

4 **2-DISTANCE COLORINGS OF INTEGER DISTANCE**
5 **GRAPHS**

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Abstract

21

A 2-distance k -coloring of a graph G is a mapping from $V(G)$ to the set of colors $\{1, \dots, k\}$ such that every two vertices at distance at most 2 receive distinct colors. The 2-distance chromatic number $\chi_2(G)$ of G is then the smallest k for which G admits a 2-distance k -coloring. For any finite set of positive integers $D = \{d_1, \dots, d_\ell\}$, the integer distance graph $G = G(D)$ is the infinite graph defined by $V(G) = \mathbb{Z}$ and $uv \in E(G)$ if and only if $|v - u| \in D$. We study the 2-distance chromatic number of integer distance graphs for several types of sets D . In each case, we provide exact values or upper bounds on this parameter and characterize those graphs $G(D)$ with $\chi_2(G(D)) = \Delta(G(D)) + 1$.

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1. INTRODUCTION

34 All the graphs we consider in this paper are simple and loopless undirected graphs.
 35 We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G ,
 36 respectively. For any two vertices u and v of G , we denote by $d_G(u, v)$ (or simply
 37 $d(u, v)$ whenever the graph G is clear from the context) the *distance* between u
 38 and v , that is the length of a shortest path joining u and v . We denote by $\Delta(G)$
 39 the maximum degree of G .

40 A (proper) k -*coloring* of a graph G is a mapping from $V(G)$ to the set of
 41 colors $\{1, \dots, k\}$ such that every two adjacent vertices receive distinct colors. The
 42 smallest k for which G admits a k -coloring is the *chromatic number* of G , denoted
 43 $\chi(G)$. A 2 -*distance* k -*coloring* of a graph G is a mapping from $V(G)$ to the set of
 44 colors $\{1, \dots, k\}$ such that every two vertices at distance at most 2 receive distinct
 45 colors. 2-distance colorings are sometimes called $L(1,1)$ -*labelings* (see [5] for a
 46 survey on $L(h, k)$ -labelings) or *square colorings* in the literature. The smallest k
 47 for which G admits a 2-distance k -coloring is the 2 -*distance chromatic number*
 48 of G , denoted $\chi_2(G)$.

49 The *square* G^2 of a graph G is the graph defined by $V(G^2) = V(G)$ and
 50 $uv \in E(G^2)$ if and only if $d_G(u, v) \leq 2$. Clearly, a 2-distance coloring of a
 51 graph G is nothing but a proper coloring of G^2 and, therefore, $\chi_2(G) = \chi(G^2)$
 52 for every graph G .

53 The study of 2-distance colorings was initiated by Kramer and Kramer [8] (see
 54 also their survey on general distance colorings in [9]). The case of planar graphs
 55 has attracted a lot of attention in the literature (see e.g. [1, 2, 3, 4, 6, 10, 14]),
 56 due to the conjecture of Wegner that suggests an upper bound on the 2-distance
 57 chromatic number of planar graphs depending on their maximum degree (see [15]
 58 for more details).

59 In this paper, we study 2-distance colorings of distance graphs. Although
 60 several coloring problems have been considered for distance graphs (see [11] for
 61 a survey), it seems that 2-distance colorings have not been considered yet. We
 62 present in Section 2 a few basic results on the chromatic number and the 2-
 63 distance chromatic number of distance graphs. We then consider specific sets D ,
 64 namely $D = \{1, a\}$, $a \geq 3$ (in Section 3), $D = \{1, a, a + 1\}$, $a \geq 3$ (in Section 4),
 65 and $D = \{1, \dots, m, a\}$, $2 \leq m < a$ (in Section 5). We finally propose some open
 66 problems in Section 6.

67

2. PRELIMINARIES

68 Let $D = \{d_1, \dots, d_\ell\}$ be a finite set of positive integers. The *integer distance*
 69 *graph* (simply called *distance graph* in the following) $G = G(D)$ is the infinite

70 graph defined by $V(G) = \mathbb{Z}$ and $uv \in E(G)$ if and only if $|v - u| \in D$. The
71 following proposition follows immediately.

72 **Proposition 1.** *For every positive integers d_1, \dots, d_ℓ with $\gcd(\{d_1, \dots, d_\ell\}) =$
73 $p > 1$, the distance graph $G(D)$ has p connected components, each of them being
74 isomorphic to the distance graph $G(D')$ with $D' = \{d_1/p, \dots, d_\ell/p\}$.*

75 In this situation, we then have $\chi_2(G(D)) = \chi_2(G(D'))$ so that we can always
76 assume $\gcd(D) = 1$ in the following.

It is easy to observe that the square of the distance graph $G(D)$ is also a distance graph, namely the distance graph $G(D^2)$ where

$$D^2 = D \cup \{d + d' : d, d' \in D\} \cup \{d - d' : d, d' \in D, d > d'\}.$$

77 For instance, for $D = \{1, 2, 5\}$, we get $D^2 = \{1, 2, 3, 4, 5, 6, 7, 10\}$. Note that if D
78 has cardinality ℓ , then D^2 has cardinality at most $\ell(\ell + 1)$.

