

The neighbour-sum-distinguishing edge-colouring game

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EXTENDED ABSTRACT

Let $\gamma : E(G) \rightarrow \mathbb{N}^*$ be an edge colouring of a graph G and $\sigma_\gamma : V(G) \rightarrow \mathbb{N}^*$ the vertex colouring given by $\sigma_\gamma(v) = \sum_{e \ni v} \gamma(e)$ for every $v \in V(G)$. A *neighbour-sum-distinguishing edge-colouring* of G is an edge colouring γ such that for every edge uv in G , $\sigma_\gamma(u) \neq \sigma_\gamma(v)$. The study of neighbour-sum-distinguishing edge-colouring of graphs was initiated by Karoński, Łuczak and Thomason [8]. They conjectured that every graph with no isolated edge admits a neighbour-sum-distinguishing edge-colouring with three colours.

We consider a game version of neighbour-sum-distinguishing edge-colouring. The *neighbour-sum-distinguishing edge-colouring game* on a graph G is a 2-player game where the two players, called Alice and Bob, alternately colour an uncoloured edge of G . Alice wins the game if, when all edges are coloured, the so-obtained edge colouring is a neighbour-sum-distinguishing edge-colouring of G . Therefore, Bob's goal is to produce an edge colouring such that two neighbouring vertices get the same sum, while Alice's goal is to prevent him from doing so. The neighbour-sum-distinguishing edge-colouring game on G with Alice having the first move will be referred to as the *A-game on G* . The neighbour-sum-distinguishing edge-colouring game on G with Bob having the first move will be referred to as the *B-game on G* .

We study the neighbour-sum-distinguishing edge-colouring game on various classes of graphs. In particular, we prove that Bob wins the game on the complete graph K_n , $n \geq 3$, whoever starts the game, except when $n = 4$. In that case, Bob wins the game on K_4 if and only if he starts the game.

General results

A *balanced edge* in a graph G is an edge $uv \in E(G)$ with $\deg_G(u) = \deg_G(v)$.

Lemma 1 *Let G be a graph containing a balanced edge.*

1. *If $|E(G)|$ is even then Bob wins the A-game on G .*
2. *If $|E(G)|$ is odd then Bob wins the B-game on G .*

For a graph G , for every vertex $v \in V(G)$, we denote by $\deg_G^p(v)$ the number of pendant neighbours of v . An *internal vertex* in G is a vertex with $\deg_G(v) > 1$.

Theorem 2 *Let G be a graph such that $\deg_G^p(v) \geq \frac{1}{2}\deg_G(v) + 1$ for every internal vertex $v \in V(G)$.*

- (1) *If $|E(G)|$ is odd then Alice wins the A-game on G .*
- (2) *If $|E(G)|$ is even then Alice wins the B-game on G .*

Theorem 2 allows us to prove that Alice wins the A-game or the B-game on some special trees. A *caterpillar* is a tree T whose set of internal vertices induces a path, called the *central path* of T . We then have:

Corollary 3 (Special caterpillars) *Let T be a caterpillar, with central path $v_1v_2 \dots v_k$, such that $\deg_T(v_1) \geq 4$, $\deg_T(v_k) \geq 4$ and $\deg_T(v_i) \geq 6$ for every i , $2 \leq i \leq k-1$. We then have:*

- (1) *If $|E(G)|$ is odd then Alice wins the A-game on G .*
- (2) *If $|E(G)|$ is even then Alice wins the B-game on G .*

Graphs families

The *double-star* $DS_{m,n}$, $m \geq n \geq 1$, is obtained from the two stars $K_{1,m}$ and $K_{1,n}$ by adding an edge joining their two centers. We prove the following:

Theorem 4 (Double-stars)

1. *For every integer $n \geq 1$, Bob wins the B-game on $DS_{n,n}$.*
2. *For every integer $n \geq 1$, Alice wins the A-game on $DS_{n,n}$.*
3. *For every integer $m > n \geq 1$, Alice wins the A-game on $DS_{m,n}$.*
4. *For every integer $m > n \geq 1$, Alice wins the B-game on $DS_{m,n}$.*

Theorem 5 (Complete graphs)

1. *For every integer $n \geq 3$, Bob wins the A-game on K_n .*
2. *For every integer $n \geq 3$, Bob wins the B-game on K_n if and only if $n \neq 4$.*

Theorem 6 (Complete bipartite graphs)

1. *For every integer $n \geq 2$, Bob wins the A-game on $K_{2,n}$.*
2. *For every integer $n \geq 2$, Alice wins the B-game on $K_{2,n}$.*

We leave as an open problem the question of determining who wins the A-game or the B-game on general complete bipartite graphs $K_{m,n}$, $3 \leq m \leq n$.

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