

Homomorphisms of signed graphs

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Abstract

A signed graph (G, Σ) is a graph G together with an assignment of signs $+$ and $-$ to all the edges of G where Σ is the set of negative edges. Furthermore (G, Σ_1) and (G, Σ_2) are considered to be equivalent if the symmetric difference of Σ_1 and Σ_2 is an edge cut of G . Naturally arising from matroid theory, several notions of graph theory, such as the theory of minors and the theory of nowhere zero flows, have been already extended to signed graphs. In an unpublished manuscript, B. Guenin introduced the notion of signed graph homomorphisms where he showed how some well-known conjectures can be captured using this notion. A signed graph (G, Σ) is said to map to (H, Σ_1) if there is an equivalent signed graph (G, Σ') of (G, Σ) and a mapping $\varphi : V(G) \rightarrow V(H)$ such that (i) if $xy \in E(G)$ then $\varphi(x)\varphi(y) \in E(H)$ and (ii) $xy \in \Sigma'$ if and only if $\varphi(x)\varphi(y) \in \Sigma_1$. The chromatic number of a signed graph (G, Σ) can then be defined as the smallest order of a homomorphic image of (G, Σ) .

Capturing the notion of graph homomorphism order, signed graph homomorphisms provide room for extensions and strengthenings of most homomorphism and coloring theories on graphs. Thus this paper is the first general study of signed graph homomorphisms. In this work our focus would be on the relation of homomorphisms of signed graphs with minors. After a thorough introduction to the concept we show that the notion of signed graph homomorphism on the set of signed graphs whose underlying graph is bipartite already captures the standard notion of graph homomorphism. We prove that the largest planar signed clique is of order 8. For the maximum chromatic number of planar signed graphs we give the lower bound of 10 and the upper bound of 48. We determine this maximum for some other families such as outerplanar signed graphs. Finally, reformulating Hadwiger's conjecture in the language of homomorphism of signed graphs whose underlying graph is bipartite, we show that while some stronger form of the conjecture holds for small chromatic number, such strengthening of the conjecture would not hold for large chromatic numbers. This could be regarded as a first indication that perhaps Hadwiger's conjecture only holds for small chromatic numbers.

keywords. Signed graph; Coloring; Homomorphism; Minor; Hadwiger's conjecture.

1 Motivation

The Four-Color Theorem has been one of the most motivational problems in developing the theory of graphs and it continues to do so especially because none of its known proofs is verified without the aid of a computer. It simply states that every map, or equivalently every simple planar graph, can be colored properly using at most four colors. Thus to understand it better, one must understand what makes a graph planar and what are the obstacles in coloring a given graph with a few number of colors. The former has given birth to the theory of graph minors. The latter has been developed to the theory of graph coloring and graph homomorphisms. Two examples of central theorems in the theory of graph coloring are Brook's Theorem and the Four-Color Theorem. Many extensions of the Four-Color Theorem have been proposed as conjectures, among which is Hadwiger's conjecture, one of the most well-known conjectures in nowadays graph theory:

Conjecture 1.1 (Hadwiger) *If G has no K_n as a minor then G is $(n - 1)$ -colorable.*

One of the important characteristics of Brook's theorem, the Four-Color theorem and Hadwiger's conjecture if it is proved, is that they provide good upper bounds on the chromatic number, which is an NP-hard parameter to compute, in terms of parameters or properties of graphs that are polynomial time to compute or verify. However none of these theorems and conjecture (if proven) provides a fixed upper bound on the chromatic number of bipartite graphs (note that it is easy to verify if a graph is bipartite). To formulate such theorems, that bound the chromatic number of say planar graphs some of whose edges are replaced by complete bipartite graphs, the theory of signed graphs and odd-minors have been introduced.

Beside the theory of minors, some other theories such as the theory of nowhere-zero flows on graphs have been extended to signed graphs. Coloring problems have also been considered for special families of signed graphs. The notion of homomorphism of signed graphs, which is the main subject of this work, was introduced by B. Guenin for its relation with an edge-coloring problem we mention below.

In [T1880] P. G. Tait proposed a (now classic) restatement of the Four-Color Theorem which claims that every bridgeless cubic planar multigraph is 3-edge colorable. Note that if a k -regular multigraph G is k -edge colorable, then for each subset X of vertices of G with $|X|$ being odd, there must be at least k edges that connect vertices in X to vertices not in X . If, in a k -regular multigraph G , for each subset X of odd size there are at least k edges joining vertices of X to vertices not in X , then G is called a k -graph. It is easily observed that if a k -regular graph is k -edge-colorable, then it is a k -graph. The Petersen graph is an example showing that not every k -graph is k -edge-colorable. Motivated by this restatement of the Four-Color Theorem, P. Seymour [S75] proposed the following conjecture in generalization of the Four-Color Theorem:

Conjecture 1.2 *Every planar k -graph is k -edge-colorable.*

We note that, for this conjecture, the fact that G is a multigraph is quite essential otherwise there is no planar k -regular graph for $k \geq 6$. For $k = 3$, Conjecture 1.2 is equivalent to the Four-Color Theorem. Conjecture 1.2 has been proved for $k = 4, 5$ by Guenin [G12], for $k = 6$ by Dvořák, Kawarabayashi and Král' [DKK], for $k = 7$ by Edwards and Kawarabayashi [EK11] and, very recently, for $k = 8$ by Chudnovsky, Edwards and Seymour [CES12].

In [N07], the first author introduced a generalization of the Four-Color Theorem and proved it to be equivalent to Conjecture 1.2 for odd values of k . B. Guenin [G05], after introducing the notion of homomorphism of signed graphs, provided a homomorphism analog for even values of

k in some stronger form. In [NRS12] we show that for $k = 2g$, Conjecture 1.2 is equivalent to the first claim of Conjecture 10.2 for $2g$.

The aim of this paper is to study colorings and homomorphisms of signed graphs with special attention to their connection with signed minors. It has come to our attention that the notion of homomorphisms of signed graphs is a special case of color switching homomorphisms of edge-colored graphs studied by Brewster and Graves in [BG09]. However it is the relation with signed minors and the possibility of extending coloring and homomorphism theories to planar and minor-closed families that makes our project special. Generally speaking, this paper is the first in developing this vast theory. It is therefore natural that we have many more questions than we have answers for. However, we do provide some exciting answers too.

The paper is organized as follows. First we settle our notation, after which we give definitions of new concepts together with examples. Then, in separate sections, we consider the possibilities of extending concepts from graph homomorphisms and graph coloring to signed graphs.

2 Notation

We use standard terminology of graph theory where a graph is considered to be simple, finite and loopless. Sometimes we allow the presence of multi-edges in which case we rather use the term multigraph. Less standard notions that we use are recalled in this section.

Given two graphs G and H , a *homomorphism* of G to H is a mapping $\phi : V(G) \rightarrow V(H)$ such that if $xy \in E(G)$ then $\phi(x)\phi(y) \in E(H)$. We will write $G \rightarrow H$ whenever there exists a homomorphism of G to H . The *homomorphic image* of G under ϕ , denoted $\phi(G)$, is the subgraph of H given by $V(\phi(G)) = \phi(V(G))$ and $xy \in E(\phi(G))$ if and only if there exists an edge $uv \in E(G)$ such that $\phi(u) = x$ and $\phi(v) = y$. A *core of a graph* G is a minimal subgraph of G to which G admits a homomorphism (see [HN04] for a proof that this is well-defined, and for more on graph homomorphisms). A *core* is a graph which is its own core. The relation $G \rightarrow H$ induces a quasi-order on the class of graphs which is a poset on the class of cores. In this order, many classical results can be restated in the language of mainstream mathematics. For example, the Four-Color Theorem is to say that the class of planar cores admits a maximum in the homomorphism order. Hadwiger's conjecture is also restated as follows:

Conjecture 2.1 (Hadwiger's conjecture reformulated [NN06]) *Every minor-closed family of graphs has a maximum with respect to the homomorphism order.*

The *chromatic number* of a graph G , denoted $\chi(G)$, is the smallest number of vertices of a homomorphic image of G . It is easily observed that $\chi(G)$ is the smallest number of colors one can assign to the vertices of G in such a way that adjacent vertices are assigned distinct colors (*proper coloring*). A graph is k -colorable if $\chi(G) \leq k$. A *bipartite graph* (k -partite graph, respectively) is a graph with at least two (k , respectively) vertices that is 2-colorable (k -colorable, respectively).

An *acyclic coloring* of a graph is a proper coloring in which every 2-colored subgraph is a forest. The *acyclic chromatic number* of a graph G is the minimum number of colors one needs for an acyclic coloring of G .

An *embedding* of a graph G on a plane is to associate distinct points of the plane to vertices of G and to associate a continuous closed curve with each edge uv which would have u and v as its endpoints and contains no other point of $V(G)$. A *planar graph* is a graph which admits an embedding on the plane such that edges intersect only at their endpoints. A *plane graph* is a

planar graph together with an embedding on the plane. An *outerplanar* graph is a planar graph which admits an embedding such that every vertex is on the outer face.

We use K_n , C_n and P_n to denote, respectively, the complete graph, the cycle and the path on n vertices. A *clique* is any complete graph. A *clique of G* is a complete graph that is a subgraph of G . The *clique number* of G , denoted $\omega(G)$, is the largest number of vertices of a clique of G .

A graph G is *connected* if for each pair x and y of vertices there is a path in G connecting x and y . The *connectivity* of a connected graph G is the minimum number of vertices of G whose removal either disconnects the remaining vertices or leaves only one vertex.

3 Definitions

3.1 Signed graphs

A *signified graph* is a graph G with an assignment of signs $+$ and $-$ to its edges. If Σ is the set of negative edges then we denote the signified graph by (G, Σ) . A *resigning* of a signified graph at a vertex v is to change the sign of each edge incident to v . We say (G, Σ_2) is a resigning of (G, Σ_1) if it is obtained from (G, Σ_1) by a sequence of resignings. Resigning then defines an equivalence relation on the set of all signified graphs over G . Each such class will be called a *signed graph* and could be denoted by any member of its class. Each member of this class is a *representation* of the signed graph. Given a signed graph (G, Σ) , Σ is called the *signature* of G . Given a signed graph (G, Σ) , we say (H, Σ_1) is a subgraph of (G, Σ) if there is a representation (G, Σ') of (G, Σ) such that (i) $V(H) \subseteq V(G)$, (ii) $E(H) \subseteq E(G)$ and (iii) $\Sigma_1 \subseteq \Sigma'$.

For a graph G on n vertices and m edges it is easy to check that if G has c connected components, then each equivalence class has 2^{n-c} elements and, therefore:

Proposition 3.1 *If G has m edges, n vertices and c components, then there are $2^{(m-n+c)}$ distinct signed graphs on G .*

In particular, we get the following:

Corollary 3.2 *There is only one signed graph on every forest.*

An *unbalanced cycle* of (G, Σ) is a cycle of G that has an odd number of negative edges. It is easily verified that this definition is independent of the choice of the signature of G . We denote by UC_k the signed graph (C_k, Σ) where Σ has an odd number of edges and we may refer to it as the *unbalanced k -cycle*.

One of the first theorems in the theory of signed graphs is that the set of unbalanced cycles uniquely determines the class of signed graphs to which a signified graph belongs. More precisely:

Theorem 3.3 (Zaslavsky [Z82]) *Two signified graphs (G, Σ_1) and (G, Σ_2) represent the same signed graph if and only if they have the same set of unbalanced cycles.*

In other words if (G, Σ_1) and (G, Σ_2) have the same set of unbalanced cycles, then the symmetric difference of Σ_1 and Σ_2 is an edge cut.

Given a signed graph (G, Σ) and k signed subgraphs $(G_1, \Sigma_1), \dots, (G_k, \Sigma_k)$ of (G, Σ) where $\Sigma_i \subseteq \Sigma$, we define their (mod 2)-sum, denoted $(G_1 \oplus \dots \oplus G_k, \Sigma')$, to be the signed subgraph of (G, Σ) induced by the set of edges that are in an odd number of the sets $E(G_1), \dots, E(G_k)$. We have the following classic and easy to prove lemma on the signature of a (mod 2)-sum of subgraphs of a signed graph.