79 As observed in the previous section, $\chi_2(G) = \chi(G^2)$ for every graph G .
80 Therefore, since $(G(D))^2 = G(D^2)$, determining the 2-distance chromatic number
81 of the distance graph $G(D)$ reduces to determining the chromatic number of the
82 distance graph $G(D^2)$. The problem of determining the chromatic number of
83 distance graphs has been extensively studied in the literature. When $|D| \leq 2$,
84 this question is easily solved, thanks to the following general upper bounds.

85 **Proposition 2.** *For every finite set of positive integers $D = \{d_1, \dots, d_\ell\}$ and
86 every positive integer p such that $d_i \not\equiv 0 \pmod{p}$ for every i , $1 \leq i \leq \ell$,
87 $\chi(G(D)) \leq p$.*

Proof. Let $\lambda : V(G(D)) \rightarrow \{1, \dots, p\}$ be the mapping defined by

$$\lambda(x) = 1 + (x \pmod{p}),$$

88 for every integer $x \in \mathbb{Z}$. Since $d_i \not\equiv 0 \pmod{p}$ for every i , $1 \leq i \leq \ell$, the
89 mapping λ is clearly a proper coloring of $G(D)$. ■

Theorem 3 Walther [13]. *For every finite set of positive integers D ,*

$$\chi(G(D)) \leq |D| + 1.$$

90

91 **Proof.** A $(|D| + 1)$ -coloring of $G(D)$ can easily be produced using the First-Fit
92 greedy algorithm, starting from vertex 0, from left to right and then from right
93 to left, since every vertex has exactly $|D|$ neighbors on its left and $|D|$ neighbors
94 on its right. ■

95 Therefore, when $|D| \leq 2$, $\chi(G(D)) = 2$ if $|D| = 1$ or all elements in D are
 96 odd (since $G(D)$ is then bipartite), and $\chi(G(D)) = 3$ otherwise (since $G(D)$ then
 97 contains cycles of odd length). The case $|D| = 3$ has been settled by Zhu [16].
 98 Whenever $|D| \geq 4$, only partial results have been obtained, namely for sets D
 99 having specific properties.

100 Another useful result is the following.

Theorem 4 (Voigt [12], cited in [7]). *For every finite set of positive integers $D = \{d_1, \dots, d_\ell\}$,*

$$\chi(G(D)) \leq \min_{n \in \mathbb{N}} n(|D_n| + 1),$$

101 where $D_n = \{d_i : n|d_i, 1 \leq i \leq \ell\}$.

102 A coloring λ of a distance graph $G(D)$ is *p-periodic*, for some integer $p \geq 1$,
 103 if $\lambda(x + p) = \lambda(x)$ for every $x \in \mathbb{Z}$. Walther also proved the following:

104 **Theorem 5** (Walther [13]). *For every finite set of positive integers D , if $\chi(G(D)) \leq$
 105 k then $G(D)$ admits a p -periodic k -coloring for some p .*

106 The sequence $\lambda(x) \dots \lambda(x + p - 1)$ of such a p -periodic coloring λ is called the
 107 *pattern* of λ . In particular, the coloring defined in the proof of Proposition 2 was
 108 p -periodic with pattern $12 \dots p$. In the following, we will describe such patterns
 109 using standard notation of Combinatorics on words. For instance, the pattern
 110 121212345 will be denoted $(12)^3 345$.

111 Finally, note that in any 2-distance coloring of a graph G , all vertices in the
 112 closed neighborhood of any vertex must be assigned distinct colors. Therefore,
 113 we have the following:

114 **Observation 6.** *For every graph G , $\chi_2(G) \geq \Delta(G) + 1$.*

115 In particular, this bound is attained by the distance graph $G(D)$ with $D =$
 116 $\{1, \dots, k\}$, $k \geq 2$:

Proposition 7. *For every $k \geq 2$,*

$$\chi_2(G(\{1, \dots, k\})) = 2k + 1 = \Delta(G(\{1, \dots, k\})) + 1.$$

117

118 **Proof.** This directly follows from Theorem 3 since $|\{1, \dots, k\}^2| = 2k$. ■

119

3. THE CASE $D = \{1, a\}$, $a \geq 3$

120 We study in this section the 2-distance chromatic number of distance graphs
 121 $G(D)$ with $D = \{1, a\}$, $a \geq 3$ (note that the case $a = 2$ is already solved by
 122 Proposition 7).

When $D = \{1, a\}$, $a \geq 3$, we have $\Delta(G(D)) = 4$ and

$$D^2 = \{1, 2, a - 1, a, a + 1, 2a\}.$$

123

The following theorem gives the 2-distance chromatic number of any such
 124 graph:

Theorem 8. *For every integer $a \geq 3$,*

$$\chi_2(G(\{1, a\})) = \begin{cases} 5 & \text{if } a \equiv 2 \pmod{5}, \text{ or } a \equiv 3 \pmod{5}, \\ 6 & \text{otherwise.} \end{cases}$$

125

126 **Proof.** Since $\{1, a\}^2 = \{1, 2, a - 1, a, a + 1, 2a\}$, we get $d \not\equiv 0 \pmod{5}$ for ev-
 127 ery $d \in \{1, a\}^2$ if and only if $a \equiv 2 \pmod{5}$ or $a \equiv 3 \pmod{5}$ and thus, by
 128 Proposition 2 and Observation 6, $\chi_2(G(\{1, a\})) = 5$.

