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[GTN LXV:1] BOUNDS FOR THE (*m,n*)-MIXED CHROMATIC NUMBER AND THE ORIENTED CHROMATIC NUMBER

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1. Introduction

A bound for the (*n*,*m*)-*mixed chromatic number* in terms of the chromatic number of the square of the underlying undirected graph is obtained. A similar bound holds when the chromatic number of the square is replaced by the *injective chromatic number*. When restricted to $n = 1$ and $m = 0$ (that is, *oriented graphs*) this provides a new bound for the *oriented chromatic number*. In this case, a slightly improved bound is obtained if the chromatic number of the square is replaced the 2-dipath chromatic number (defined in Section 4). In all cases, the method of proof generalizes an argument that has been used to obtain Brooks-type theorems for injective oriented colorings **[1]–[8]**. Similar, although not identical, arguments have appeared in the work of Sopena **[9]**, and Nešetril and Raspaud **[10][11]**.

Colorings of mixed graphs were first studied by Nešetril and Raspaud **[11]**. They gave a bound on the (*n*,*m*)-mixed chromatic number in terms of the *acyclic chromatic number* of the underlying undirected graph. Bounds for the (0, 2)-mixed chromatic number of planar graphs and other graph families have been obtained **[12]**. Some of these have been extended to arbitrary *n* and *m* **[13]**.

2. Definitions

For basic definitions in graph theory, see the text by Bondy and Murty **[14]**.

Definition 2.1: A (*n*,*m*)-*colored mixed graph* is an $(m + n + 1)$-tuple

$$G = (V, A_1, A_2, \ldots, A_n, E_1, E_2, \ldots, E_m),$$

where:

(1) V is a set of vertices;
(2) for $i = 1, 2, \ldots, n$, A_i is a set of ordered pairs of distinct vertices of G called the arcs of color i;
(3) for $j = 1, 2, \ldots, m$, E_j is a set of unordered pairs of distinct vertices of G called the edges of color j; and
(4) the underlying undirected graph $U[G]$, with vertex set $V(U[G]) = V(G)$ and an edge joining x to y for every i for which the ordered pair $xy \in A_i$ and for every j for which the unordered pair $xy \in E_j$, is simple. ■

Observe that a (0,1)-colored mixed graph is a simple graph, and a (1,0)-colored mixed graph is an oriented graph.

Homomorphisms between (*n*,*m*)-colored mixed graphs are defined in the natural way. For (*n*,*m*)-colored mixed graphs G and H, a *homomorphism* of G to H is a function $f: V(G) \to V(H)$ such that for $1 \le i \le n$,

$f(x)f(y) \in A_i(H)$ whenever $xy \in A_i(G)$, and for $1 \le j \le m$, $f(x)f(y) \in E_j(H)$ whenever $xy \in E_j(G)$. For more on homomorphisms, see **[15]**.

Observe that homomorphisms of (n,m)-colored mixed graphs compose. That is, if F, G, and H are (n,m)-colored mixed graphs, f is a homomorphism of F to G, and g is a homomorphism of G to H, then $g \circ f$ is a homomorphism of F to H.

By analogy with the corresponding definitions for other coloring concepts for oriented graphs, a k-coloring of a (n,m)-colored mixed graph is defined to be a homomorphism to a (n,m)-colored mixed graph on k vertices. The smallest k for which there is a k-coloring of a (n,m)-colored graph G is called the (n,m)-*mixed chromatic number* of G and is denoted by $\chi_{(n,m)}(G)$.

Informally, a k-coloring of an (n,m)-colored mixed graph is a partition of the vertices of G into k independent sets (sets containing no edges or arcs of any color), $X_1, X_2, \ldots, X_k$, such that, for any two independent sets X_p and X_q, there is only one type of adjacency; that is, only arcs of the same color and orientation; or only edges of the same color, between vertices in X_p and X_q.

Example 1: Consider a path on four vertices v_1, v_2, v_3, v_4 with arcs v_1v_2 and v_2v_3 of color 1 and an edge v_3v_4 of color 1. Call this (1, 1)-colored mixed graph G. Then $\chi_{(n,m)}(G) = 3$; color v_1 and v_4 with color 1, v_2 with color 2, and v_3 with color 3.

