

Packing colouring of some classes of cubic graphs

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Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that its set of vertices $V(G)$ can be partitioned into k disjoint subsets V_1, \dots, V_k , in such a way that every two distinct vertices in V_i are at distance greater than i in G for every i , $1 \leq i \leq k$.

We determine the exact value of, or upper and lower bounds on the packing chromatic number of several classes of cubic graphs, namely circular ladders, H-graphs and generalised H-graphs.

Keywords: Packing colouring; Packing chromatic number; Circular ladder; H-graph; Generalised H-graph.

MSC 2010: 05C15, 05C12, 05C70.

1 Introduction

All the graphs we consider are simple and loopless. For a graph G , we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. The *distance* $d_G(u, v)$ between vertices u and v in G is the length (number of edges) of a shortest path joining u and v . The *diameter* of G is the maximum distance between two vertices of G . We denote by P_n , $n \geq 1$, the path of order n and by C_n , $n \geq 3$, the cycle of order n .

A *packing k -colouring* of G is a mapping $\pi : V(G) \rightarrow \{1, \dots, k\}$ such that, for every two distinct vertices u and v , $\pi(u) = \pi(v) = i$ implies $d_G(u, v) > i$. The *packing chromatic number* $\chi_\rho(G)$ of G is then the smallest k such that G admits a packing k -colouring. In other words, $\chi_\rho(G)$ is the smallest integer k such that $V(G)$ can be partitioned into k disjoint subsets V_i , $1 \leq i \leq k$, in such a way that every two vertices in V_i are at distance greater than i in G for every i , $1 \leq i \leq k$. A packing colouring of G is *optimal* if it uses exactly $\chi_\rho(G)$ colours.

The packing colouring of graphs was introduced by Goddard, Hedetniemi, Hedetniemi, Harris and Rall in [19, 20], under the name *broadcast colouring*, and studied by many authors since then [2–18, 21–31].

In the seminal paper of Goddard *et al.* [20], the question of determining the maximum packing chromatic number in the class of cubic graphs of a given order is posed. In [26], Sloper proved that the packing chromatic number is unbounded in the class of k -ary trees for every $k \geq 3$, from which it follows that the packing chromatic number is unbounded in the class of graphs with maximum degree 4.

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In [18], Gastineau and Togni observed that each cubic graph of order at most 20 has packing chromatic number at most 10. They also observed that the largest cubic graph with diameter 4 (this graph has 38 vertices and is described in [1]) has packing chromatic number 13, and asked whether there exists a cubic graph with packing chromatic number larger than 13 or not. This question was answered positively by Brešar, Klavžar, Rall and Wash [9] who exhibited a cubic graph on 78 vertices with packing chromatic number at least 14. Recently, Balogh, Kostochka and Liu finally proved in [5] that the packing chromatic number is unbounded in the class of cubic graphs.

On the other hand, the packing chromatic number is known to be upper bounded in several classes of graphs with maximum degree 3, e.g. complete binary trees [26], hexagonal lattices [6, 14, 21], base-3 Sierpiński graphs [7], subdivisions of subcubic graphs [8, 18], or several subclasses of outerplanar subcubic graphs [17].

We prove in this paper that the packing chromatic number is bounded in other classes of cubic graphs, extending in particular partial results given in [30] and [31]. More precisely, we determine the exact value of, or upper and lower bounds on the packing chromatic number of circular ladders (in Section 3), H-graphs (in Section 4) and generalised H-graphs (in Section 5).

2 Preliminary results

We give in this section a few results that will be useful in the sequel.

Let G be a graph. A subset S of $V(G)$ is an i -packing, for some integer $i \geq 1$, if any two vertices in S are at distance at least $i + 1$ in G . Note that such a set S is a 1-packing if and only if S is an independent set. A packing colouring of G is thus a partition of $V(G)$ into k disjoint subsets V_1, \dots, V_k , such that V_i is an i -packing for every i , $1 \leq i \leq k$.