Note that for every $x \in \mathbb{Z}$, the set of vertices

$$C(x) = \{x - a, x - 1, x, x + 1, x + a\}$$

129 induces a 5-clique in $G(\{1, a\}^2)$ (see Figure 1). We now claim that every 2-
 130 distance 5-coloring λ of $G(\{1, a\})$ is necessarily 5-periodic, that is $\lambda(x+5) = \lambda(x)$
 131 for every $x \in \mathbb{Z}$. To show that, it suffices to prove that any five consecutive vertices
 132 $x, \dots, x + 4$ must be assigned distinct colors. Assume to the contrary that this is
 133 not the case and, without loss of generality, let $x = 0$. Since vertices 0, 1 and 2
 134 necessarily get distinct colors, we only have two cases to consider:

1. $\lambda(0) = \lambda(3) = 1$, $\lambda(1) = 2$, $\lambda(2) = 3$.

Since $C(1)$ induces a 5-clique in $G(\{1, a\}^2)$ (depicted in bold in Figure 1),
 we have

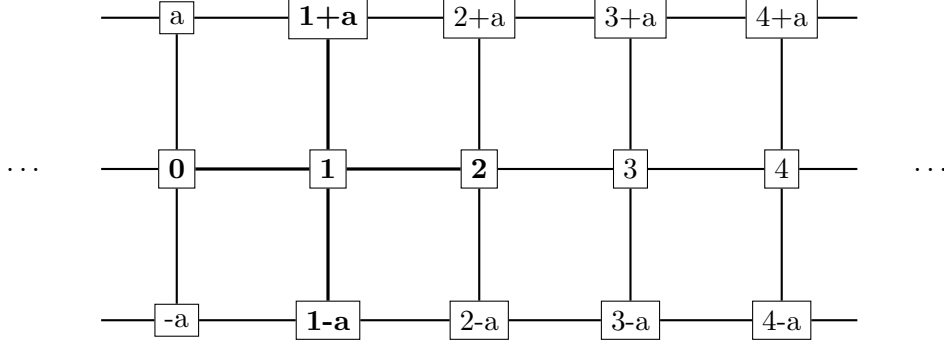
$$\{\lambda(1 - a), \lambda(1 + a)\} = \{4, 5\},$$

which implies

$$\{\lambda(2 - a), \lambda(2 + a)\} = \{4, 5\}.$$

135

(More precisely, $\lambda(2 - a) = 9 - \lambda(1 - a)$ and $\lambda(2 + a) = 9 - \lambda(1 + a)$). This
 136 implies $\lambda(3 - a) = \lambda(3 + a) = 2$, a contradiction since $d(3 - a, 3 + a) = 2$.

Figure 1. Subgraph of the distance graph $G(\{1, a\})$, $a \geq 3$

2. $\lambda(0) = \lambda(4) = 1$, $\lambda(1) = 2$, $\lambda(2) = 3$, $\lambda(3) = 4$.

As in the previous case we have

$$\{\lambda(1-a), \lambda(1+a)\} = \{4, 5\},$$

which implies

$$\{\lambda(2-a), \lambda(2+a)\} = \{1, 5\}.$$

137 We then get $\lambda(3-a) = \lambda(3+a) = 2$, again a contradiction.

138 Therefore, $\chi_2(G(\{1, a\})) = 5$ if and only if 5 does not divide any element of
 139 $\{1, a\}^2 = \{1, 2, a-1, a, a+1, 2a\}$. This is clearly the case if and only if $a \equiv 2$
 140 $(\text{mod } 5)$ or $a \equiv 3 (\text{mod } 5)$.

141 We finally prove that there exists a 2-distance 6-coloring of $G(\{1, a\})$ for any
 142 value of a . We consider three cases, according to the value of $(a \text{ mod } 3)$:

1. $a = 3k$, $k \geq 1$.

Let λ be the $(2a-1)$ -periodic mapping defined by the pattern

$$(123)^k(456)^{k-1}45.$$

If $\lambda(x) = \lambda(y) = c$, $1 \leq c \leq 5$, then

$$|x-y| \in \{3q, 0 \leq q \leq k-1\} \cup \{(2a-1)p+3q, p \geq 1, 1-k \leq q \leq k-1\}.$$

If $\lambda(x) = \lambda(y) = 6$ (which occurs if and only if $k \geq 2$), then

$$|x-y| \in \{3q, 0 \leq q \leq k-2\} \cup \{(2a-1)p+3q, p \geq 1, 2-k \leq q \leq k-2\}.$$

143 Therefore, in both cases, $|x-y| \notin \{1, 2, a-1, a, a+1, 2a\}$, and thus λ is a
 144 2-distance 6-coloring of $G(\{1, a\})$.