Example 2: Consider a path on five vertices v_1, v_2, v_3, v_4, v_5 with arcs v_1v_2 and v_2v_3 of color 1, an edge v_3v_4 of color 1, and an edge v_4v_5 of color 2. Call this (1,2)-colored mixed graph G. Then $\chi_{(n,m)}(G) = 4$; color v_1 and v_4 with color 1, v_2 with color 2, v_3 with color 3, and v_5 with color 4.

We next define a (n,m)-colored mixed graph, Let $t \ge 3$. For $k = 1, 2, \ldots, t$, let I_k be the set of all sequences of length t in which the k^{th} element is "$\cdot$" and every other entry is a "$+_i$" or "$-_i$" ($1 \le i \le n$), or "$\sim_j$" ($1 \le j \le m$).

Definition 2.2: The graph $H^t_{(n,m)}$ is the (n,m)-*colored mixed graph* with vertex set $V(H^t_{(n,m)}) = I_1 \cup I_2 \cup \ldots \cup I_t$ and
(1) an arc of color i joining sequence $\alpha \in I_k$ to sequence $\beta \in I_l$ if and only if the l^{th} entry of α is $+_i$ and the k^{th} entry of β is $-_i$, and
(2) an edge of color j joining sequence $\alpha \in I_k$ to sequence $\beta \in I_l$ if and only if the l^{th} entry of α and the k^{th} entry of β are both $\sim_j$. ■

In the next section, we bound the (n,m)-mixed chromatic number of $H^t_{(n,m)}$.

3. Main Result

The graph $H^t_{(1,0)}$ is used to prove bounds for injective oriented colorings **[1]**–**[5]**. Young proved that an oriented graph can be properly t-colored so that any two vertices joined by a directed path of length two get different colors if and only if it has an homomorphism to $H^t_{(1,0)}$ **[8]** (Min and Wang **[16]** call this a 2-*dipath k-coloring*.) Our bound on the (n,m)-mixed chromatic number of a (n,m)-mixed colored graph G will be obtained by coloring the square of the underlying graph of G with t colors, and then refining this coloring to obtain a homomorphism of G to $H^t_{(n,m)}$.

Lemma 3.1 (The Refinement Lemma):

$$\chi_{(n,m)}(H^t_{(n,m)}) \le 1 + (2n+m) + (2n+m)^2 + \ldots + (2n+m)^{t-1}$$
$$= \begin{cases} t & \text{if } n = 0 \text{ and } m = 1 \\ \dfrac{(2n+m)^t - 1}{2n+m-1} & \text{otherwise.} \end{cases}$$

Proof: Let $B = 1 + (2n+m) + (2n+m)^2 + \ldots + (2n+m)^{t-1}$. We describe a (n,m)-mixed B-coloring of $H_{(n,m)}$. For $k = 1, 2, \ldots, t$, partition the independent set I_k into $(2n+m)^{k-1}$ subsets $I_{k,l}$, $1 \le l \le (2n+m)^{k-1}$, such that all sequences in $I_{k,l}$ agree in the first k places. The total number of sets $I_{p,q}$ is B.

By definition of these sets, if $p < r$ then:
(1) all arcs joining vertices in $I_{p,q}$ and $I_{r,s}$ are of color i and
  (a) are oriented toward $I_{p,q}$ if the p^{th} symbol of all sequences in $I_{r,s}$ is $+_i$ and the r^{th} symbol in all sequences in $I_{r,s}$ is $-_i$;
  (b) oriented towards $I_{r,s}$ if the p^{th} symbol of all sequences in $I_{r,s}$ is $-_i$ and the r^{th} symbol in all sequences in $I_{r,s}$ is $+_i$;
(2) all edges joining vertices in $I_{p,q}$ and $I_{r,s}$ are of color j if the p^{th} symbol of all sequences in $I_{r,s}$ and the r^{th} symbol of all sequences in $I_{p,q}$ are both $\sim_j$.