For every integer $i \geq 1$, we denote by $\rho_i(G)$ the maximum cardinality of an i -packing in G . Since at most $\rho_i(G)$ vertices can be assigned colour i in any packing colouring of G , we have the following result.

Proposition 1 *If G is a graph with $\chi_\rho(G) = k$, then*

$$\sum_{i=1}^{i=k} \rho_i(G) \geq |V(G)|.$$

Let H be a subgraph of G . Since $d_G(u, v) \leq d_H(u, v)$ for any two vertices $u, v \in V(H)$, the restriction to $V(H)$ of any packing colouring of G is a packing colouring of H . Hence, having packing chromatic number at most k is a hereditary property:

Proposition 2 (Goddard, Hedetniemi, Hedetniemi, Harris and Rall [20])

Let G and H be two graphs. If H is a subgraph of G , then $\chi_\rho(H) \leq \chi_\rho(G)$.

In particular, Proposition 2 allows to get a lower bound on the packing chromatic number of graph G whenever G contains a subgraph H whose packing chromatic number is known. As we will see later, all the cubic graphs we consider in this paper contain a corona of a cycle as a subgraph. Recall that the *corona* $G \odot K_1$ of a graph G is the graph obtained from G by adding a degree-one neighbour to every vertex of G . In [23], we have determined the packing chromatic number of coronae of cycles.

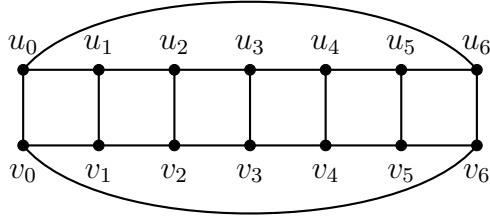


Figure 1: The circular ladder CL_7 .

Theorem 3 (Laïche, Bouchemakh, Sopena [23])

The packing chromatic number of the corona graph $C_n \odot K_1$ is given by:

$$\chi_\rho(C_n \odot K_1) = \begin{cases} 4 & \text{if } n \in \{3, 4\}, \\ 5 & \text{if } n \geq 5. \end{cases}$$

This result will thus provide a lower bound on the packing chromatic number of each cubic graph considered in this paper.

3 Circular ladders

Recall that the Cartesian product $G \square H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, two vertices (u, u') and (v, v') being adjacent if and only if either $u = v$ and $u'v' \in E(H)$ or $u' = v'$ and $uv \in E(G)$.

The *circular ladder* CL_n of length $n \geq 3$ is the Cartesian product $CL_n = C_n \square K_2$. Note that CL_n is a bipartite graph if and only if n is even.

For every circular ladder CL_n , we let

$$V(CL_n) = \{u_0, \dots, u_{n-1}\} \cup \{v_0, \dots, v_{n-1}\},$$

and

$$E(CL_n) = \{u_i v_i \mid 0 \leq i \leq n-1\} \cup \{u_i u_{i+1}, v_i v_{i+1} \mid 0 \leq i \leq n-1\}$$

(subscripts are taken modulo n). Figure 1 depicts the circular ladder CL_7 .

Note that for every $n \geq 3$, the corona graph $C_n \odot K_1$ is a subgraph of the circular ladder CL_n . Therefore, by Proposition 2, Theorem 3 provides a lower bound on the packing chromatic number of circular ladders. More precisely, $\chi_\rho(CL_n) \geq 4$ if $n \in \{3, 4\}$, and $\chi_\rho(CL_n) \geq 5$ if $n \geq 5$.

William and Roy [31] proved that the packing chromatic number of a circular ladder of length $n = 6q$, $q \geq 1$, is at most 5. In Theorem 7 below, we extend this result and determine the packing chromatic number of every circular ladder.

We first need the following technical lemma, which will also be useful in Section 5.