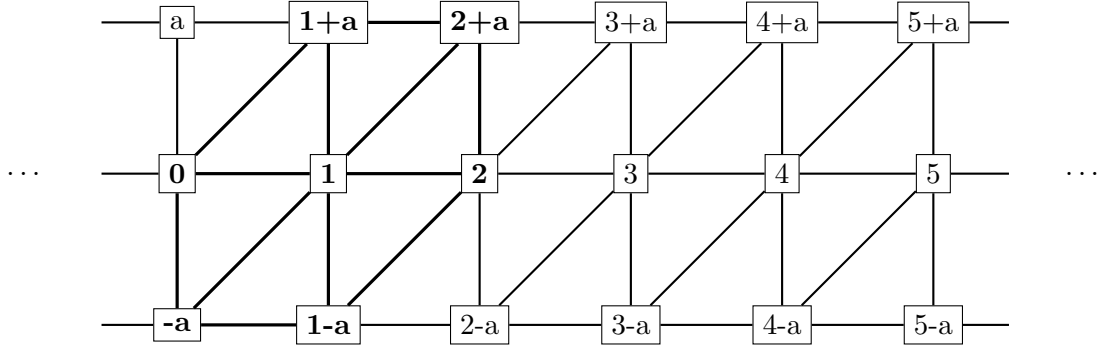


Figure 2. Subgraph of the distance graph $G(\{1, a, a + 1\})$, $a \geq 3$

145 2. $a = 3k + 1$, $k \geq 1$.

146 In that case, the result follows from Theorem 4 (taking $n = 3$), since the
 147 only element divisible by 3 in $\{1, 2, a - 1, a, a + 1, 2a\}$ is $a - 1$.

148 3. $a = 3k + 2$, $k \geq 1$.

149 Again, the result follows from Theorem 4 (taking $n = 3$), since the only
 150 element divisible by 3 in $\{1, 2, a - 1, a, a + 1, 2a\}$ is $a + 1$.

151 This concludes the proof. ■

152 4. THE CASE $D = \{1, a, a + 1\}$, $a \geq 3$

153 We study in this section the 2-distance chromatic number of distance graphs
 154 $G(D)$ with $D = \{1, a, a + 1\}$, $a \geq 3$ (note that the case $a = 2$ is already solved
 155 by Proposition 7).

When $D = \{1, a, a + 1\}$, $a \geq 3$, we have $\Delta(G(D)) = 6$ and

$$D^2 = \{1, 2, a - 1, a, a + 1, a + 2, 2a, 2a + 1, 2a + 2\}.$$

156 We first prove the following:

Theorem 9. *For every integer a , $a \geq 3$,*

$$\chi_2(G(\{1, a, a + 1\})) = 7 = \Delta(G(\{1, a, a + 1\})) + 1$$

157 *if and only if $a \equiv 2 \pmod{7}$ or $a \equiv 4 \pmod{7}$.*

158 **Proof.** Since $\{1, a, a+1\}^2 = \{1, 2, a-1, a, a+1, a+2, 2a, 2a+1, 2a+2\}$, we get
 159 $d \not\equiv 0 \pmod{7}$ for every $d \in \{1, a, a+1\}^2$ if and only if $a \equiv 2 \pmod{7}$ or $a \equiv 4$
 160 $\pmod{7}$ and thus, by Proposition 2 and Observation 6, $\chi_2(G(\{1, a, a+1\})) = 7$.

Note that for every $x \in \mathbb{Z}$, the set of vertices

$$C(x) = \{x - a - 1, x - a, x - 1, x, x + 1, x + a, x + a + 1\}$$

161 induces a 7-clique in $G(\{1, a, a+1\}^2)$. We now claim that every 2-distance 7-
 162 coloring λ of $G(\{1, a, a+1\})$ is necessarily 7-periodic, that is $\lambda(x+7) = \lambda(x)$ for
 163 every $x \in \mathbb{Z}$. To show that, it suffices to prove that any 7 consecutive vertices
 164 $x, \dots, x+6$ must be assigned distinct colors. Assume to the contrary that this is
 165 not the case and, without loss of generality, let $x = 0$. Since vertices 0, 1 and 2
 166 necessarily get distinct colors, we only have four cases to consider (see Figure 2):

- 167 1. Vertices 0, 1, 2, 3 are colored with the colors 1, 2, 3 and 1, respectively.
 168 We consider two subcases:

- (a) $\lambda(4) = 2$.

Since $C(1)$ induces a 7-clique in $G(\{1, a, a+1\}^2)$ (depicted in bold in
 Figure 2), we have

$$\{\lambda(-a), \lambda(1-a), \lambda(1+a), \lambda(2+a)\} = \{4, 5, 6, 7\}.$$

Since $C(3)$ is also a 7-clique, we also have

$$\{\lambda(2-a), \lambda(3-a), \lambda(3+a), \lambda(4+a)\} = \{4, 5, 6, 7\}.$$

169 This implies $\lambda(-a) = \lambda(4-a)$ or $\lambda(1+a) = \lambda(5+a)$. Each of these
 170 cases thus corresponds to case 2 below.

- 171 (b) $\lambda(4) \neq 2$.

172 Note that we necessarily have $\lambda(4) \neq 3$ and $\lambda(4) \neq 1$ since vertex 4
 173 is at distance 2 and 1 from vertices 2 and 3, respectively. We can
 174 thus assume $\lambda(4) = 4$, without loss of generality. Since $d(5, 4) = 1$
 175 and $d(5, 3) = 2$, we have $\lambda(5) \notin \{1, 4\}$. Moreover, if $\lambda(5) = 2$, we get
 176 $\lambda(2) = \lambda(5)$, which corresponds to case 2 below. We can thus suppose
 177 either $\lambda(5) = 3$ or $\lambda(5) > 4$, say $\lambda(5) = 5$ without loss of generality.
 178 We consider these two cases separately.

- i. $\lambda(5) = 3$.