Lemma 3.4 *Given signed subgraphs $(G_1, \Sigma_1), \dots, (G_k, \Sigma_k)$ of (G, Σ) , if their (mod 2)-sum is isomorphic to the disjoint union of cycles C_1, \dots, C_ℓ , then the number of unbalanced cycles among C_1, \dots, C_ℓ is congruent to $|\Sigma_1| + \dots + |\Sigma_\ell| \pmod{2}$.*

3.2 Signed minors

A *signed minor* of a signed graph (G, Σ) is a signed graph (H, Σ') obtained from (G, Σ) by a sequence of deleting vertices, deleting edges and contracting *positive edges*, in any order. We note that at any step of this process we can replace a signed graph with one of its equivalent forms or, equivalently, we may add a fourth operation in producing a minor that is “resigning”. Furthermore, it is important to note that though originally we are not allowed to resign a negative edge, we can do so after a resigning at (only) one end of it. Though, ordinarily, in the study of signed minors we allow existence of parallel edges, one of each sign, in this paper we do not allow parallel edges at all. Therefore, each contraction of an edge which generates parallel edges is associated with the deletion of all but one of these edges. In particular, contracting an edge uv such that there exists a 2-path uvw having one positive and one negative edge allows to keep either a positive or a negative edge from the new vertex to w .

The following lemma indicates the importance of signed minors from an algebraic point of view:

Lemma 3.5 *If (H, Σ') is a signed minor of (G, Σ) which is obtained only by contracting edges (i.e., nothing is deleted), then the image of an unbalanced cycle of (G, Σ) is an unbalanced cycle in (H, Σ') .*

Corollary 3.6 *If (H, Σ_1) is a signed minor of (G, \emptyset) then $(H, \Sigma_1) = (H, \emptyset)$.*

3.3 Homomorphisms of signed graphs

Given two signed graphs (G, Σ_1) and (H, Σ_2) , we say there is a *homomorphism* of (G, Σ_1) to (H, Σ_2) if there is a presentation (G, Σ'_1) of (G, Σ_1) and a presentation (H, Σ'_2) of (H, Σ_2) together with a mapping $\phi : V(G) \rightarrow V(H)$ such that every edge of (G, Σ'_1) is mapped to an edge of (H, Σ'_2) of the same sign. We will write $(G, \Sigma_1) \rightarrow (H, \Sigma_2)$ whenever there is a homomorphism of (G, Σ_1) to (H, Σ_2) . An *automorphism* of a signed graph (G, Σ) is a homomorphism of (G, Σ) to itself that is both surjective and one-to-one, when considered as a function from $V(G)$ to $V(G)$, and such that the induced function on the edge set is surjective. A signed graph (G, Σ) is called *vertex-transitive* if for each pair x and y of vertices there is an automorphism ρ of (G, Σ) such that $\rho(x) = y$. Similarly, (G, Σ) is called *edge-transitive* if for each pair $e_1 = xy$ and $e_2 = uv$ of edges there is an automorphism ρ of (G, Σ) such that $\{\rho(x), \rho(y)\} = \{u, v\}$. The unbalanced cycle UC_k is an example of a signed graph which is both vertex-transitive and edge-transitive. We say a signed graph (G, Σ) is *isomorphic* to (H, Σ') if there is a homomorphism of (G, Σ) to (H, Σ') which is one-to-one and onto both as a vertex function and an edge function.

Suppose $\phi : V(G) \rightarrow V(H)$ is a homomorphism of (G, Σ_1) to (H, Σ_2) using the representations (G, Σ'_1) and (H, Σ'_2) of (G, Σ_1) and (H, Σ_2) respectively. Let S be the set of vertices one must resign at to get (H, Σ'_2) from (H, Σ_2) . Let (G, Σ''_1) be the resigning of (G, Σ'_1) at all vertices of $\phi^{-1}(S)$. Then we can easily check that ϕ is also a homomorphism with respect to representations (G, Σ''_1) and (H, Σ_2) . Therefore, when checking for the existence of a homomorphism between two signed graphs, the choice of equivalent signatures is not important for the image. However, as shown by the easy example of Figure 1, the choice of signature is important for the domain graph.

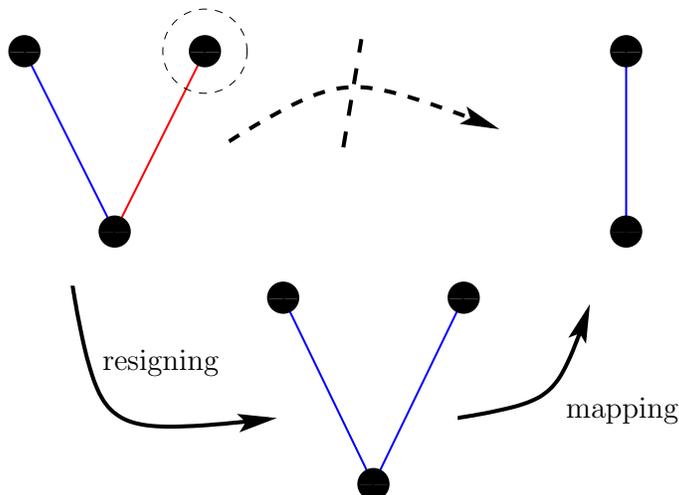


Figure 1: Resigning at the domain can be necessary for mapping

We define a core and core of a signed graph analogously to their graph counterpart. A *signed core* is a signed graph that admits no homomorphism to a proper signed subgraph of itself. In other words, (G, Σ) is a core if every homomorphism of (G, Σ) to (G, Σ) is an automorphism. A *core of a signed graph* (G, Σ) is a minimal subgraph of (G, Σ) to which (G, Σ) admits a homomorphism. The first theorem on the notion of cores, which is proved in Section 5, is to show that the concept of a core is well defined (see Theorem 5.1).

The fact that the choice of the signature in the target graph is free allows us to show, easily, that the binary relation of existence of a homomorphism on signed graphs is associative. Thus, homomorphisms of signed graphs define a quasi-order on the class of all signed graphs which is a poset when restricted to the class of all signed cores. This order will be called the *signed homomorphism order*. Hence we may interchange our notions freely and say (H, Σ_2) *bounds* (G, Σ_1) or that (G, Σ_1) is *smaller* than (H, Σ_2) for indicating that there is a homomorphism of (G, Σ_1) to (H, Σ_2) . Furthermore, if \mathcal{C} is a class of signed graphs, we say that a signed graph (H, Σ_2) *bounds* \mathcal{C} if (H, Σ_2) bounds every member of \mathcal{C} .

By taking all signed graphs with empty signature or by taking all signed graphs of the form $(G, E(G))$ we observe that the signed homomorphism order of signed graphs indeed contains the homomorphism order of graphs and, therefore, contains an isomorphic copy of every finite or countable poset (see [PT80]). This work is a first step in extending results from this usual order to the new order we introduce here. As we will see in Section 6, the class of signed graphs whose underlying graph is bipartite is also of special importance. In particular the sub-order induced on this set of signed graphs contains a natural isomorphic copy of the homomorphism order of graphs. A signed graph whose underlying graph is bipartite will be called a *signed bipartite graph*.

3.4 Signed graph coloring and signed chromatic number

One of the first natural questions to ask in the poset we have just introduced is: given a signed graph (G, Σ) what is the smallest order of a signed graph which bounds (G, Σ) ? The answer to this question in the usual homomorphism order is called the chromatic number of the graph. Thus we define the *signed chromatic number* of a signed graph, denoted $\chi(G, \Sigma)$, to be the

answer to this question. Analogously one can define signed graph coloring and, therefore, the signed chromatic number of a signed graph as follows: a *proper coloring* of a signed graph (G, Σ) is an assignment of colors to the vertices of G such that adjacent vertices receive distinct colors and there is a representation (G, Σ') of (G, Σ) such that whenever the two colors associated with the vertices of an edge e_1 are the same as those of another edge e_2 , the two edges e_1 and e_2 have same signs. The signed chromatic number of (G, Σ) is then the minimum number of colors needed for a proper coloring of (G, Σ) .

The signed chromatic number provides a first test for the possibility of the existence of a homomorphism of (G, Σ_1) to (H, Σ_2) . These kinds of tests are called “no homomorphism lemmas”. More precisely, by the associativity of the homomorphism order we have:

Lemma 3.7 *If $(G, \Sigma_1) \rightarrow (H, \Sigma_2)$, then $\chi(G, \Sigma_1) \leq \chi(H, \Sigma_2)$.*

3.5 Signed cliques and signed clique numbers

Using the terminology of signed chromatic number we define a signed clique as follows: a signed graph (G, Σ) is called a *signed clique*, or simply an *S-clique*, if its signed chromatic number is equal to the number of its vertices. In other words, an S-clique is a signed graph (G, Σ) whose homomorphic images are all isomorphic to itself. The following lemma shows how to check whether a signed graph is a signed clique or not:

Lemma 3.8 *A signed graph (G, Σ) is an S-clique if and only if for each pair u and v of vertices either $uv \in E(G)$ or u and v are vertices of an unbalanced cycle of length 4.*

Proof. Clearly, if every non-adjacent pair of vertices of (G, Σ) belongs to an unbalanced 4-cycle, then no pair of vertices can be identified in a homomorphic image of (G, Σ) .

For the other direction, let x and y be a pair of non-adjacent vertices in G . If $d_G(x, y) \geq 3$ then, by identifying x and y , we get a simple graph G' . The graph G' together with the signature induced by Σ is a signed graph of order $n - 1$ which is a homomorphic image of (G, Σ) , a contradiction. Therefore $d_G(x, y) = 2$. Let u be a vertex adjacent to both x and y . We can assume xu and yu are both of the same sign, as otherwise we may resign at x . If in the current signature, for every other vertex v adjacent to both x and y both edges xv and yv are of the same sign, then we get a contradiction just as before by identifying x and y and deleting the multiple edges. Finally, if there is a vertex v which is adjacent to x and y with two edges of different signs, then the cycle induced by x, u, y and v is an unbalanced 4-cycle, just as claimed. \square

The signed clique number (S-clique number) of a signed graph could be defined in two natural ways. The *absolute S-clique number* of (G, Σ) , denoted $\omega_{sa}(G, \Sigma)$, is the order of the largest subgraph (H, Σ_1) of (G, Σ) such that (H, Σ_1) itself is an S-clique. The *relative S-clique number* of (G, Σ) , denoted $\omega_{sr}(G, \Sigma)$, is the number of vertices of a largest subgraph (H, Σ_1) of (G, Σ) such that in every homomorphic image $\phi(G, \Sigma)$ of (G, Σ) , we have $|\phi(H)| = |V(H)|$. It is easy to verify that these definitions are independent of resigning.

We note that the difference between the absolute S-clique number and the relative S-clique number can be arbitrarily large. For example, take a set of n independent vertices and, for each pair x, y of them, add a new pair of vertices u_{xy}, v_{xy} and form an unbalanced 4-cycle on x, y, u_{xy}, v_{xy} such that xy is not an edge. Let (G, Σ) be the graph obtained in this way. Then $\omega_{sa}(G, \Sigma) = 4$ while $\omega_{sr}(G, \Sigma) = n$.

It is again easy to check that each of these two terms provides another no homomorphism lemma:

Lemma 3.9 *If $(G, \Sigma_1) \rightarrow (H, \Sigma_2)$, then $\omega_{sa}(G, \Sigma_1) \leq \omega_{sa}(H, \Sigma_2)$ and $\omega_{sr}(G, \Sigma_1) \leq \omega_{sr}(H, \Sigma_2)$.*

These two parameters and the signed chromatic number are related by the following theorem whose proof directly follows from the definitions.

Theorem 3.10 *For every signed graph (G, Σ) , $\omega_{sa}(G, \Sigma) \leq \omega_{sr}(G, \Sigma) \leq \chi(G, \Sigma)$.*

We should also note that the problem of computing S-clique number(s) and the signed chromatic number of a general signed graph includes, in particular, the problem of finding the clique number and the chromatic number for graphs by setting $\Sigma = \emptyset$. Thus each of these parameters is NP-hard to compute.

4 Homomorphisms versus minors

The concepts of homomorphisms and minors can be regarded as dual concepts: in producing a minor of a (signed) graph we identify pairs of adjacent vertices, one pair at a time, whereas in producing a homomorphic image of a (signed) graph we identify pairs of non-adjacent vertices, again one pair at a time.

Hadwiger's conjecture is to claim that the largest clique one can produce from a graph G by minor operations is at least as big as the smallest homomorphic image one can produce from G . Besides Hadwiger's conjecture there are many other challenging questions, some in direct extension of the Four-Color Theorem, that are about relations between minors and homomorphisms. For example what can be said about an optimal \mathcal{Q} -bound for a subclass \mathcal{C} of a minor-closed family of graphs, each having some homomorphism property \mathcal{P} , where the \mathcal{Q} -bound has some homomorphism property \mathcal{Q} ?