Lemma 4 *Let X be the graph depicted in Figure 2, and π be a packing 5-colouring of X . If $\pi(u_i) \neq 1$ and $\pi(v_i) \neq 1$ for some integer i , $3 \leq i \leq 5$, then either u_i or v_i has colour 2, and its three neighbours have colours 3, 4 and 5 (the three corresponding edges are the vertical edges surrounded by the dashed box).*

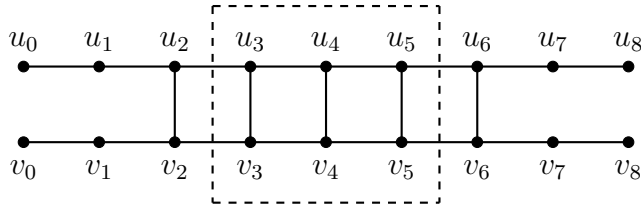


Figure 2: The graph X .

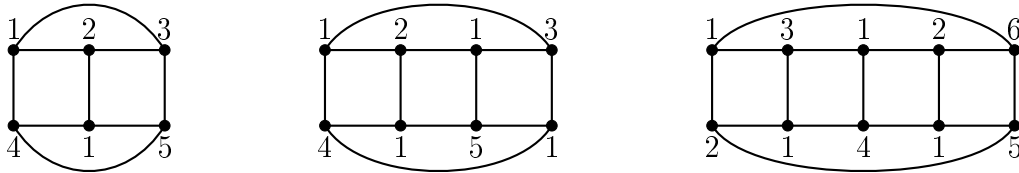


Figure 3: Optimal packing colouring of CL_3 , CL_4 and CL_5 .

Proof. The proof is done by case analysis and is given in Appendix A. \square

Observe now that for every integer $n \geq 9$, the subgraph of CL_n induced by the set of vertices $\{u_i, v_i \mid 0 \leq i \leq 8\}$ contains the graph X of Figure 2 as a subgraph. Moreover, every packing 5-colouring π of CL_n , $5 \leq n \leq 8$, can be “unfolded” to produce a packing 5-colouring of X , by setting $\pi(u_{n-1+i}) = \pi(u_{i-1})$ and $\pi(v_{n-1+i}) = \pi(v_{i-1})$ for every i , $1 \leq i \leq 9 - n$. Therefore, thanks to the symmetries of CL_n for every $n \geq 5$, Proposition 2 and Lemma 4 give the following corollary.

Corollary 5 *Let CL_n , $n \geq 5$, be a circular ladder with $\chi_\rho(CL_n) \leq 5$, and π be a packing 5-colouring of CL_n . For every integer i , $0 \leq i \leq n - 1$, if $\pi(u_i) \neq 1$ and $\pi(v_i) \neq 1$, then either u_i or v_i has colour 2, and its three neighbours have colours 3, 4 and 5.*

Let CL_n be a circular ladder satisfying the hypothesis of Corollary 5, and π be a packing 5-colouring of CL_n . From Corollary 5, it follows that if $\pi(u_i) \neq 1$ and $\pi(v_i) \neq 1$ for some edge $u_i v_i$ of CL_n , then the colour 2 has to be used on the edge $u_i v_i$ and, since the neighbours of the 2-coloured vertex are coloured with 3, 4 and 5, the colour 2 can be replaced by colour 1. Therefore, we get the following corollary.

Corollary 6 *If CL_n , $n \geq 5$, is a circular ladder with $\chi_\rho(CL_n) \leq 5$, then there exists a packing 5-colouring of CL_n such that the colour 1 is used on each edge of CL_n .*

Note that from Corollary 6, it follows that for every integer $n \geq 5$, $\chi_\rho(CL_n) \leq 5$ implies that CL_n is a bipartite graph. Hence, $\chi_\rho(CL_n) \geq 6$ for every odd $n \geq 5$.

We are now able to prove the main result of this section.