In that case, we necessarily have

$$\{\lambda(-a), \lambda(1+a)\} \subseteq \{4, 5, 6, 7\}, \quad \{\lambda(1-a), \lambda(2+a)\} \subseteq \{4, 5, 6, 7\},$$

$$\{\lambda(2-a), \lambda(3+a)\} \subseteq \{5, 6, 7\}, \quad \{\lambda(3-a), \lambda(4+a)\} \subseteq \{2, 5, 6, 7\},$$

$$\{\lambda(4 - a), \lambda(5 + a)\} \subseteq \{5, 6, 7\}.$$

By setting $\{x, y, z\} = \{5, 6, 7\}$, we get

$$\{\lambda(-a), \lambda(1 + a)\} = \{x, y\}, \quad \{\lambda(1 - a), \lambda(2 + a)\} = \{4, z\},$$

$$\{\lambda(2 - a), \lambda(3 + a)\} = \{x, y\}, \quad \{\lambda(3 - a), \lambda(4 + a)\} = \{2, z\},$$

$$\{\lambda(4 - a), \lambda(5 + a)\} = \{x, y\}.$$

179 Since $\lambda(-a), \lambda(2 - a), \lambda(4 - a) \in \{x, y\}$ and $\lambda(-a) \neq \lambda(2 - a)$,
 180 $\lambda(2 - a) \neq \lambda(4 - a)$, it follows that $\lambda(-a) = \lambda(4 - a)$. That case
 181 corresponds to case 2 below.

ii. $\lambda(5) = 5$.

In that case, we necessarily have

$$\{\lambda(-a), \lambda(1 + a)\} \subseteq \{4, 5, 6, 7\}, \quad \{\lambda(1 - a), \lambda(2 + a)\} \subseteq \{4, 5, 6, 7\},$$

$$\{\lambda(2 - a), \lambda(3 + a)\} \subseteq \{5, 6, 7\}, \quad \{\lambda(3 - a), \lambda(4 + a)\} \subseteq \{2, 6, 7\},$$

$$\{\lambda(4 - a), \lambda(5 + a)\} \subseteq \{3, 6, 7\}.$$

By setting $\{x, y\} = \{6, 7\}$, we get

$$\{\lambda(-a), \lambda(1 + a)\} = \{5, x\}, \quad \{\lambda(1 - a), \lambda(2 + a)\} = \{4, y\},$$

$$\{\lambda(2 - a), \lambda(3 + a)\} = \{5, x\}, \quad \{\lambda(3 - a), \lambda(4 + a)\} = \{2, y\},$$

$$\{\lambda(4 - a), \lambda(5 + a)\} = \{3, x\}.$$

182 We then necessarily have either $\lambda(1 + a) = \lambda(5 + a)$ or $\lambda(-a) =$
 183 $\lambda(4 - a)$ and, in both cases, we are in the situation of case 2 below.

2. Vertices $0, 1, 2, 3, 4$ are colored with the colors $1, 2, 3, 4$ and 1 , respectively. Again considering the 7-cliques $C(1)$, $C(2)$ and $C(3)$ in $G(\{1, a, a + 1\}^2)$, we get

$$\{\lambda(1 - a), \lambda(2 + a)\} \subseteq \{5, 6, 7\},$$

and

$$\{\lambda(2 - a), \lambda(3 + a)\} \subseteq \{5, 6, 7\},$$

184 a contradiction since vertices $1 - a, 2 - a, a + 2$ and $a + 3$ induce a 4-clique
 185 in $G(\{1, a, a + 1\}^2)$.

3. Vertices $0, 1, 2, 3, 4, 5$ are colored with the colors $1, 2, 3, 4, 5$ and 1 , respectively.

Considering the 7-cliques $C(1)$, $C(2)$ and $C(3)$ in $G(\{1, a, a + 1\}^2)$, we get

$$\{\lambda(-a), \lambda(1 - a), \lambda(1 + a), \lambda(2 + a)\} = \{4, 5, 6, 7\},$$

$$\begin{aligned}\{\lambda(2-a), \lambda(3+a)\} &\subseteq \{1, \lambda(-a), \lambda(1+a)\} \setminus \{4, 5\}, \\ \{\lambda(3-a), \lambda(4+a)\} &\subseteq \{2, \lambda(1-a), \lambda(2+a)\} \setminus \{4, 5\},\end{aligned}$$

and thus

$$\{\lambda(2-a), \lambda(3+a)\} \subseteq \{1, 6, 7\} \quad \text{and} \quad \{\lambda(3-a), \lambda(4+a)\} \subseteq \{2, 6, 7\}.$$

186 Assuming that none of cases 1 or 2 occurs, we have two subcases to consider:

(a) $\lambda(6) = 2$.

Considering the 7-clique $C(4)$ in $G(\{1, a, a+1\}^2)$, we get

$$\{\lambda(4-a), \lambda(5+a)\} \subseteq \{3, \lambda(2-a), \lambda(3+a)\} \setminus \{1, 2\} = \{3, 6, 7\}.$$

If $\{\lambda(4-a), \lambda(5+a)\} = \{3, 6\}$, then

$$\begin{aligned}\{\lambda(3-a), \lambda(4+a)\} &= \{2, 7\}, \\ \{\lambda(2-a), \lambda(3+a)\} &= \{1, 6\}, \\ \{\lambda(1-a), \lambda(2+a)\} &= \{5, 7\}\end{aligned}$$

and

$$\{\lambda(-a), \lambda(1+a)\} = \{4, 6\}.$$

187 If $\lambda(-a) = 6$ then $\lambda(2-a) = 1$ and thus $\lambda(4-a) = \lambda(-a) = 6$ which
188 corresponds to case 2. If $\lambda(1+a) = 6$ then $\lambda(3+a) = 1$ and thus
189 $\lambda(5+a) = \lambda(1+a) = 6$ which again corresponds to case 2.