For an example of these kind of questions and results we have the following theorem of J. Nešetřil and P. Ossona De Mendez. For any set \mathcal{X} of graphs, let $Forb_h(\mathcal{X})$ denote the set of graphs that admit no homomorphism of a member of \mathcal{X} , and $Forb_m(\mathcal{X})$ denote the set of graphs that admit no member of \mathcal{X} as a minor. Then we have:

Theorem 4.1 (Nešetřil and Ossona De Mendez [NO08]) *For every set of graphs \mathcal{M} and every set of connected graphs \mathcal{H} , the class $Forb_m(\mathcal{M}) \cap Forb_h(\mathcal{H})$ is bounded by a graph in $Forb_h(\mathcal{H})$.*

Finding a bound as in Theorem 4.1 with smallest possible number of vertices proves to be a very difficult question in general. For the simplest case of $\mathcal{M} = \mathcal{H} = \{K_n\}$ finding the smallest bound in terms of number of vertices will, in particular, solve Hadwiger's conjecture. For the case $\mathcal{M} = \{K_5, K_{3,3}\}$ and $\mathcal{H} = \{C_{2k-1}\}$ it is conjectured by the first author [N07] that the projective cube of dimension $2k$ is the optimal solution (we refer to [N12] and [NRS12] for definitions and details).

The following is a more general related question that is introduced in [N12]. The question surprisingly captures or relates to many theories on planar graphs such as the theory of edge-coloring, fractional coloring, circular coloring and, furthermore, it gives ideas to develop further interesting theories.

Problem 4.2 *What is the smallest graph of odd-girth $2k + 1$ which bounds the class of planar graphs of odd-girth at least $2r + 1$ ($r \geq k$)?*

The main goal of this paper is to investigate relations between signed minors and homomorphisms of signed graphs. For this reason we will mainly focus on minor or signed minor-closed families such as planar and outerplanar graphs. While in this paper we mainly extend results from graphs to signed graphs, we hope that in the future the more algebraic notion and structure of signed graphs will help to settle some of these difficult questions in relation with minors and homomorphisms. Using the terminology of signed graphs we can extend Problem 4.2 for even values (see Problem 10.3 and [NRS12]). It is also natural to consider families of signed graphs to which Theorem 4.1 can be extended. For suggestions of such extensions see Problem 10.1.

5 Examples and basic results

At first we prove, as promised, that the notion of the core of a signed graph is well defined.

Theorem 5.1 *Given a signed graph (G, Σ) , the core of (G, Σ) is unique up to isomorphism (of signed graphs).*

Proof. Assume (H_1, Σ_1) and (H_2, Σ_2) are two cores of (G, Σ) . Since (H_1, Σ_1) is a subgraph of (G, Σ) we have $(H_1, \Sigma_1) \rightarrow (H_2, \Sigma_2)$. Let φ be such a homomorphism. We show that φ is a one-to-one and onto mapping of both $V(H_1)$ and $E(H_1)$ to $V(H_2)$ and $E(H_2)$ (respectively).

The fact that φ is onto follows from the composition $(G, \Sigma) \rightarrow (H_1, \Sigma_1) \rightarrow (H_2, \Sigma_2)$ and the fact that (H_2, Σ_2) is a core. Similarly any homomorphism of (H_2, Σ_2) to (H_1, Σ_1) must be onto. To see that φ is one-to-one as a vertex mapping suppose, by contradiction, that two vertices x and y of H_1 are mapped to a same vertex of H_2 . Then in the composition $(H_2, \Sigma_2) \rightarrow (H_1, \Sigma_1) \rightarrow (H_2, \Sigma_2)$ the nonempty preimages of x and y in (H_2, Σ_2) are mapped to a same vertex of (H_2, Σ_2) . This implies that a proper subgraph of (H_2, Σ_2) is a homomorphic image of (G, Σ) , this is in contradiction with (H_2, Σ_2) being a core of (G, Σ) . \square

As a consequence, we get for instance that every S-clique is a core and, since there is only one signed graph on a given tree T , that the core of any signed tree (T, Σ) is (K_2, \emptyset) . Therefore, we have:

Corollary 5.2 *If G is a tree (forest) then $\omega_{sa}(G, \Sigma) = \omega_{sr}(G, \Sigma) = \chi(G, \Sigma) = 2$.*

Furthermore we can easily classify the set of all 2-colorable signed graphs:

Theorem 5.3 *A signed graph (G, Σ) is 2-colorable if and only if (i) G is bipartite and (ii) there is no unbalanced cycle in G (in other words, (G, Σ) can be presented by (G, \emptyset)).*

For a given k the problem k -COLORING-SIGNED-GRAPHS is the following:

k -COLORING-SIGNED-GRAPHS

Input: A signed graph (G, Σ) .

Question: Is $\chi(G, \Sigma) \leq k$?

By Theorem 5.3 and since the problem k -COLORING-SIGNED-GRAPHS contains, in particular, the problem k -COLORING-GRAPHS, we have the following dichotomy.

Corollary 5.4 *The problem k -COLORING-SIGNED-GRAPHS is polynomial-time for $k = 1, 2$ and NP-complete for $k \geq 3$.*

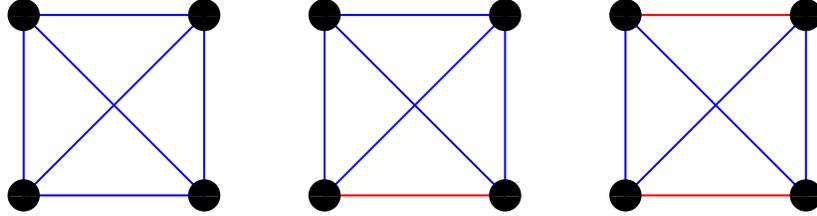


Figure 2: Signed complete graphs on 4 vertices

A *signed complete graph* is a complete graph with a signature. Thus every signed complete graph is an S-clique but the converse is not true. There are 2^{n-1} elements in each class of a signed complete graph and, therefore, there are $2^{\binom{n}{2}-n+1}$ signed complete graphs on n labeled vertices, however many of them are isomorphic. We do not know the exact number of non-isomorphic signed complete graphs. There are three such graphs on four vertices. This can be seen by considering a presentation with minimum number of negative edges. Hence there is one with no negative edge, one with exactly one negative edge and the third has two negative edges that are not adjacent. They are depicted in Figure 2. In all the figures, blue edges are positive and red edges are negative.

The class $\mathcal{C} = \{(G, \Sigma) \mid G \text{ has no } K_4\text{-minor}\}$ of signed graphs is, therefore, exactly the class of signed graphs which have none of the three signed complete graphs of Figure 2 as a signed minor. Similarly the class of planar signed graphs can be characterized by means of signed minors: a signed graph (G, Σ) is planar if it has no (K_5, Σ) or $(K_{3,3}, \Sigma')$ as a signed minor (for any choice of Σ or Σ'). There are exactly seven non-isomorphic such signed graphs on K_5 and three on $K_{3,3}$.

It follows from the definitions that the underlying graph of every S-clique must be at least 2-connected:

Lemma 5.5 *An S-clique cannot have a cut-vertex.*

Proof. By contradiction, assume that u is a cut-vertex. Thus there should be non-adjacent vertices x and y connected only through u but, in a signed clique, every pair of non-adjacent vertices belongs to an unbalanced 4-cycle. \square

This lemma implies, in particular, that an S-clique of order at least 3 cannot have a vertex of degree 1. However an S-clique (of large order) may have a vertex of degree 2. Hence an S-clique is not necessarily 3-connected. An example of such an S-clique is built as follows: for a fixed $n \geq 2$ consider the signed graph (K_n, \emptyset) with x and y being two different vertices. Add a new vertex v and join v to x and to y with a negative and a positive edge, respectively. The new signed graph is still an S-clique with v being a vertex of degree 2.

Example 5.6 Let $K_{n,n}$ be the complete bipartite graph on vertices $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$. Let M be the matching $\{x_1y_1, \dots, x_ny_n\}$. The signed graph $(K_{2,2}, M)$ is isomorphic to the balanced C_4 and thus admits (K_2, \emptyset) as a core. We prove below that for $n \geq 3$ the signed graph $(K_{n,n}, M)$ is a signed clique and therefore a core.

Proposition 5.7 *For $n \geq 3$ the signed graph $(K_{n,n}, M)$ is an S-clique.*

Proof. Since every x_i is adjacent to every y_j , all we need to prove is that every pair $\{x_i, x_j\}$, $i \neq j$, lies in an unbalanced 4-cycle (a similar argument will then work for pairs of the form

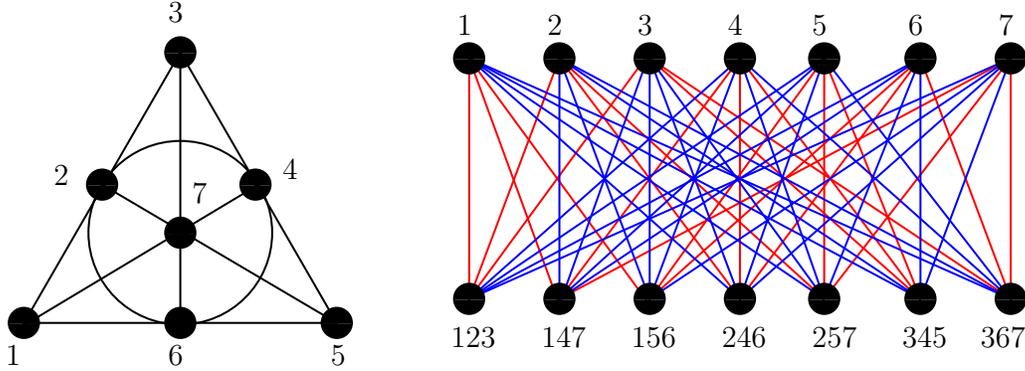


Figure 3: The Fano plane and the signed graph $Fano$

$\{y_i, y_j\}$). Since $n \geq 3$, there is an index $\ell \notin \{i, j\}$. The cycle induced by $\{x_i, x_j, y_i, y_\ell\}$ is a 4-cycle with $x_i y_i$ as the only negative edge. \square

It is also not hard to check that $(K_{n,n}, M)$ is vertex-transitive, however this graph is not edge-transitive for $n \geq 3$. This can be seen by counting the number of unbalanced 4-cycles an edge belongs to.

Corollary 5.8 *For every graph G and every signature Σ of G , $\chi(G, \Sigma) \geq \chi(G)$. The difference $\chi(G, \Sigma) - \chi(G)$ can be arbitrarily large.*

Proof. The inequality follows from the definition. From Proposition 5.7, we get for every $n \geq 3$ that $\chi(K_{n,n}, M) - \chi(K_{n,n}) = 2n - 2$. \square

In a similar way we can prove that the following signed bipartite graphs are cores.

Example 5.9 Let X be a set of size k and Y be a set of size 2^{k-1} whose elements are labeled with distinct unordered pairs of the form $\{A, \bar{A}\}$ where A is any subset of X and \bar{A} is the complement of A in X . Let $K_{k,2^{k-1}}$ be the complete bipartite graph on $X \cup Y$.

Let $CB_k = (K_{k,2^{k-1}}, \Sigma)$ be the signed bipartite graph where $x\{A, \bar{A}\} \in \Sigma$ if and only if $x \in A$. Similarly to the signed bipartite graph of Example 5.6 one can show that this signed bipartite graph is an S-clique. We will use it later to define the bipartite chromatic number of signed bipartite graphs. This signed graph is well defined because we can resign at $\{A, \bar{A}\}$.

Example 5.10 The *Fano plane* is a finite geometry composed of seven points and seven lines arranged as depicted in the left side of Figure 3. The *Fano signed graph*, denoted $Fano$, is the signed graph $(K_{7,7}, \Sigma)$, where Σ is defined as follows: first we label the vertices of $K_{7,7}$ by points and lines of the Fano plane. We use the seven points for one part and the seven lines for the other part. Then, for any line L and any point x we have $xL \in \Sigma$ if and only if $x \in L$. This signed graph is depicted in the right side of Figure 3.

We then have:

Proposition 5.11 *The signed graph $Fano$ is an S-clique.*

Proof. Since each pair of vertices from different parts are adjacent, and by Lemma 3.8, it is enough to show that each pair of vertices in a same part lies in an unbalanced 4-cycle.

Considering the symmetries between lines and points and the symmetries between pairs of points in the Fano plane, it is enough to check this only for one pair say $\{1, 2\}$, of non-adjacent vertices of *Fano*. Observe that together with lines 156 and 345, this pair induces an unbalanced 4-cycle in *Fano*. \square

The *Fano* signed graph is also vertex-transitive but it is not edge-transitive.

By Proposition 3.1 there are exactly two signed graphs on a cycle C_k . The balanced cycle, which can be represented as (C_k, \emptyset) , and the unbalanced cycle UC_k which can be represented by a signature with one negative edge only. The next lemma is about the existence of a homomorphism between two unbalanced cycles. Even though it is easy to prove, it is quite essential.