Hence, by identifying all vertices in each set $I_{k,l}$, $1 \le k \le t$, $1 \le l \le (2n+m)^{k-1}$, a homomorphism of $H^t_{(n,m)}$ onto the (n,m)-colored mixed graph $O_{(n,m)}$ on B vertices is obtained. Therefore, $\chi_{(n,m)}(H^t_{(n,m)}) \le B$. ■

Lemma 3.1 is dubbed the *Refinement Lemma* because the independent sets I_k are refined to obtain the homomorphism to $O_{(n,m)}$.

Let G be an (n,m)-colored mixed graph. Define $S(G)$ to be square of the underlying undirected graph of G. That is, $S(G)$ has the same vertex set as G and $xy \in E(S(G))$ if and only if $1 \le \text{dist}_G(x,y) \le 2$.

> **Lemma 3.2:** Let G be an (n,m)-colored mixed graph. If $S(G)$ is t-colorable, then there is a homomorphism from G to $H^t_{(n,m)}$.

Proof: Let $C_1, C_2, \ldots, C_t$ be a t-coloring of $S(G)$. Let $x \in C_k$. Define the sets

$$S_{x,i} = \{p : xy \in A_i(G) \text{ for some } y \in C_p\},\ R_{x,i} = \{p : yx \in A_i(G) \text{ for some } y \in C_p\}, \text{ and}$$
$$T_{x,j} = \{p : xy \in E_j(G) \text{ for some } y \in C_p\}.$$

These are, respectively, the sets of (vertex) colors to which x sends an arc of color i in D, from which x receives an arc of color i in D, or to which x is joined by an edge of color j in D. Since C_k is an independent set, $x \notin S_{x,i} \cup R_{x,i'} \cup T_{x,j}$ for any i, i', and j. By construction of $S(G)$, for all i, i', and j the intersection between any two of $S_{x,i}$, $R_{x,i'}$, and $T_{x,j}$ is empty.

Thus, each vertex in C_k can be associated with a sequence with k^{th} entry "$\cdot$" and in which the l^{th} entry is "$+_i$" if $k \in S_{x,i}$, is "$-_i$" if $k \in R_{x,i}$, is "$\sim_j$" if $k \in T_{x,j}$, and is "$\sim_t$" otherwise. This is a homomorphism of G to $H^t_{(n,m)}$. ■

> **Corollary 3.3:** Let G be an (n,m)-colored mixed graph. If $S(G)$ is t-colorable, then
>
> $$\chi_{(n,m)}(G) \le 1 + (2n+m) + (2n+m)^2 + \ldots + (2n+m)^{t-1}$$
> $$= \begin{cases} t & \text{if } n = 0 \text{ and } m = 1 \\ \dfrac{(2n+m)^t - 1}{2n+m-1} & \text{otherwise.} \end{cases}$$
>
> ■

An *injective k-coloring* of a graph G is an assignment of k colors to the vertices of G so that vertices at distance two are assigned different colors. Adjacent vertices may be assigned the same color. The *injective chromatic number* of G is the least k for which there exists an injective k-coloring of G. The injective chromatic number is bounded above by the chromatic number of $S(G)$ (see, for example, **[17]**).

The previous arguments work essentially as given with a small modification to take into account the fact that vertices of the same color may be adjacent. The graph $H^t_{(n,m)}$ is redefined by replacing replace each "$\cdot$" by one of "$+_i$", "$-_i$", or "$\sim_j$". This increases both the number of vertices of $H^t_{(n,m)}$ and the bound on its (n,m)-mixed chromatic number by a factor of $(2n+m)$. Thus, one obtains:

Corollary 3.4: Let G be an (n,m)-colored mixed graph. If G has an injective t-coloring, then

$$\chi_{(n,\,m)}(G) \le (2n+m) + (2n+m)^2 + \ldots + (2n+m)^t$$
$$= \begin{cases} t & \text{if } n = 0 \text{ and } m = 1 \\ (2n+m)\frac{(2n+m)^t - 1}{2n+m-1} & \text{otherwise.} \end{cases}$$

■

4. Oriented Colorings

When $n = 1$ and $m = 0$ we have an *oriented graph*, and the (n,m)-mixed chromatic number is equal to the oriented chromatic number. Hence, the above results imply bounds for the oriented chromatic number. In particular, if $S(G)$ has a t-coloring, then $\chi_o(G) \le 2^t - 1$.