190 The case $\{\lambda(4-a), \lambda(5+a)\} = \{3, 7\}$ is similar and leads to the same
191 conclusion.

192 Finally, if $\{\lambda(4-a), \lambda(5+a)\} = \{6, 7\}$ then $\lambda(3-a) = \lambda(4+a) = 2$,
193 a contradiction since $d(3-a, 4+a) = 2$.

(b) $\lambda(6) = 6$.

Considering the 7-clique $C(4)$ in $G(\{1, a, a+1\}^2)$, we get

$$\{\lambda(4-a), \lambda(5+a)\} \subseteq \{3, \lambda(2-a), \lambda(3+a)\} \setminus \{1, 6\} = \{3, 7\}.$$

This implies

$$\begin{aligned}\{\lambda(3-a), \lambda(4+a)\} &= \{2, 6\}, \\ \{\lambda(2-a), \lambda(3+a)\} &= \{1, 7\}, \\ \{\lambda(1-a), \lambda(2+a)\} &= \{5, 6\}\end{aligned}$$

and

$$\{\lambda(-a), \lambda(1+a)\} = \{4, 7\}.$$

194 If $\lambda(-a) = 7$ then $\lambda(2-a) = 1$ and thus $\lambda(4-a) = \lambda(-a) = 7$ which
195 corresponds to case 2. If $\lambda(1+a) = 7$ then $\lambda(3+a) = 1$ and thus
196 $\lambda(5+a) = \lambda(1+a) = 7$ which again corresponds to case 2.

4. Vertices $0, 1, 2, 3, 4, 5, 6$ are colored with the colors $1, 2, 3, 4, 5, 6$ and 1 , respectively.

Again considering the 7-cliques $C(1)$, $C(2)$ and $C(3)$ in $G(\{1, a, a + 1\}^2)$, we get

$$\begin{aligned}\{\lambda(-a), \lambda(1 - a), \lambda(1 + a), \lambda(2 + a)\} &= \{4, 5, 6, 7\}, \\ \{\lambda(2 - a), \lambda(3 + a)\} &\subseteq \{1, \lambda(-a), \lambda(1 + a)\} \setminus \{4, 5\},\end{aligned}$$

and thus

$$\{\lambda(3 - a), \lambda(4 + a)\} \subseteq \{2, \lambda(1 - a), \lambda(2 + a)\} \setminus \{4, 5, 6\} = \{2, 7\}.$$

This implies

$$\begin{aligned}\{\lambda(2 - a), \lambda(3 + a)\} &= \{1, 6\}, \\ \{\lambda(1 - a), \lambda(2 + a)\} &= \{5, 7\}\end{aligned}$$

and

$$\{\lambda(-a), \lambda(1 + a)\} = \{4, 6\}.$$

Therefore,

$$\{\lambda(4 - a), \lambda(5 + a)\} \subseteq \{3, \lambda(2 - a), \lambda(3 + a)\} \setminus \{1, 6\} = \{3\},$$

197 a contradiction since $d(4 - a, 5 + a) = 2$.

198 Therefore, every 2-distance 7-coloring λ of $G(\{1, a, a + 1\})$ is necessarily 7-
199 periodic, and thus $\chi_2(G(\{1, a, a + 1\})) = 7$ if and only if 7 does not divide any
200 element of $\{1, 2, a - 1, a, a + 1, a + 2, 2a, 2a + 1, 2a + 2\}$. This is clearly the case
201 if and only if $a \equiv 2 \pmod{7}$ or $a \equiv 4 \pmod{7}$. ■

202 The following result provides an upper bound on $\chi_2(G(\{1, a, a + 1\}))$ for any
203 value of a .

Theorem 10. *For every integer a , $a \geq 3$,*

$$\chi_2(G(\{1, a, a + 1\})) \leq 9 = \Delta(G(\{1, a, a + 1\})) + 3.$$

204

205 **Proof.** Recall first that $\{1, a, a + 1\}^2 = \{1, 2, a - 1, a, a + 1, a + 2, 2a, 2a + 1, 2a + 2\}$.
206 We consider three cases, according to the value of $(a \pmod{3})$:

- 207 1. $a = 3k$, $k \geq 1$.

208 Since the only elements divisible by 3 in $\{1, a, a + 1\}^2$ are a and $2a$, the
209 result follows by Theorem 4 (taking $n = 3$).

2. $a = 3k + 1$, $k \geq 1$.