Lemma 5.12 *There is a homomorphism of UC_k to UC_ℓ if and only if $k \geq \ell$ and $k = \ell \pmod{2}$.*

Thus we have another “no homomorphism lemma”:

Corollary 5.13 *If $(G, \Sigma_1) \rightarrow (H, \Sigma_2)$, then the shortest unbalanced cycle of odd length (even length, respectively) in (G, Σ_1) is at least as large as the shortest unbalanced cycle of odd length (even length, respectively) in (H, Σ_2) .*

The shortest length of an unbalanced cycle of (G, Σ) will be called the *unbalanced girth* of (G, Σ) . In fact this corollary proposes two separate terminology of shortest unbalanced girth of odd and even length, but we will not use them in this paper.

Example 5.14 Given a prime power $q = 1 \pmod{4}$, let \mathbb{F}_q be a finite field of order q . The *signed Paley graph* of order q , denoted $SPal_q$, is the signed complete graph with vertex set \mathbb{F}_q , with the edge xy being positive if and only if $x - y$ is a square in \mathbb{F}_q . This is, of course, a particular representation of the signed Paley graph but because of its importance we will call this representation the *signified Paley graph* of order q . We will then use $SPal_q$ to denote both the signed Paley graph and the signified Paley graph of order q . The important property of the signified $SPal_q$ for q large enough is that, given a small but arbitrary set $\{v_1, \dots, v_k\}$ of vertices and almost any sequence $A := a_1, \dots, a_k$ of signs, there is a vertex x for which the sign of the edge xv_i is a_i (with a possibility of resigning at x only). This will be called property *Prop_k*.

For example, the signified Paley graph $SPal_5$ depicted in Figure 4 has property *Prop₂*. That means that for every pair $\{u, v\}$ of vertices, if a_1 and a_2 are not both of the same sign as the sign of the edge uv , then there is a vertex x where xu and xv have the signs a_1 and a_2 respectively.

This property of $SPal_q$ will help us to prove the existence of a homomorphism to $SPal_q$ from signed graphs on partial k -trees. In particular we use this idea to prove, in Section 8, that for every K_4 -minor-free graph G and any signature Σ , the signed graph (G, Σ) has signed chromatic number at most 5. It can be checked that $SPal_{13}$ and $SPal_{17}$ both have property *Prop₃* and that $SPal_{29}$ has property *Prop₄*.

Our last examples in this section are *signed projective cubes*. Using projective cubes some of the most outstanding problem in Combinatorics can be translated or related to homomorphism of signed graphs, see Problem 10.2 and [G05, NRS12].

Example 5.15 The *projective cube* of dimension d , denoted \mathcal{PC}_d , is the graph with \mathbb{Z}_2^d as vertices where vertices u and v are adjacent if $u - v \in \{e_1, e_2, \dots, e_d\} \cup \{J\}$. Here e_i 's are standard basis and J is the all 1 vector. This graph can be built from hypercubes in two different ways: either by identifying antipodal vertices of hypercube of dimension $d + 1$ or by

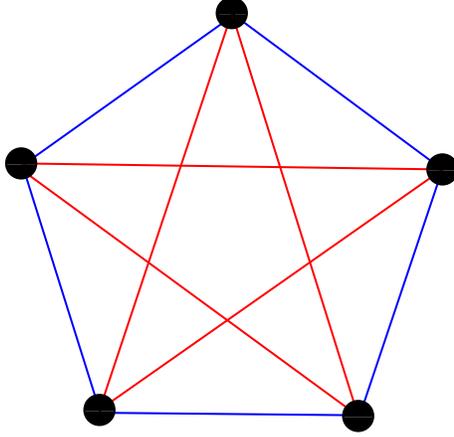


Figure 4: Signed Payley graph of order 5

adding an edge between pairs of antipodal vertices in hypercube of dimension d . The *signed projective cube* of dimension d , \mathcal{SPC}_d , is the signed graph (\mathcal{PC}_d, Σ) where Σ is the set of edges corresponding to J .

The signed graph \mathcal{SPC}_d is of unbalanced girth $d + 1$. If d is even, then \mathcal{SPC}_d is equivalent to $(\mathcal{PC}_d, E(\mathcal{PC}_d))$. For even values of d the graph \mathcal{PC}_d is bipartite. For proofs, the importance of these graphs and for more details we refer to [NRS12].

6 Two important subclasses

Lemma 5.12 is an indication of the importance of studying the homomorphism order restricted to two subfamilies of signed graphs: signed graphs in which all the unbalanced cycles are odd and signed graphs in which all the unbalanced cycles are even. For similar reasons, it is also natural to consider classes of graphs in which all balanced cycles have the same parity. However if a graph is well connected, then in the symmetric difference of two balanced cycles there will be a cycle which is both balanced and of even length. Thus we consider the following two cases:

I. *Signed graphs (G, Σ) for which every balanced cycle is even and every unbalanced cycle is odd.*

Then the set of odd cycles of G is the set of all unbalanced cycles of (G, Σ) and, therefore, by Theorem 3.3, (G, Σ) can be represented by $(G, E(G))$. Thus this is the class of signed graphs in which all the edges are negative. Such a signed graph will be called an *odd signed graph*. The problem of the existence of a homomorphism of (G, Σ_1) to (H, Σ_2) in the class of odd signed graphs is reduced to the existence of a homomorphism of G to H . Thus the homomorphism order induced on this set of signed graphs is trivially isomorphic to the homomorphism order of graphs. However it is the difference between the concept of minor and signed minor that allows to establish or conjecture stronger results in the class of odd signed graphs. The most outstanding such example is the following extension of Hadwiger's conjecture, known as *odd Hadwiger's conjecture*, proposed by B. Gerard and P. Seymour [JT95].

Conjecture 6.1 (Odd Hadwiger's conjecture) *If $(G, E(G))$ does not have $(K_n, E(K_n))$ as a signed minor, then $\chi(G) = \chi(G, E(G)) \leq n - 1$.*

To see the strength of this conjecture let us examine the case $n = 3$. The class of K_3 -minor-free graphs is exactly the class of forests and Hadwiger's conjecture is easily true: every forest

is 2-colorable. The class of $(K_3, E(K_3))$ -signed minor-free odd signed graphs is exactly the class of signed graphs $(G, E(G))$ where G is bipartite. Thus the conjecture is again true, but where the original Hadwiger's conjecture is bounding the chromatic number of forests, odd Hadwiger's conjecture gives the same bound of 2 for the class of all bipartite graphs. The conjecture was proposed based on a proof by P. A. Catlin [C79] for $n = 4$. For $n = 5$ a proof was presented by B. Guenin in 2005 but we do not know of a reference for this proof.

II. *Signed graphs (G, Σ) for which every balanced cycle is even and every unbalanced cycle is also even.*

This will be the case if and only if G is bipartite. Thus such a signed graph is called a *signed bipartite graph* but, in contrast to the previous case, we may equivalently use the term *even signed graph* for members of this class.

We show, by means of a simple construction, that most homomorphism problems for the class of odd signed graphs, and, therefore, homomorphism problems for graphs rather than signed graphs, are captured by the homomorphism problems for the class of even signed graphs. This is an indication that this class deserves special attention.

We first define the following construction. Let G be a graph; the signed graph $S(G) = (G^*, \Sigma)$ is obtained by replacing each edge uv of G by an unbalanced 4-cycle on four vertices $ux_{uv}vy_{uv}$, where x_{uv} and y_{uv} are new and distinct vertices. See Figure 5 for an example.

The following theorem shows how to define $\chi(G)$ in the homomorphism order induced on the set of signed bipartite graphs (the signed graph $(K_{k,k}, M)$ has been defined in Example 5.6).

Theorem 6.2 *For every $k \geq 3$ and every graph G , $\chi(G) \leq k$ if and only if $S(G) \rightarrow (K_{k,k}, M)$.*

Proof. It would be enough to prove the theorem for connected graphs. Let $\varphi : G \rightarrow K_k$ be a k -coloring of G . Label vertices in one part of $K_{k,k}$ with vertices of K_k (or equivalently with k colors). We can then regard φ as a partial mapping of $S(G)$ to $(K_{k,k}, M)$. We extend this mapping to the remaining vertices of $S(G)$ as follows: for each pair u and v of adjacent vertices of G , φ is extended to x_{uv} and y_{uv} in such a way that the image of the unbalanced cycle $ux_{uv}vy_{uv}$ is an unbalanced 4-cycle in $(K_{k,k}, M)$. This is possible simply because $k \geq 3$. It is then straightforward to check that this extension is a homomorphism of $S(G)$ to $(K_{k,k}, M)$.

For the converse, assume there is a homomorphism ϕ of $S(G) = (G^*, \Sigma)$ to $(K_{k,k}, M)$. Then ϕ is, in particular, a homomorphism of the bipartite graph G^* to the complete bipartite graph $K_{k,k}$. In the bipartition of G^* , one part is formed by $V(G)$ and the other part is the set of new vertices. Thus, the restriction of ϕ on $V(G)$ is a mapping of $V(G)$ to k vertices of one side of $K_{k,k}$. Furthermore, if uv is an edge of G , then u and v must be mapped to distinct vertices because of the unbalanced 4-cycle $ux_{uv}vy_{uv}$. Hence this restriction of ϕ is a k -coloring of G . \square

In a similar way we show below that the problem of the existence of a homomorphism of a graph G into a graph H is captured by the notion of homomorphism between signed bipartite graphs.

Theorem 6.3 *For every two graphs G and H , $G \rightarrow H$ if and only if $S(G) \rightarrow S(H)$.*

Proof. Any homomorphism of G to H can easily be extended to a homomorphism of $S(G)$ to $S(H)$.

For the converse, suppose that ϕ is a homomorphism of $S(G)$ to $S(H)$. If G has no edge, then there is nothing to prove. If G is bipartite with at least one edge, then H must also have at least one edge for ϕ to exist and, therefore, G maps to H . Thus we may assume G has at

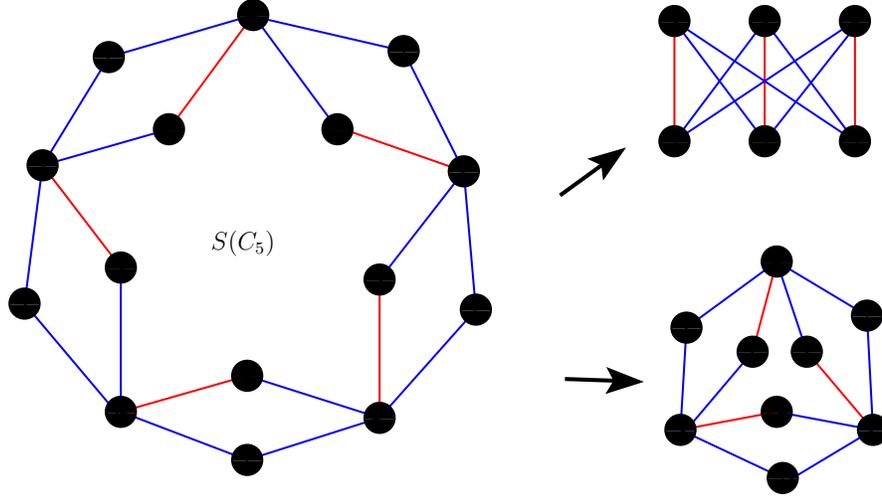


Figure 5: $S(C_5) \rightarrow (K_{3,3}, M)$ and $S(C_5) \rightarrow S(C_3)$

least one odd cycle. Furthermore, we may assume that both G and H are connected as we can easily compose homomorphisms on connected components.

We claim that in the mapping ϕ from $V(S(G))$ to $V(S(H))$ the set $V(G)$ must be mapped into $V(H)$. Since $V(G)$ is a part in the bipartition of $S(G)$, and $V(H)$ is a part in the bipartition of $V(S(H))$, and since G and H are both connected, ϕ either maps all vertices in $V(G)$ to vertices in $V(H)$ or none of them. Let C_{2r+1} be an odd cycle of G and let w_1, \dots, w_{2k+1} be its vertices, connected in this cyclic order. To complete the proof of our claim we show that vertices of C_{2k+1} must be mapped into $V(H)$. By contradiction suppose that a vertex w_i of C_{2k+1} is mapped to a vertex of the form x_{uv} in $S(H)$. Then, because of the unbalanced 4-cycle associated to the edge $w_i w_{i+1}$ (addition of the index is taken mod $2k+1$) in $S(G)$, w_{i+1} is mapped to the vertex y_{uv} . Continuing this process we obtain a 2-coloring of C_{2k+1} using x_{uv} and y_{uv} which is a contradiction.

Thus ϕ maps $V(G)$ to $V(H)$. To show that the restriction of ϕ on these two sets is a homomorphism of G to H , let uv be an edge in G and let UC' be the unbalanced 4-cycle associated with this edge in $S(G)$. The image of UC' under ϕ then must be another unbalanced 4-cycle containing $\phi(u), \phi(v)$ and the other two vertices must be vertices not from $V(H)$. This means we have constructed an unbalanced 4-cycle on $\phi(u)$ and $\phi(v)$, but the condition for having such a cycle is to have an edge between $\phi(u)$ and $\phi(v)$ in H . Hence ϕ induces a homomorphism of G to H . \square

Since the homomorphism order on signed bipartite graphs captures the homomorphism order on graphs, it is natural to look for extensions of many known coloring and homomorphism results on graphs to signed bipartite graphs. In particular we will consider some possible extensions of Hadwiger's conjecture in Section 9.