An important property of a coloring of $S(G)$, or an injective t-coloring of G, used in the above arguments is that vertices joined by a directed path of length two must be assigned different colors. The *2-dipath chromatic number* of G is the smallest k for which there is a 2-dipath k-coloring of G. The 2-dipath chromatic number is at most the chromatic number of the square **[16]**.

The argument in Section 3 goes through unchanged for oriented graphs if a k-coloring of $S(G)$ is replaced by a 2-dipath k-coloring of G. This is the same as replacing $S(G)$ by the square of G as a directed graph (two vertices are adjacent if they are joined by a directed path of length at most two) in the argument. Doing so, one obtains the following bound:

Corollary 4.1: Let G be an oriented graph. If G has an 2-dipath t-coloring, then

$$\chi_o(G) = \chi_{(1,\,0)}(G) \le 2^t - 1 .$$ ■

Finally, if the 2-dipath t-coloring need not assign different colors to adjacent vertices, then proceeding as discussed at the end of the previous section one obtains $\chi_o(G) = \chi_{(1,\,0)}(G) \le 2^{t+1} - 1$ (see also **[7]**).

References

[1] R. Campbell; *Oriented Colorings that are Injective on Neighbourhoods*. M.Sc. Thesis, Department of Mathematics and Statistics, University of Victoria, Canada (2009).

[2] N. Clarke, and G. MacGillivray; Injective oriented colourings: In- and out-neighbourhoods separately. Manuscript (2009).

[3] G. MacGillivray, A. Raspaud, and J. Swarts: *Injective Oriented Colourings*. Lecture Notes in Computer Science 5911, Montpellier, France. Proceedings of WG (2009).

[4] G. MacGillivray, A. Raspaud, and J. Swarts; Obstructions to injective oriented colouring. *Electronic Notes in Discrete Math*., **38**, 597–605 (2011).

[5] G. MacGillivray, A. Raspaud, and J. Swarts; Obstructions to locally injective oriented improper colorings. Manuscript (2011).

[6] G. MacGillivray and J. Swarts; The complexity of locally injective homomorphisms. *Discrete Mathematics*, **310**, 2685–2696 (2010).

[7] J. Swarts; *The Complexity of Digraph Homomorphisms: Local Tournaments, Injective Homomorphisms and Polymorphisms*. Ph.D. Thesis, Department of Mathematics and Statistics, University of Victoria, Victoria, BC, Canada (2008).

[8] K.M. Young; *2-Dipath and Proper 2-Dipath Colourings*. M.Sc. Thesis, University of Victoria (2011).

[9] E. Sopena; The chromatic number of oriented graphs. *J. Graph Theory*, **25**, 191–205 (1997).

[10] J. Nešetril and A. Raspaud; Antisymmetric ows and strong colorings of oriented graphs. *Ann. Inst. Fourier, Grenoble,* **49**, 1037–1056 (1999).

[11] J. Nešetril and A. Raspaud; Coloured homomorphisms of colored mixed graphs. *J. Combinatorial Theory, Series B*, **80**, 147–155 (2000).

[12] A. Montejano, P. Ochem, A. Pinlou, A. Raspaud, and E. Sopena; Homomorphisms of 2-edge-colored graphs. *Discrete Appiled Mathematics*, **158**, 1365–1379 (2010).

[13] A. Montejano, A. Pinlou, A. Raspaud, and E. Sopena; Chromatic number of sparse colored planar graphs. *Electronic Notes in Discrete Math*., **34**, 363–367 (2009).

[14] J.A. Bondy and U. S. R.Murty; *Graph Theory*. Springer (2007).

[15] P. Hell and J. Nešetril; *Graphs and Homomorphisms*. Oxford University Press (2004).

[16] C. Min and W. Wang, The 2-dipath chromatic number of Halin graphs. *Information Processing Letters*, **99**, 47–53 (2006).

[17] G. Hahn, J. Kratochovil, J. Siran, and D. Sotteau; On the injective chromatic number of graphs. *Discrete Math*., **256**, 179–192 (2002).

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