Let λ be the $(3a + 2)$ -periodic mapping defined by the pattern

$$(123)^k(456)^k7123(789)^{k-1}4568.$$

If $\lambda(x) = \lambda(y) = c$, $1 \leq c \leq 6$, then

$$\begin{aligned} |x - y| \in & \{3q, 0 \leq q \leq k - 1\} \\ & \cup \{3q + 2a - 1, 1 - k \leq q \leq 0\} \\ & \cup \{(3a + 2)p + 2a - 1, p > 0\} \\ & \cup \{(3a + 2)p - 2a + 1, p > 0\} \\ & \cup \{(3a + 2)p + 3q, p > 0, 1 - k \leq q < 0\} \\ & \cup \{(3a + 2)p + 3q + 2a - 1, p > 0, 1 - k \leq q < 0\} \\ & \cup \{(3a + 2)p + 3q, p > 0, 0 < q \leq k - 1\} \\ & \cup \{(3a + 2)p + 3q - 2a + 1, p > 0, 0 < q \leq k - 1\}. \end{aligned}$$

If $\lambda(x) = \lambda(y) = 7$ (which occurs if and only if $k \geq 2$), then

$$\begin{aligned} |x - y| \in & \{3q, 0 \leq q \leq k - 2\} \\ & \cup \{3q + 4, 0 \leq q \leq k - 2\} \\ & \cup \{(3a + 2)p + 3q - 4, p > 0, 2 - k \leq q \leq 0\} \\ & \cup \{(3a + 2)p + 3q + 4, p > 0, 0 \leq q \leq k - 2\} \\ & \cup \{(3a + 2)p + 3q, p > 0, 2 - k \leq q \leq k - 2\}. \end{aligned}$$

If $\lambda(x) = \lambda(y) = 8$ (which occurs if and only if $k \geq 2$), then

$$\begin{aligned} |x - y| \in & \{3q, 0 \leq q \leq k - 2\} \\ & \cup \{3q + a - 2, 2 - k \leq q \leq 0\} \\ & \cup \{(3a + 2)p + a - 2, p > 0\} \\ & \cup \{(3a + 2)p - a + 2, p > 0\} \\ & \cup \{(3a + 2)p + 3q, p > 0, 2 - k \leq q < 0\} \\ & \cup \{(3a + 2)p + 3q + a - 2, p > 0, 2 - k \leq q < 0\} \\ & \cup \{(3a + 2)p + 3q, p > 0, 0 < q \leq k - 2\} \\ & \cup \{(3a + 2)p + 3q - a + 2, p > 0, 0 < q \leq k - 2\}. \end{aligned}$$

If $\lambda(x) = \lambda(y) = 9$ (which occurs if and only if $k \geq 2$), then

$$|x - y| \in \{3q, 0 \leq q \leq k - 2\} \cup \{(3a + 2)p + 3q, p \geq 1, 2 - k \leq q \leq k - 2\}.$$

210 Therefore, in all these cases, $|x - y| \notin \{1, 2, a - 1, a, a + 1, a + 2, 2a, 2a +$
 211 $1, 2a + 2\}$, and thus λ is a 2-distance 9-coloring of $G(\{1, a, a + 1\})$.

212 3. $a = 3k + 2$, $k \geq 1$.

213 Since the only elements divisible by 3 in $\{1, a, a + 1\}^2$ are $a + 1$ and $2a + 2$,
 214 the result follows by Theorem 4 (taking $n = 3$).

215 This concludes the proof. ■

216 From Theorems 9 and 10, we thus get:

Corollary 11. *For every integer a , $a \geq 3$, $a \not\equiv 2, 4 \pmod{7}$,*

$$8 \leq \chi_2(G(\{1, a, a+1\})) \leq 9.$$

217

218 5. THE CASE $D = \{1, \dots, m, a\}$, $2 \leq m < a$

219 We study in this section the 2-distance chromatic number of distance graphs
220 $G(D)$ with $D = \{1, \dots, m, a\}$, $2 \leq m < a$ (note that the case $a = m+1$ is
221 already solved by Proposition 7).

When $D = \{1, \dots, m, a\}$, we have $\Delta(G(D)) = 2m+2$ and

$$D^2 = \{1, 2, \dots, 2m\} \cup \{a-m, a-m+1, \dots, a+m\} \cup \{2a\}.$$

222 We first prove the following:

Theorem 12. *For all integers m and a , $2 \leq m < a$,*

$$\chi_2(G(\{1, \dots, m, a\})) = 2m+3 = \Delta(G(\{1, \dots, m, a\})) + 1$$

223 *if and only if $a \equiv m+1 \pmod{2m+3}$ or $a \equiv m+2 \pmod{2m+3}$.*

224 **Proof.** Since $\{1, \dots, m, a\}^2 = \{1, \dots, 2m\} \cup \{a-m, a-m+1, \dots, a+m\} \cup \{2a\}$,
225 $d \not\equiv 0 \pmod{2m+3}$ for every $d \in \{1, \dots, m, a\}^2$ if and only if $a \equiv m+1$
226 $\pmod{2m+3}$ or $a \equiv m+2 \pmod{2m+3}$, and thus, by Proposition 2 and
227 Observation 6, $\chi_2(G(\{1, \dots, m, a\})) = 2m+3$.