7 S-clique numbers of planar signed graphs

In this section we consider the problem of determining the S-clique number of a planar signed graph. We show that the largest planar S-clique one can build is of order 8, which gives the maximum of the absolute S-clique number of planar signed graphs. We do not know the maxi-

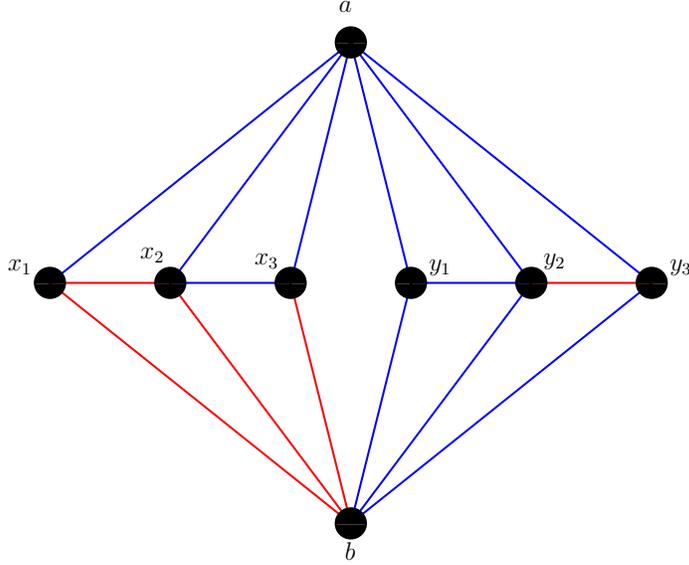


Figure 6: A planar S-clique on 8 vertices

imum of the relative S-clique number of planar signed graphs, though we obtain an upper bound through the bounds for the signed chromatic number of planar graphs in Section 8.

Theorem 7.1 *The maximum order of a planar S-clique is 8.*

Proof. An example of a planar S-clique on eight vertices is given in Figure 6. To see that this signed graph is an S-clique, it is enough to observe that every pair of non-adjacent vertices lies on an unbalanced 4-cycle.

Assume now that (G, Σ) is an S-clique of order 9 or more. Furthermore, we may assume without loss of generality that G is a triangulation. Recall first that, by Lemma 3.8, each pair of non-adjacent vertices of G lies on an unbalanced 4-cycle. This will be a key tool for our proof. We prove, through several claims, that $K_{2,3}$ cannot be a subgraph of G . Using this we will get a contradiction at the end.

When referring to $K_{2,i}$, we use a and b to denote the vertices from the part with two vertices and x_1, \dots, x_i to denote the vertices from the other part, ordered from left to right with respect to a given embedding of G in the plane. By $K_{2,i}^+$ we denote the graph obtained from $K_{2,i}$ by adding the edge ab . Furthermore when we speak of faces of these subgraphs we refer to their planar embedding induced by the planar embedding of G .

Claim 1 $K_{2,7}$ cannot be a subgraph of a planar S-clique of order at least 9.

Suppose $K_{2,7} \subseteq G$. Consider a cyclic ordering of x_1, x_2, \dots, x_7 . With respect to Σ , each path $ax_i b$ is either positive or negative. Hence four of these seven paths are of the same sign. Suppose $ax_{i_1} b, ax_{i_2} b, ax_{i_3} b, ax_{i_4} b$ are of the same sign. Then, in the cyclic order of x_1, x_2, \dots, x_7 , at least two of $x_{i_1}, x_{i_2}, x_{i_3}$ and x_{i_4} , say x_{i_1} and x_{i_2} , are at distance 3. To see this, we build a graph on x_1, x_2, \dots, x_7 by joining vertices at distance 3 in the above cyclic order. The graph built is isomorphic to C_7 whose independence number is 3, thus if we choose a set of four vertices, two of them will be adjacent. Finally we note that there is no possibility for the non-adjacent pair x_{i_1} and x_{i_2} of vertices of (G, Σ) to be in an unbalanced 4-cycle. Together with Lemma 3.8 this

proves Claim 1.

We note that this claim holds generally, i.e., if (G, Σ) is a planar S -clique then $K_{2,7} \not\subseteq G$. But the next claims are only true because we have assumed that G has nine or more vertices.

Claim 2 $K_{2,5}$ cannot be a subgraph of a planar S -clique of order at least 9.

By contradiction suppose $K_{2,5} \subseteq G$ and consider the cyclic order on x_1, x_2, \dots, x_5 . Furthermore, sums in the indices are taken modulo 5. Let u be a vertex of G which is not in $K_{2,5}$. Suppose u is in the face $ax_i bx_{i+1}$ of $K_{2,5}$. Then x_{i+3} is not adjacent to u . For these two vertices to be in a common unbalanced 4-cycle, u must be adjacent to both a and b . Since u was an arbitrary vertex, every vertex not in $K_{2,5}$ must be joined to both a and b . Because we assume G has at least 9 vertices, this would imply that $K_{2,7} \subseteq G$ which contradicts Claim 1.

Claim 3 $K_{2,4}^+$ cannot be a subgraph of a planar S -clique of order at least 9.

Let x_1, x_2, x_3 and x_4 be the four vertices of the part of size 4. By considering an imaginary vertex x_5 on the edge ab we could repeat the same argument as in the previous case to get a contradiction.

Claim 4 $K_{2,4}$ cannot be a subgraph of a planar S -clique of order at least 9.

Assume $K_{2,4} \subseteq G$. Suppose, by symmetry, that there is a vertex u in the outer face of $K_{2,4}$. By Claim 2, u is adjacent to at most one of a and b . Suppose u is not adjacent to b . By Lemma 3.8, there could be no other vertex in the face $ax_2 bx_3$. Since u is not adjacent to x_2 and x_3 , by Lemma 3.8, x_2 and u (and similarly x_3 and u) must be in a common 4-cycle. The only way for this to happen is that u is adjacent to x_1, a and x_4 .

In similar way each vertex in the outer face of $K_{2,4}$ must be adjacent to either x_1, a and x_4 or to x_1, b and x_4 . However, by planarity of G , for each of these triples there can be at most one vertex in the outer face of $K_{2,4}$ joined to all three of them. First we consider the case when there are two such vertices and let v be the vertex joined to x_1, b and x_4 . In this case we prove that there is no vertex of G on the faces $ax_1 bx_2$ and $ax_3 bx_4$ of $K_{2,4}$. For a contradiction, suppose t is a vertex on the face $ax_1 bx_2$ of $K_{2,4}$. Then, to be in a 4-cycle with u , t must be adjacent to a and, to be in a 4-cycle with v , t must be adjacent to b . Thus, G contains a $K_{2,5}$ as a subgraph which contradicts Claim 2. This leaves us with at most 8 vertices which contradicts the order of G . Hence we may assume there is at most one vertex in each of the faces of $K_{2,4}$ and that u is one such vertex in the outer face. Thus, there is no vertex in the face $ax_2 bx_3$ of $K_{2,4}$. To complete the proof of the claim we show that faces $ax_1 bx_2$ and $ax_3 bx_4$ of $K_{2,4}$ cannot contain vertices at the same time. That is simply true because such vertices must both be connected to a and b in order to be in a same 4-cycle.

Claim 5 $K_{2,3}^+$ cannot be a subgraph of a planar S -clique of order at least 9.

Assume $K_{2,3}^+ \subseteq G$. Suppose that, in the planar embedding of G , the subgraph $K_{2,3}^+$ is embedded as in Figure 7. We first show that the faces f_1 and f_2 of $K_{2,3}^+$ are also faces of G . For a contradiction suppose there is a vertex t on the face abx_3 of $K_{2,3}^+$. Then for the non-adjacent

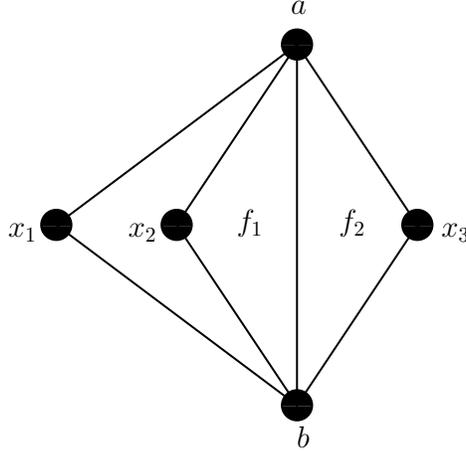


Figure 7: $K_{2,3}^+$ subgraph of a planar S-clique

pair t and x_2 of vertices of G to be in a 4-cycle, t must be connected to both a and b . Hence $K_{2,4}$ is a subgraph of G which contradicts Claim 4. The proof for the abx_2 -cycle is similar.

Let now z be a vertex in the outer face of $K_{2,3}^+$. Then, by Claim 4, z cannot be adjacent to both a and b . Suppose, by symmetry, that z is not adjacent to b . Since z is not adjacent to x_2 , by Lemma 3.8, they must be in a common 4-cycle. For this to be possible x_2 must be adjacent to x_1 . Furthermore, z also must be adjacent to both a and x_1 . Similarly, any other vertex of G is either adjacent to both a and x_1 or adjacent to both b and x_1 . Since there are at least four vertices in G which are not in the $K_{2,3}^+$, there are at least two vertices, say u and v , adjacent to the same pair, say a and x_1 without loss of generality. Then u, v, x_2 and b together with a and x_1 form a $K_{2,4}^+$ subgraph of G which contradicts Claim 3.

Claim 6 *If K_4^- is a subgraph of G , then the two triangles of this subgraph are faces of G .*

Let a, x, b and y be the four vertices of K_4^- with ab being the missing edge (this edge might exist in G). Let t be a vertex in the triangle axy separated from b . Thus t and b are not adjacent and, therefore, by Lemma 3.8, they are in a common 4-cycle. By symmetry of a and b , we consider two cases: either t is adjacent to both x and y , in which case $\{x, y\}$ and $\{a, t, b\}$ induce a $K_{2,3}^+$; or t is adjacent to both a and x , in which case $\{a, x\}$ and $\{b, y, t\}$ induce a $K_{2,3}^+$. In both cases we have a contradiction with Claim 5.

Claim 7 *The graph H of Figure 8 admits no signature with respect to which it would be an S-clique.*

By Lemma 3.8, all we need is to prove that there is no signature on H such that each pair of nonadjacent vertices is contained in an unbalanced 4-cycle. To this end we note that each of the following nine pairs are in a unique 4-cycle of H : (i) p and s in the cycle $C_1 = ptsr$, (ii) y and p in the cycle $C_2 = yxpa$, (iii) y and t in the cycle $C_3 = yatx$, (iv) y and s in the cycle $C_4 = yzsb$, (v) y and r in the cycle $C_5 = ybrx$, (vi) p and z in the cycle $C_6 = pazt$, (vii) p and b in the cycle $C_7 = pxbr$, (viii) s and a in the cycle $C_8 = szat$, (ix) s and x in the cycle $C_9 = sbxr$. Thus each of the cycles C_1, \dots, C_9 is an unbalanced 4-cycle. Therefore, by Lemma 3.4, the cycle $patzsbxr$, which is the (mod 2)-sum $(C_2 \oplus C_3) \oplus (C_4 \oplus C_5)$, is balanced.

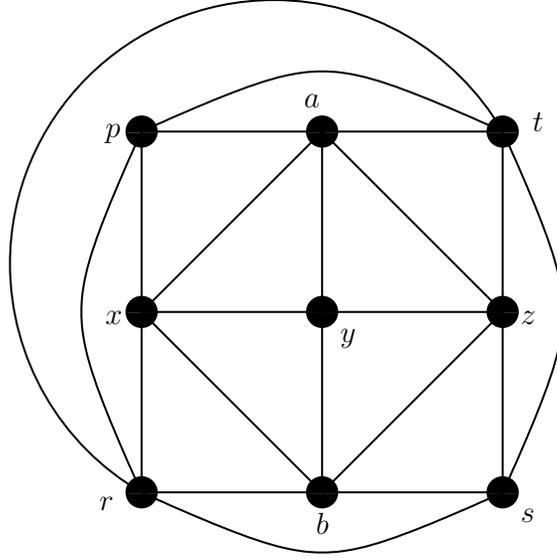


Figure 8: Graph H , a candidate on 9 vertices for being a signed clique

Since the triangles pat and tzs are two connected components of the $(\text{mod } 2)$ -sum $C_6 \oplus C_8$, they are of the same balance by Lemma 3.4. Similarly, considering $(\text{mod } 2)$ -sum $C_7 \oplus C_9$ we conclude that the triangles sbr and rxp are of the same balance. Therefore, by Lemma 3.4, the $(\text{mod } 2)$ -sum $pat \oplus tzs \oplus sbr \oplus rxp \oplus patzsbrx$ is balanced. However this $(\text{mod } 2)$ -sum is C_1 which is supposed to be unbalanced, a contradiction.

