectivity, rainbow k -connectivity, k -rainbow index and rainbow vertex connection. See [7] for a survey about these different parameters.

The notions of rainbow connection and strong rainbow connection readily extend to digraphs, using arc-colouring instead of edge-colourings and directed paths instead of paths. The study of rainbow connection in oriented graphs was initiated by Dorbec et al. in [6] and then studied by Alva-Samos and Montellano-Ballesteros in [2, 3, 4]. Rainbow connectivity in digraphs was considered by Ananth, Nasre and Sarpatwar in [5] from the computational point of view.

A digraph G is *strongly connected* (*strong* for short) if there exists a uv -path in G for every two vertices u and v . The digraph G is *minimally strongly connected* if G is strong and, for every arc xy in G , the digraph $G - xy$ (obtained from G by removing the arc xy) is not strong.

Let G be a digraph. A k -arc-colouring of G , $k \geq 1$, is a mapping $\varphi : A(G) \rightarrow \{1, \dots, k\}$. Note that adjacent arcs may receive the same colour. An *arc-coloured* digraph is then a pair (G, φ) where G is a digraph and φ an arc-colouring of G . A path P in (G, φ) is *rainbow* if no two arcs of P are coloured with the same colour. An arc-coloured digraph (G, φ) is *rainbow connected* (or, equivalently, φ is a *rainbow arc-colouring* of G) if any two vertices in G are connected by a rainbow path.

For any two vertices u, v of G , a *rainbow uv -geodesic* is a rainbow uv -path of length $\text{dist}_G(u, v)$. An arc-coloured digraph (G, φ) is *strongly rainbow connected* (or, equivalently, φ is a *strong rainbow arc-colouring* of G) if there exists a rainbow uv -geodesic for any two vertices u and v in G .

Note that in order to admit a rainbow arc-colouring, or a strong rainbow arc-colouring, a digraph must be strong. The *rainbow connection number* of a strong digraph G , denoted by $\vec{rc}(G)$, is the smallest number k such that G admits a rainbow k -arc-colouring. The *strong rainbow connection number* of a strong digraph G , denoted $s\vec{rc}(G)$, is the smallest number k such that G admits a strong rainbow k -arc-colouring. Observe that for any strong digraph G we have $\vec{rc}(G) \leq s\vec{rc}(G)$.

Another easy observation is that $s\vec{rc}(G) \leq n$ for every digraph G of order n . Let $V(G) = \{x_1, \dots, x_n\}$. We define an n -arc-colouring φ of G by setting $\varphi(x_i x_j) = j$ for every arc $x_i x_j$ in $A(G)$. Obviously, every elementary path in G is rainbow and, therefore, φ is a strong rainbow arc-colouring of G , which gives $s\vec{rc}(G) \leq n$.

Moreover, we clearly have $\text{diam}(G) \leq \vec{rc}(G)$, since any rainbow uv -path such that (u, v) is an antipodal pair of vertices must use at least $\text{diam}(G)$ colours. We thus have the following proposition.

Proposition 1 *If G is a strong digraph of order n with diameter d , then $d \leq \vec{rc}(G) \leq s\vec{rc}(G) \leq n$.*

We prove that the value of $s\vec{r}c(G) - \vec{r}c(G)$ can be arbitrarily large:

Theorem 2 *For every integer k there exists a strong digraph G_k with $s\vec{r}c(G) - \vec{r}c(G) = k$.*

It follows from the definitions that if H is a strong spanning subdigraph of G (that is, $V(H) = V(G)$), then $\vec{r}c(G) \leq \vec{r}c(H)$. However, such an equality does not hold for the strong rainbow connection number of digraphs. Alva-Samos and Montellano-Ballesteros [2] showed that there exists digraphs G and H , such that H is a spanning subdigraph of G and $s\vec{r}c(G) > s\vec{r}c(H)$. The digraph G constructed in [2] contains pairs of opposite arcs. We can prove that this property also holds for oriented graphs:

Proposition 3 *There exist strong oriented graphs G and H , such that H is a spanning subdigraph of G and $s\vec{r}c(G) > s\vec{r}c(H)$.*

The upper bound in Proposition 1 is tight since $\vec{r}c(C_n) = s\vec{r}c(C_n) = n$ for the cycle C_n on n vertices. Moreover, we prove the following theorem.

Theorem 4 *Let G be a minimally strongly connected digraph of order n . Then $s\vec{r}c(G) = n$ if and only if G is a cycle.*

Since $\vec{r}c(G) \leq \vec{r}c(H)$ whenever H is a spanning subdigraph of an oriented graph G , it follows from Theorem 4 that $\vec{r}c(G) \leq n - 1$ for every non-Hamiltonian oriented graph G of order n (this was already observed in [6]). Although the inequality $\vec{r}c(G) \leq \vec{r}c(H)$ does not hold in general for digraphs (see Proposition 3), we can prove a similar result for non-Hamiltonian digraphs, with an additional assumption on the diameter. However, we conjecture that this diameter condition can be dropped.

Theorem 5 *Let G be a strong digraph of order n . If G is non-Hamiltonian and $\text{diam}(G) = n - 1$, then $s\vec{r}c(G) \leq n - 1$.*

Conjecture 6 *Let G be a strong digraph of order n . If G is non-Hamiltonian then $s\vec{r}c(G) \leq n - 1$.*

Furthermore, we study the strong rainbow connection number of strong tournaments. We first prove the following result.

Theorem 7 *Let T be a strong tournament of order $n \geq 4$. Then $3 \leq s\vec{r}c(T) \leq n - 1$.*

Both bounds in Theorem 7 are tight and, moreover, $s\vec{r}c(T)$ can take any value between 3 and $n - 1$, as shown by the following result.

Theorem 8 *For every two integers n and k , $3 \leq k \leq n - 1$, there exists a strong tournament $T_{n,k}$ of order n with $s\vec{r}c(T_{n,k}) = k$.*

In [6], it has been proved that the rainbow connection number of every tournament is bounded by a function of its diameter, namely $\vec{r}c(T) \leq \text{diam}(T) + 2$. However, such a result does not hold for the strong rainbow connection number, as shown by the next theorem.

Theorem 9 *For every two integers d and k , $3 \leq d \leq k$, there exists a strong tournament $F_{d,k}$ with $\text{diam}(F_{d,k}) = d$ and $s\vec{r}c(F_{d,k}) = k$.*

Theorem 9 says that the strong rainbow connection number of a tournament with diameter at least 3 can be arbitrarily large. We believe that this is no longer true for tournaments with diameter 2 and thus propose the following:

Conjecture 10 *There exists a constant t such that for every strong digraph G with $\text{diam}(G) = 2$, $s\vec{r}c(G) \leq t$.*

References

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