228 We now claim that every 2-distance $(2m+3)$ -coloring λ of $G(\{1, \dots, m, a\})$
229 is necessarily $(2m+3)$ -periodic, that is $\lambda(x+2m+3) = \lambda(x)$ for every $x \in \mathbb{Z}$. To
230 show that, it suffices to prove that any $2m+3$ consecutive vertices $x, \dots, x+2m+2$
231 must be assigned distinct colors. Assume to the contrary that this is not the case
232 and, without loss of generality, let $x = 0$. Since vertices $0, 1, \dots, 2m$ necessarily
233 get distinct colors, we only have two cases to consider:

1. Vertices $0, 1, \dots, 2m+1$ are colored with the colors $1, 2, \dots, 2m+1$ and 1 , respectively.

Note that vertices $m-a$ and $m+a$ are both adjacent to all vertices $0, 1, \dots, 2m$. Hence,

$$\{\lambda(m-a), \lambda(m+a)\} = \{2m+2, 2m+3\},$$

which implies

$$\{\lambda(m+1-a), \lambda(m+1+a)\} = \{2m+2, 2m+3\}$$

234 (more precisely, $\lambda(m+1-a) = 4m+5 - \lambda(m-a)$ and $\lambda(m+1+a) =$
 235 $4m+5 - \lambda(m+a)$). This implies $\lambda(m+2-a) = \lambda(m+2+a) = 2$, a
 236 contradiction since $d(m+2-a, m+2+a) = 2$.

2. Vertices $0, 1, \dots, 2m+2$ are colored with the colors $1, 2, \dots, 2m+2$ and 1 , respectively.

As in the previous case we have

$$\{\lambda(m-a), \lambda(m+a)\} = \{2m+2, 2m+3\},$$

which implies

$$\{\lambda(m+1-a), \lambda(m+1+a)\} = \{1, 2m+3\}.$$

237 We thus get $\lambda(m+2-a) = \lambda(m+2+a) = 2$, again a contradiction.

238 Therefore, every 2-distance $(2m+3)$ -coloring λ of $G(\{1, \dots, m, a\})$ is neces-
 239 sarily $(2m+3)$ -periodic, and thus $\chi_2(G(\{1, \dots, m, a\})) = 2m+3$ if and only if
 240 $2m+3$ does not divide any element of $\{1, 2, \dots, 2m\} \cup \{a-m, a-m+1, \dots, a+$
 241 $m\} \cup \{2a\}$. This is clearly the case if and only if $a \equiv m+1 \pmod{2m+3}$ or
 242 $a \equiv m+2 \pmod{2m+3}$. ■

243 For other values of a , we propose the following general upper bound.

Theorem 13. *For all integers m and a , $2 \leq m < a$,*

$$\chi_2(G(\{1, \dots, m, a\})) \leq 4m+2 = 2\Delta(G(\{1, \dots, m, a\})) - 2.$$

244

245 **Proof.** Note first that $\{1, \dots, m, a\}^2 = \{1, \dots, 2m\} \cup \{a-m, \dots, a+m\} \cup \{2a\}$.
 246 Therefore, if $2m+1$ does not divide a , the set $\{1, \dots, m, a\}^2$ contains only one
 247 element e divisible by $2m+1$ (with $e \in \{a-m, \dots, a+m\}$). In that case, the
 248 result then follows by Theorem 4 (taking $n = 2m+1$).

Suppose now that $a = k(2m+1)$, with $k \geq 1$. Let λ be the $(2a-m)$ -periodic mapping defined by the pattern

$$[12 \dots (2m+1)]^k [(2m+2)(2m+3) \dots (4m+2)]^{k-1} (2m+2)(2m+3) \dots (3m+2).$$

If $\lambda(x) = \lambda(y) = c$, $1 \leq c \leq 3m+2$, then

$$\begin{aligned} |x-y| \in & \{q(2m+1), 0 \leq q \leq k-1\} \\ & \cup \{p(2a-m) + q(2m+1), p \geq 1, 1-k \leq q \leq k-1\}. \end{aligned}$$

If $\lambda(x) = \lambda(y) = c$, $3m + 3 \leq c \leq 4m + 2$ (which occurs if and only if $k \geq 2$), then

$$|x - y| \in \{q(2m + 1), 0 \leq q \leq k - 2\} \\ \cup \{p(2a - m) + q(2m + 1), p \geq 1, 2 - k \leq q \leq k - 2\}.$$

249 Therefore, in both cases, $|x - y| \notin \{1, \dots, m, a\}^2$, and thus λ is a 2-distance
250 $(4m + 2)$ -coloring of $G(\{1, \dots, m, a\})$.

251 This concludes the proof. ■

252 From Theorems 12 and 13, we thus get:

Corollary 14. *For all integers m and a , $2 \leq m < a$, $a \not\equiv m+1, m+2 \pmod{2m+3}$,*

$$2m + 4 \leq \chi_2(G(\{1, \dots, m, a\})) \leq 4m + 2.$$

253

254 6. DISCUSSION

255 In this paper, we studied 2-distance colorings of several types of distance graphs.
256 In each case, we characterized those distance graphs that admit an optimal 2-
257 distance coloring, that is distance graphs $G(D)$ with $\chi_2(G(D)) = \Delta(G(D)) + 1$.
258 We also provided general upper bounds for the 2-distance chromatic number of
259 the considered graphs. Note here that all our results can be extended to a larger
260 class of integer distance graphs, thanks to Proposition 1, by multiplying all the
261 elements of the set D by a same constant $k > 1$.

262 We leave as open problems the question of completely determining the 2-
263 distance chromatic number of distance graphs $G(D)$ when $D = \{1, a, a + 1\}$,
264 $a \geq 3$, or $D = \{1, \dots, m, a\}$, $2 \leq m < a$.

265 Considering other types of sets D would certainly be also an interesting
266 direction for future research.

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