Claim 8 $K_{2,3}$ cannot be a subgraph of a planar S -clique of order at least 9.

Towards a contradiction, let $K_{2,3}$ be a subgraph of G . Suppose $K_{2,3}$ is a plane subgraph of G as depicted in the left side of Figure 9. As the first step we show that at least one of the three pairs x_1x_2 , x_1x_3 , x_2x_3 should be an edge of G . By contradiction, suppose none of them is an edge of G . Then, since G is a triangulation, and because of Claim 5, there should be a vertex in each face of $K_{2,3}$. Let t be a vertex on the outer face of $K_{2,3}$. Then t is not adjacent to x_2 , so they must be in a common 4-cycle, but to this end either t is adjacent to both a and b , which contradicts Claim 4, or x_2 is adjacent to one of x_1 or x_3 as we wanted. Without loss of generality we now assume that $x_1x_2 \in E(G)$, as depicted in the right side of Figure 9.

We show as the next step that either $x_1x_3 \in E(G)$ or $x_2x_3 \in E(G)$. Assume neither x_1x_3 nor x_2x_3 is an edge of G . So, just as in the previous step, we assume t is a vertex on the outer face of $K_{2,3}$. Furthermore let t' be a vertex on the face ax_2bx_3 of $K_{2,3}$. Since t and t' are not adjacent, they must be in a common unbalanced 4-cycle. By Claim 4 neither of t and t' can be adjacent to both a and b . Hence they both are adjacent either to a and x_3 or to b and x_3 . By symmetry of a and b , we assume t and t' both are adjacent to a and x_3 . By Claim 6, any other vertex must be either inside $at'x_3bx_2$ or outside of atx_3bx_1 . Let u be such a vertex and, by symmetry of these two cycles, we assume it is inside $at'x_3bx_2$. Thus u is not adjacent to t and therefore it should be in common 4-cycle with t . To this end it should be adjacent to a and x_3 . This would induce a $K_{2,3}^+$ on $\{a, x_3\} \cup \{t, t', u\}$, contradicting Claim 5. Hence, for every $K_{2,3}$ subgraph of G , there must be at least two edges induced by vertices of the part of size 3.

Finally, to complete the proof of this claim, we show that if $K_{2,3} \subseteq G$, then G is isomorphic to the graph of Figure 8. By the previous step, and by symmetry, we may assume $x_1x_2 \in E(G)$

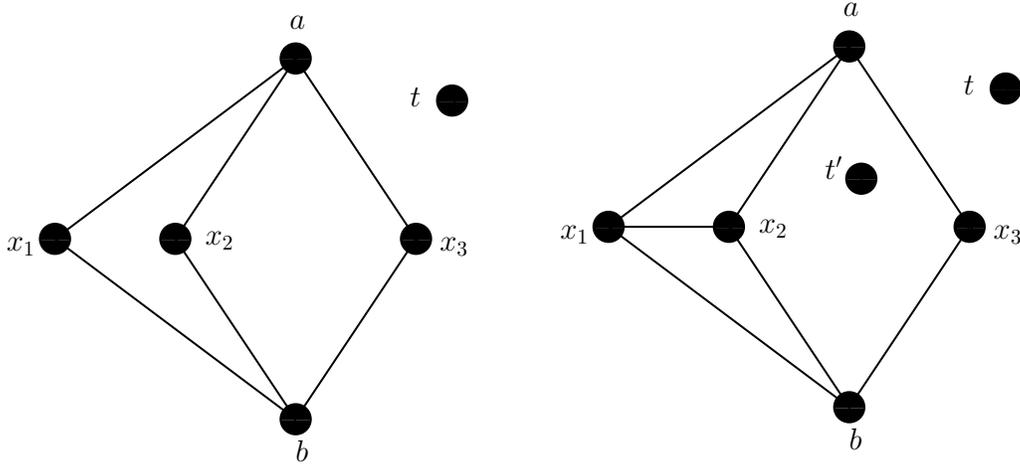


Figure 9: Possible situations for $K_{2,3}$ in G

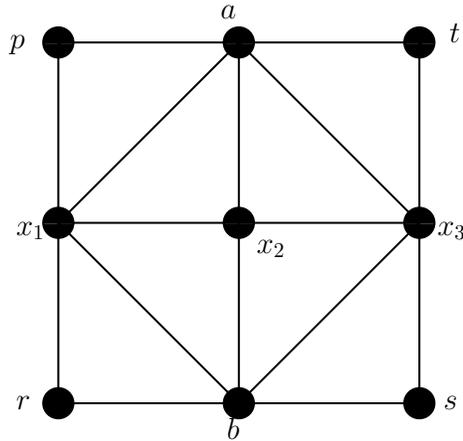


Figure 10: Partial extension of $K_{2,3}$ subgraph

and $x_2x_3 \in E(G)$. First we note that by Claim 6 any other vertex of G must be in the outside of ax_3bx_1 (see Figure 10).

Let t be any such vertex. Since t is not adjacent to x_2 , in order for t and x_2 to be in a common 4-cycle, t should be adjacent to at least two neighbors of x_2 . However it cannot be adjacent to both a and b as otherwise we would have $K_{2,4} \subseteq G$ which contradicts Claim 4. Similarly t cannot be adjacent to both x_1 and x_3 . Thus it must be adjacent to both ends of an edge of the ax_3bx_1 -cycle. Furthermore for each edge of this cycle there can be at most one vertex, other than x_2 , adjacent to both ends. Because if there were two such vertices, together with x_2 , they would produce a $K_{2,3}^+$ subgraph of G , contradicting Claim 5. Since G has at least nine vertices, this implies that G has exactly nine vertices and we have the graph of Figure 10 as a subgraph of G . We note that connecting a to b would produce $K_{2,3}^+$ and connecting a to r or s would produce $K_{2,4}$ both of which were proved to be forbidden subgraphs of G . Hence, in the graph of Figure 10, the vertex a is already adjacent to all its neighbors in G . The same holds for b , x_1 and x_3 . Thus to form a triangulation of the graph of Figure 10, and by the symmetry of

t , p , r and s , we must have a graph isomorphic to the graph of Figure 8. However, by Claim 7, this graph admits no signature under which it would form an S-clique.

Claim 9 $\delta(G) \geq 5$ (and thus $\delta(G) = 5$).

Since G is a triangulation with more than 3 vertices, it has no vertex of degree 2 or less. If x is a vertex of degree 3, then, together with its neighbors, it will induce a K_4 and by Claim 6 all the faces of K_4 are also faces of G ; hence G has only 4 vertices. If x is a vertex of degree 4, then, since G is a triangulation, together with its neighbors it will create a $K_{2,3}$ subgraph which contradicts Claim 8.

We now complete the proof of the theorem. Let v be a vertex of degree 5. Since G is a triangulation, its neighbors form a 5-cycle C_5 . Each vertex not adjacent to v must be joined to two vertices of this C_5 , but no two of them can be adjacent to a same pair as otherwise we contradict Claim 8. Therefore, by planarity of G , there can be at most seven such vertices. On the other hand, by Claim 9 and by the Euler formula, either G has twelve vertices all of degree 5 or thirteen vertices which all but one are of degree 5 and the last one is of degree 6. It follows from the degree conditions for the vertices of the C_5 induced by $N(v)$ that there are at most eleven edges connecting neighbors of v to non-neighbors of v , but each such non-neighbor is joined to at least two neighbors of v . Hence there are a total of at most five non-neighbors of v and hence G has at most eleven vertices, which is a contradiction. \square

Corollary 7.2 *The absolute S-clique number of a planar signed graph is at most 8. This bound is tight.*

Some bounds on the relative S-clique number of planar signed graphs follow from bounding their signed chromatic number, but we do not know the optimal bound for the relative S-clique number of planar signed graphs.

8 The signed chromatic number of minor closed families of signed graphs

Towards generalization of the Four-Color Theorem (or the Five-Color Conjecture at that time), K. Wagner [W64] proved that the chromatic number of any proper minor-closed family is bounded by a constant. Hadwiger's conjecture is to find the best such constant for certain minor-closed families of graphs. We note that such a general result is not true for signed graphs. For example, the class \mathcal{C} of all signed graphs not containing $(K_3, E(K_3))$ as a signed minor contains all the signed graphs (G, \emptyset) and, therefore, admits no bound on its signed chromatic number. In this section we show that some stronger minor condition would imply a constant bound on the signed chromatic number. We start with signed graphs (G, Σ) where G is K_4 -minor-free, in which case we give the best possible bound. Recall that $SPal_5$ is the graph of Figure 4.

Theorem 8.1 *Let (G, Σ) be a signed graph where G is a K_4 -minor-free graph. Then $(G, \Sigma) \rightarrow SPal_5$. Therefore $\chi(G, \Sigma) \leq 5$ and, moreover, this bound is tight.*

Proof. Without loss of generality we may assume that G is an edge-maximal K_4 -minor-free graph. A classical decomposition theorem for edge-maximal K_4 -minor-free graphs states that

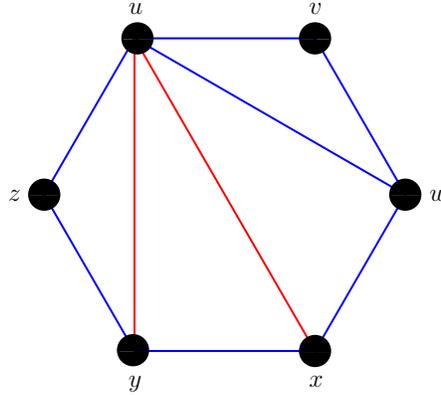


Figure 11: A 5-chromatic planar signed graph

such a graph is built from a sequence of triangles, starting by one triangle and pasting each new triangle to the graph previously built along an edge. Let T_1, \dots, T_ℓ denote the corresponding sequence of triangles. Consider the first triangle T_1 . If all the edges are of the same sign, then we resign at a vertex. Now T_1 has at least one negative and at least one positive edge. So it can easily be homomorphically mapped to $SPal_5$. Inductively, assume that (G_i, Σ_i) , defined as the signed subgraph induced by T_1, \dots, T_i , $i < \ell$, is mapped to $SPal_5$. Consider T_{i+1} . If all the edges are of the same sign, then resign at the vertex of T_{i+1} which is not in G_i . Now, since T_{i+1} is a triangle of possible form in $SPal_5$ and by the main property of $SPal_5$ (see Example 5.14), we can extend the homomorphism of G_i to $SPal_5$ to a homomorphism of G_{i+1} to $SPal_5$. We note that resigning happens only when a vertex is added to the previously built part of the graph, so the process is well defined.

We thus have $\chi(G, \Sigma) \leq 5$. Let us show that this bound is tight. For that, consider the planar signed graph of Figure 11. By contradiction, suppose f is a 4-coloring of this graph. Since $uvwxy$ is an unbalanced 4-cycle, we may assume $f(u) = 1$, $f(v) = 2$, $f(w) = 3$ and $f(x) = 4$. Since $uxyz$ is also an unbalanced 4-cycle, y and z must be colored 2 and 3. But then the balanced triangle uvw and the unbalanced triangle uyz receive the same set of colors, which is a contradiction. \square

Since every outerplanar graph is K_4 -minor-free and since the example of Figure 11 is outerplanar, we get:

Corollary 8.2 *The signed chromatic number of every outerplanar signed graph is at most 5 and this bound is tight.*

For the class of planar signed graphs we do not know the maximum possible value of the signed chromatic number, but using the bound on the acyclic chromatic number of planar graphs and techniques similar to that of [RS94] and [AM98], we obtain an upper bound of 48. In [AM98], Alon and Marshall proved that every m -edge-colored graph whose underlying graph has acyclic chromatic number at most k admits a homomorphism to an m -edge-colored graph of order at most km^{k-1} . This result has been generalized to colored mixed graphs by Nešetřil and Raspaud [NR00] (see also Montejano *et al.* [MOPRS10]). In case of signed graphs, thanks to resigning, we obtain an improved bound as follows.

Theorem 8.3 *If G is acyclically k -colorable and Σ is any signature on G , then $\chi_s(G, \Sigma) \leq \lceil \frac{k}{2} \rceil 2^{k-1}$.*

Proof. The result is immediate when $k \leq 2$. Hence that we assume $k \geq 3$. Let $\varphi : V(G) \rightarrow \{0, \dots, k-1\}$ be an acyclic k -coloring of G . For any two colors i and j , $0 \leq i < j \leq k-1$, let $F_{i,j}$ denote the forest induced by vertices of color i or j .

Let now (G, Σ) be any signed graph with underlying graph G . We first resign the $\lfloor k/2 \rfloor$ vertex-disjoint forests $\{F_{2p, 2p+1}, 0 \leq p \leq \lfloor k/2 \rfloor - 1\}$ in such a way that all their edges become positive (this can be done according to Corollary 3.2). We denote by (G, Σ') the so-obtained signed graph.

Let (H_k, Θ_k) be the signed graph defined as follows. The vertices of H_k are the $(k+1)$ -tuples $[\alpha; a_0, \dots, a_{k-1}]$ where α is one of the k colors of the acyclic coloring of G and $a_i \in \{*, 0, 1\}$, for every i , $0 \leq i \leq k-1$, satisfying the following rules:

1. $\alpha \in \{0, \dots, k-1\}$,
2. $a_\alpha = *$,
3. if α is even and $\alpha < k-1$ then $a_{\alpha+1} = *$,
4. if α is odd then $a_{\alpha-1} = *$,
5. $a_i \in \{0, 1\}$ otherwise.

Note that the number of vertices of H_k is precisely $k2^{k-2}$ if k is even, and $(k+1)2^{k-2}$ if k is odd.

There is an edge in H_k linking vertices $[\alpha; a_0, \dots, a_{k-1}]$ and $[\beta; b_0, \dots, b_{k-1}]$ if and only if $\alpha \neq \beta$. The set of negative edges Θ_k of H_k is then the set of pairs $\{[\alpha; a_0, \dots, a_{k-1}], [\beta; b_0, \dots, b_{k-1}], \}$ such that either $\alpha \neq \beta$ and $\lfloor \alpha/2 \rfloor = \lfloor \beta/2 \rfloor$, or $\lfloor \alpha/2 \rfloor \neq \lfloor \beta/2 \rfloor$ and $a_\beta = b_\alpha$. It is not difficult to observe that (H_k, Θ_k) is indeed an S-clique.

We claim that (G, Σ') admits a homomorphism to (H_k, Θ_k) which will prove the Theorem.

Let $F_{i,j}$ be any forest not belonging to the set $\{F_{2p, 2p+1}, 0 \leq p \leq \lfloor k/2 \rfloor - 1\}$. We claim that there exists a mapping $\lambda_{i,j} : V(F_{i,j}) \rightarrow \{0, 1\}$ such that for every edge uv in $F_{i,j}$, $uv \in \Sigma'$ if and only if $\lambda_{i,j}(u) = \lambda_{i,j}(v)$. Such a mapping can be inductively constructed as follows. Take any connected component $T_{i,j}$ of $F_{i,j}$, any arbitrary vertex u_0 of $T_{i,j}$, and set $\lambda_{i,j}(u_0) = 0$. Assume that the mapping $\lambda_{i,j}$ has been defined for all vertices $\{u_0, \dots, u_{i-1}\}$ of a connected subtree of $T_{i,j}$ and let u_i be any vertex of $T_{i,j}$ linked by an edge to some (unique) $u_j \in \{u_0, \dots, u_{i-1}\}$. We then set $\lambda_{i,j}(u_i) = \lambda_{i,j}(u_j)$ if $u_i u_j \in \Sigma'$ and $\lambda_{i,j}(u_i) = 1 - \lambda_{i,j}(u_j)$ otherwise. Repeating this procedure for every connected component of $F_{i,j}$, we clearly obtain the desired mapping.

For every i , $0 \leq i \leq k-1$, let $\lambda_{i,i}$ be the mapping defined by $\lambda_{i,i}(u) = *$ for every $u \in V(G)$. Similarly, for every i , $0 \leq i \leq k-1$, i odd (resp. i even and $i < k-1$) let $\lambda_{i,i-1}$ (resp. $\lambda_{i,i+1}$) be the mapping defined by $\lambda_{i,i-1}(u) = *$ (resp. $\lambda_{i,i+1}(u) = *$) for every $u \in V(G)$.

For convenience, we let $\lambda_{j,i} = \lambda_{i,j}$ for every i and j , $0 \leq i, j \leq k-1$.

We now claim that the mapping $h : V(G) \rightarrow V(H_k)$ defined by

$$h(u) = [\varphi(u); \lambda_{0, \varphi(u)}(u), \dots, \lambda_{k-1, \varphi(u)}(u)]$$

is a homomorphism of (G, Σ') to (H_k, Θ_k) . Note first that, thanks to the definition of the mappings $\lambda_{i,j}$, $h(u) \in V(H_k)$ for every $u \in V(G)$. Moreover, since φ is an acyclic coloring, and thus a proper coloring, every edge uv of G is mapped to an edge of H (the first components of

$h(u)$ and $h(v)$ are distinct and, therefore, $h(u)$ and $h(v)$ are linked by an edge in H_k . It remains to show that an edge uv of G belongs to Σ' if and only if its image $h(u)h(v)$ belongs to Θ_k .

If $uv \in E(F_{i,j})$ for some $F_{i,j} \in \{F_{2p,2p+1}, 0 \leq p \leq \lfloor k/2 \rfloor - 1\}$ then $uv \notin \Sigma'$ and, by definition of $\lambda_{\varphi(u),\varphi(v)}$, $h(u)h(v) \notin \Theta_k$.

Otherwise, thanks to the property of $\lambda_{\varphi(u),\varphi(v)}$, we have $uv \in \Sigma'$ if and only if $h(u)h(v) \in \Theta_k$, which concludes the proof. \square

A k -tree is a graph obtained from the complete graph K_k by adding a sequence v_1, v_2, \dots, v_r of vertices where each v_i is joined to a set of k vertices that form a k -clique in the subgraph induced by vertices of the original K_k and v_1, v_2, \dots, v_{i-1} . A subgraph of a k -tree is a *partial k -tree*. In particular, K_4 -minor free graphs are exactly partial 2-trees. Since every k -tree is obviously acyclically $(k+1)$ -colorable, we have:

Corollary 8.4 *If G is a partial k -tree and Σ any subset of $E(G)$, then $\chi(G, \Sigma) \leq \lceil \frac{k+1}{2} \rceil 2^k$.*

This, in particular, gives an upper bound of 8 (respectively 16) for the signed chromatic number of (G, Σ) where G is a K_4 -minor-free graph (respectively a partial 3-tree). The former was improved in Theorem 8.1 using $SPal_5$ and the latter can be improved to 13 with $SPal_{13}$ as the target with the same method as in the proof of Theorem 8.1, using stronger properties of $SPal_{13}$.

Using the bounds on the acyclic chromatic number of planar graphs we also have the following:

Theorem 8.5 *If (G, Σ) is a planar signed graph, then $\chi(G, \Sigma) \leq 48$. There is a planar S -clique of order 8 and there is a planar signed graph with signed chromatic number 10.*

Proof. The upper bound of 48 follows from Theorem 8.3 and the fact that every planar graph is acyclically 5-colorable [B79]. An example of a 10-chromatic planar signed graph is given in Figure 12. Note that this signed graph is built from the S -clique of Figure 6 by adding two pairs of vertices, one on the right and one on the left. It is then not difficult to check that for each pair we need at least one more new color and that the pair on the right needs distinct colors than that on the left. \square

We further note that if the maximum signed chromatic number of planar signed graphs is say k , then there exists a signed graph of order k to which every planar signed graph admits a signed homomorphism. Perhaps it would be possible to prove, directly, that every planar signed graph admits a homomorphism to a fixed signed Paley graph.

9 Hadwiger's conjecture for signed bipartite graphs

We saw that odd Hadwiger's conjecture proposes a possible strengthening of Hadwiger's conjecture for the class of odd signed graphs. In this section we examine possibilities of such a strengthening for the class of even signed graphs. i.e., signed bipartite graphs. Recall that for every graph G , the signed graph $S(G)$ is obtained from G by replacing each edge uv of G by an unbalanced 4-cycle $ux_{uv}vy_{uv}$, where x_{uv} and y_{uv} are new and distinct vertices.

We first prove the following minor relation between graphs and signed bipartite graphs:

Theorem 9.1 *For every integer n and every graph G , G has a K_n -minor if and only if $S(G)$ has a (K_n, Σ) -minor for some Σ (equivalently for any Σ).*

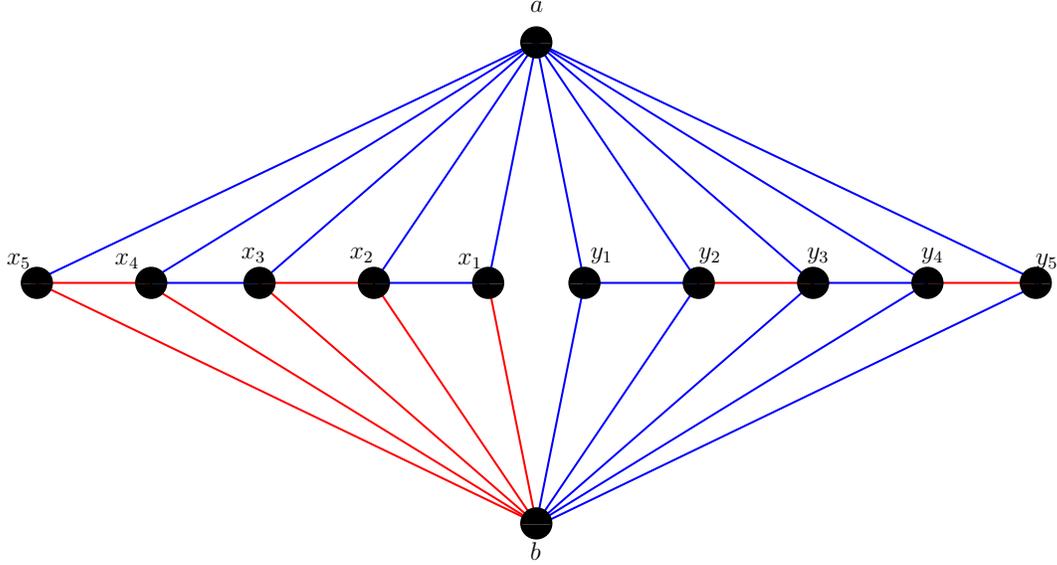


Figure 12: A 10-chromatic planar signed graph on 12 vertices

Proof.

First assume (K_n, Σ) is a signed minor of $S(G)$ for some Σ . We would like to prove that K_n is a minor of G . This is clear for $n = 1, 2$. So we assume $n \geq 3$. Thus, in producing (K_n, Σ) as a signed minor of $S(G)$ each vertex of degree 2 in $S(G)$ is either deleted or identified with one of its neighbours as a result of contracting an incident edge. We define a minor of G as follows: For each edge uv of G , if the corresponding unbalanced 4-cycle is deleted in the process of producing (K_n, Σ) as a signed minor of $S(G)$, then delete uv . If u and v are identified through contraction of edges in producing (K_n, Σ) as a signed minor of $S(G)$, then contract the edge uv . Otherwise uv remains an edge. The resulting minor then must be K_n .

For the opposite direction, suppose K_n is a minor of G . Let uv be an edge of G . If the edge uv is deleted in producing K_n -minor from G , then delete all the four edges of corresponding unbalanced 4-cycle. If uv is contracted, then contract two positive edges of the corresponding unbalanced 4-cycle in $S(G)$ in such a way that u and v are identified after these contractions and delete the other two edges of the unbalanced 4-cycle. Otherwise contract two positive edges of the corresponding unbalanced 4-cycle in such a way that there are two new parallel edges between u and v , one positive and one negative. Finally delete all isolated vertices. By allowing multiple edges at the end of this process we get a signed minor of $S(G)$ which has n vertices and for each pair x and y of vertices two xy edges, one positive and one negative. For each such pair we delete the negative edge unless $xy \in \Sigma$ in which case we delete the positive edge. The result is (K_n, Σ) obtained as a signed minor of G . \square

By Theorem 6.2, Hadwiger’s conjecture can be restated as follows:

Conjecture 9.2 (Hadwiger’s conjecture restated) *Given $n \geq 4$, the class $\mathcal{C} = \{S(G) \mid G \text{ is } K_n\text{-minor-free}\}$ of signed bipartite graphs is bounded by $(K_{n-1, n-1}, M)$ in the signed graph homomorphism order.*

If the conjecture holds, then the next question would be: what is a natural superclass of \mathcal{C} which is still bounded by $(K_{n-1, n-1}, M)$?

Hadwiger's conjecture is known to be true for $n \leq 6$, thus Conjecture 9.2 is also true for $n \leq 6$. For $n = 4$ we have the following generalization.

Theorem 9.3 *If G is a bipartite graph with no K_4 -minor and Σ is any signature on G , then $(G, \Sigma) \rightarrow (K_{3,3}, M)$.*

Proof. By adding more edges, if needed, we may assume that G is edge maximal with respect to being bipartite and having no K_4 -minor. Obviously it is enough to prove the theorem for such edge maximal graphs.

As mentioned before, a classical decomposition theorem for edge-maximal K_4 -minor-free graphs states that every such graph is built from a sequence of triangles starting by one triangle and pasting each new triangle to the graph previously built along an edge. To use the decomposition theorem we add new edges to G , of green color, until we reach a maximal K_4 -minor-free graph G' , which obviously is not bipartite anymore. Let G'' be the edge-colored graph obtained from G' by coloring original positive edges of (G, Σ) in blue, original negative edges of (G, Σ) in red and keeping the green color for edges not in G .

We claim that there is no triangle in G'' with exactly two green edges. To see this, suppose that v_1v_2 and v_1v_3 are both green and that v_2v_3 is an edge of G . Since G is bipartite v_2 and v_3 are in two different parts and thus v_1 is in a different part with respect to one of them. Without loss of generality assume v_1 and v_2 are in different parts. Consider the graph $G + \{v_1v_2\}$. By the choice of v_2 this graph is bipartite and since it is a subgraph of G' , it has also no K_4 -minor but this contradicts the edge maximality of G .

We now build a new edge-colored graph F from $(K_{3,3}, M)$. The blue and red edges of F are defined as before and we add green edges between every pair of vertices non adjacent in $(K_{3,3}, M)$. The edge-colored graph F has three types of triangles: (i) triangles with three green edges, (ii) triangles with one green edge and two blue edges, and (iii) triangles with no two edges of the same color. Furthermore it is not hard to verify that each red edge only belongs to triangles of type (iii), each blue edge belongs to triangles of type (ii) or (iii) and each green edge is contained in triangles of each of the three types.

To prove the theorem we now prove the following stronger statement: there exists a suitable "resigning" G^* of G'' such that G^* admits a color-preserving homomorphism to F . By resigning here we mean exchanging the colors red and blue on edges of an edge cut, this can be regarded as a sequence of vertex resigning.

To prove this stronger statement, let T_1, \dots, T_k be the sequence of triangles obtained from the decomposition of G'' mentioned above. Note that since G was bipartite, each such triangle contains a green edge. Consider the triangle T_1 . Either it is one of the three types (i), (ii) or (iii), in which case we simply map it to F , or it has one green and two red edges. Let u be the common vertex of these two red edges. After resigning at u we have a triangle of type (ii) and thus we can map it to F .

By induction, assume now that the graph G''_i , obtained by pasting the triangles T_1, \dots, T_i , $i < k$, is mapped to F and assume that T_{i+1} is pasted to G''_i along the edge e . Let v be the vertex of T_{i+1} not incident to e . If T_{i+1} is a triangle of one the three types, because of the above mentioned property of F , we can extend the mapping of G''_i to G''_{i+1} , where the colors of the two edges of T_{i+1} incident with v are preserved. Otherwise T_{i+1} has exactly two red edges and one green edge. By resigning at v we get a triangle that has either one or no red edge, thus obtaining a triangle of type (ii) or (iii). We now extend the homomorphism thanks to the properties of F . In this process, resigning a vertex would be done at most once, when it is added to the already built part of the graph, so our process is well-defined and the stronger claim is proved. \square

We note that our proof has an algorithmic feature. Given a signed bipartite graph (G, Σ) , where G is a K_4 -minor-free graph, we can find, in polynomial time, a homomorphism of (G, Σ) to $(K_{3,3}, M)$.

Furthermore, we believe that the following stronger statement should also be true:

Conjecture 9.4 *If G is bipartite and (G, Σ) has no $(K_4, E(K_4))$ as a signed minor, then $(G, \Sigma) \rightarrow (K_{3,3}, M)$.*

For $n = 4$ it is shown in [NRS12] that the following holds.

Theorem 9.5 *If G is a bipartite planar graph and Σ is any signature on G , then $(G, \Sigma) \rightarrow (K_{4,4}, M)$.*

This theorem is indeed stronger than the Four-Color Theorem and it does use the Four-Color Theorem in its proof. We believe that using Wagner's decomposition theorem of edge-maximal K_5 -minor-free graphs and with a method similar to that of [NNS09] the condition of planarity can be replaced with the more relaxed condition of having no K_5 -minor. However the following extension, proposed by B. Guenin [G05] is a lot more challenging:

Conjecture 9.6 *Suppose G is a bipartite graph and Σ is any signature on G . If (G, Σ) does not have $(K_5, E(K_5))$ as a signed minor then $(G, \Sigma) \rightarrow (K_{4,4}, M)$.*

For large values of n ($n \geq 7$) we show that no such simple conjecture would hold. This could be regarded as a first negative indication for Hadwiger's conjecture for $n \geq 7$.

Theorem 9.7 *There exists no value of n for which $Fano$ (the signed bipartite graph of Figure 3) admits a homomorphism to $(K_{n,n}, M)$.*

Proof. Since $Fano$ is an S-clique, any homomorphic image of $Fano$ is isomorphic to itself. Thus, if $Fano$ maps to $(K_{n,n}, M)$, then its image should be of the form $(K_{7,7}, M')$ where M' is a matching of size 7 or less induced by M on $K_{7,7}$. If there are two vertices of the same part of $K_{7,7}$ not matched by M' , then identifying them would result in a signed homomorphic image of order at most 13 of $Fano$ which is a contradiction.

Thus we consider two cases, $|M'| = 7$ or $|M'| = 6$. In each case, by counting the number of unbalanced 4-cycles containing a pair of non-adjacent vertices, we show that $Fano$ cannot be isomorphic to $(K_{7,7}, M')$. Note that there are exactly 12 unbalanced 4-cycles containing an arbitrary pair of non-adjacent vertices of $Fano$. For $(K_{7,7}, M')$ with $|M'| = 7$ the number of unbalanced 4-cycles containing any pair of non-adjacent vertices is 10. For $(K_{7,7}, M')$ with $|M'| = 6$, this number is either 10 or 6. \square

Corollary 9.8 *The class $\mathcal{C} = \{(G, \Sigma) \mid G \text{ is bipartite and has no } H\text{-minor}\}$ is not bounded by $(K_{n,n}, M)$ (for no values of n) if H is a graph on at least 15 vertices.*

This shows that for $n \geq 15$ the reformulation of Hadwiger's conjecture given in Conjecture 9.2 cannot be extended to a general minor closed class of signed bipartite graphs. Even though such an extension was possible for small values of n .

We note that to prove Hadwiger's conjecture for a K_n -minor free graph G , using a restatement in the suborder of signed bipartite graphs, one does not need to map the whole $S(G)$ to $(K_{n-1, n-1}, M)$. It is rather enough to map $S(G)$ to any signed bipartite graph in which the part which is the image of the vertices of G is of size at most $n - 1$. This leads us to the following

definition of bipartite chromatic number and a relaxation of Hadwiger's conjecture. Given a signed bipartite graph (G, Σ) the *bipartite chromatic number* of (G, Σ) , denoted $\chi_b(G, \Sigma)$, is the smallest n such that $(G, \Sigma) \rightarrow CB_n$ (see Example 5.9 for the definition of the signed graph CB_n). Intuitively speaking, the bipartite chromatic number is the smallest number of vertices on one part of a signed bipartite graph to which (G, Σ) admits a homomorphism. We propose the following relaxation of Hadwiger's conjecture:

Conjecture 9.9 *If G is a K_n -minor free graph then $\chi_b(S(G)) \leq n - 1$.*

It is then natural to consider the problem of finding

$$f(n) = \max\{\chi_b(G, \Sigma) \mid G \text{ is a } K_n - \text{minor free bipartite graph}\}.$$

If $f(n)$ was equal to $n - 1$ it would imply Hadwiger's conjecture. This is indeed the case for $n = 4$ (using Theorem 9.3). Perhaps using Theorem 9.5 and Wagner's decomposition of K_5 -minor free graphs it would not be too difficult to verify that $f(5) = 4$. However as the following example shows, in general $f(n)$ is far from $n - 1$. This is another indication that perhaps Hadwiger's conjecture is true only for small chromatic numbers.

Example 9.10 Let S_1 and S_2 be two vertex disjoint copies of CB_{n-2} . Note that K_{n-1} is the largest clique minor of $K_{n-2, 2^{n-3}}$ (underlying graph of CB_{n-2}). Consider two vertices x and y from S_1 and S_2 such that x is from the larger part of S_1 and y is from the smaller part of S_2 . Let (S, Σ) be the signed graph obtained from S_1 and S_2 by identifying vertices x and y . It is easy to check that S is a K_n -minor free bipartite graph. Let (B, Σ') be a signed bipartite graph to which (S, Σ) admits a homomorphism to and let φ be such a homomorphism. Let B_1 and B_2 be the two parts of B . As a homomorphism of S to B , φ preserves the bipartition of S . Since the larger part of S_1 and S_2 are in different parts of S , each part B_i of B is a range for a larger part of CB_{n-2} for some mapping of CB_{n-2} to B . But since CB_{n-2} is an S-clique each part of B must be of size at least 2^{n-3} .

10 Prospects

We have just opened a door to an ocean of problems in direction of some of the most motivational problems in graph theory such as the Four-Color Theorem and Hadwiger's conjecture. Hence it is not possible to list all the problems we would like to continue working on. But beside the questions we asked in the text, there are a few more questions which we think should be mentioned here.

Problem 10.1 *How far can Theorem 4.1 be extended to signed graphs? In particular does the straightforward extension hold for the families of odd signed graphs and of signed bipartite graphs? Furthermore, when there is such an extension, what is the optimal bound in terms of number of vertices?*

As a special case to the previous question we introduce the following conjecture which is the bipartite analog of the (odd) graph homomorphism problem studied in [N12].

Conjecture 10.2 *Every planar signed bipartite graph of unbalanced girth $2g$ admits a homomorphism to SPC_{2g-1} . Furthermore SPC_{2g-1} is the smallest signed bipartite graph of unbalanced girth $2g$ which bounds the class of all planar signed bipartite graphs of unbalanced girth $2g$.*

This question is related to several other well-known results and conjectures. We refer to [G05] and [NRS12] for further study on this question.

A bipartite analog of Problem 4.2 is the following problem which contains Conjecture 10.2 as a particular case:

Problem 10.3 *What is the smallest signed bipartite graph of unbalanced girth $2k$ to which every planar signed bipartite graph of unbalanced girth $2r$ ($r \geq k$) admits a homomorphism?*

We think the answer in each case should be a subgraph of \mathcal{SPC}_{2k-1} . While for the extreme case of $k = r$ we propose the signed projective cubes to be the answer, for the other extreme, i.e., when r is large enough with respect to k , a simple discharging method would imply that UC_{2k} is the answer. The exact value of r for which UC_{2k} is the answer for this question is the subject of the next conjecture which can also be regarded as the bipartite analog of Jaeger-Zhang’s conjecture. For further references and for the best current result on Jaeger-Zhang’s conjecture we refer to [BKKW04].

Conjecture 10.4 *Every planar signed bipartite graph of unbalanced girth $4g - 2$ admits a homomorphism to UC_{2g} .*

A positive answer for Conjecture 10.2 for $g = 2$ given in [NRS12] implies that every planar signed bipartite graph admits a homomorphism to $(K_{4,4}, M)$, and thus a bound of 8 for the signed chromatic number of this family of graphs. We do not know if 8 is the best bound for this. Furthermore, it would be interesting to give a proof of this weaker statement without using the Four-Color Theorem.

Problem 10.5 *What is the largest chromatic number of a planar signed bipartite graph?*

We would also like to ask if the reformulation given by Conjecture 2.1 of Hadwiger’s conjecture can be extended to the odd Hadwiger’s conjecture:

Problem 10.6 *Is Conjecture 6.1 equivalent to saying that every signed minor-closed family of odd signed graphs have a maximum with respect to the signed homomorphism order?*

At the end we should also mention the algorithmic point of view. The problem (G, Σ) -COLORING-OF-SIGNED-GRAPHS can be difficult from two aspects: sometimes it is difficult to find a required mapping, sometimes it is difficult to find an equivalent signature of the input graph which would provide the homomorphism, but most of the time it is difficult to do either of the two tasks. In general it is conjectured in [FN12] that the following dichotomy holds:

Conjecture 10.7 *The problem (G, Σ) -COLORING-OF-SIGNED-GRAPHS is NP-complete unless $\chi(G, \Sigma) = 2$.*

This would extend the dichotomy result of [HN90] and propose a new extension of the dichotomy conjecture of [FV98] through an extension of the definitions from signed graphs to signed structural relations. It has been shown in [FN12] that the problem UC_k -COLORING is NP-complete even if the input signed graph is restricted to be in the class of planar signed graphs.

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