Theorem 7 *For every integer $n \geq 3$,*

$$\chi_\rho(CL_n) = \begin{cases} 5 & \text{if } n = 3, \text{ or } n \text{ is even and } n \notin \{8, 14\}, \\ 7 & \text{if } n \in \{7, 8, 9\}, \\ 6 & \text{otherwise.} \end{cases}$$

Proof. We first consider the case $n \leq 5$. Figure 3 describes a packing 5-colouring of CL_3 and CL_4 , and a packing 6-colouring of CL_5 . We claim that these three packing colourings are optimal. To see that, observe that $\rho_1(CL_3) = 2$, $\rho_i(CL_3) = 1$ for every $i \geq 2$, $\rho_1(CL_4) = \rho_1(CL_5) = 4$, $\rho_2(CL_4) = \rho_2(CL_5) = 2$, and $\rho_i(CL_4) = \rho_i(CL_5) = 1$ for every $i \geq 3$. The optimality for CL_3 and CL_5 then follows from Proposition 1. The optimality for CL_4 also follows, with the additional observation that colour 2 can be used at most once if colour 1 is used four times.

Assume now $n \geq 6$. Since $n \geq 6$ and every circular ladder CL_n contains the corona graph $C_n \odot K_1$ as a subgraph, we get $\chi_\rho(CL_n) \geq \chi_\rho(C_n \odot K_1) \geq 5$ by Theorem 3 and Proposition 2. Moreover, by Corollary 6, we have $\chi_\rho(CL_n) \geq 6$ if n is odd.

We now consider two general cases.

1. n is even and $n \notin \{8, 14\}$.

As observed above, in that case, it is enough to exhibit a packing 5-colouring of CL_n to prove $\chi_\rho(CL_n) = 5$.

If $n \equiv 0 \pmod{6}$, a packing 5-colouring of CL_n is obtained by repeating the following circular pattern (the first row gives the colours of vertices u_i , $0 \leq i \leq n-1$, the second row gives the colours of vertices v_i , $0 \leq i \leq n-1$, according to the value of $(i \pmod{6})$):

$$\left\| \begin{array}{cccccc} 1 & 3 & 1 & 2 & 1 & 5 \\ 2 & 1 & 4 & 1 & 3 & 1 \end{array} \right\|$$

If $n \equiv 2 \pmod{6}$, which implies $n \geq 20$, a packing 5-colouring of CL_n is obtained by repeating the previous circular pattern $\frac{n-20}{6}$ times and adding a pattern of length 20, as illustrated below:

$$\left\| \begin{array}{cccccc} 1 & 3 & 1 & 2 & 1 & 5 \\ 2 & 1 & 4 & 1 & 3 & 1 \end{array} \right\| \left\| \begin{array}{cccccccccccccccccccc} 1 & 3 & 1 & 2 & 1 & 3 & 1 & 4 & 1 & 5 & 1 & 3 & 1 & 2 & 1 & 3 & 1 & 4 & 1 & 5 \\ 2 & 1 & 4 & 1 & 5 & 1 & 2 & 1 & 3 & 1 & 2 & 1 & 4 & 1 & 5 & 1 & 2 & 1 & 3 & 1 \end{array} \right\|$$

Finally, if $n \equiv 4 \pmod{6}$, which implies $n \geq 10$, a packing 5-colouring of CL_n is obtained by repeating the same circular pattern $\frac{n-10}{6}$ times and adding a pattern of length 10:

$$\left\| \begin{array}{cccccc} 1 & 3 & 1 & 2 & 1 & 5 \\ 2 & 1 & 4 & 1 & 3 & 1 \end{array} \right\| \left\| \begin{array}{cccccc} 1 & 3 & 1 & 2 & 1 & 3 & 1 & 4 & 1 & 5 \\ 2 & 1 & 4 & 1 & 5 & 1 & 2 & 1 & 3 & 1 \end{array} \right\|$$

2. n is odd and $n \geq 11$.

As observed above, in that case, it is enough to exhibit a packing 6-colouring of CL_n to prove $\chi_\rho(CL_n) = 6$.

Similarly as in the previous case, if $n \equiv 1, 3$ or $5 \pmod{6}$, a packing 6-colouring of CL_n is obtained by repeating the previous circular pattern $\frac{n-7}{6}$, $\frac{n-9}{6}$ or $\frac{n-5}{6}$ times, respectively, and adding a pattern of length 7, 9 or 5, respectively, as illustrated below:

$$\left\| \begin{array}{cccccc} 1 & 3 & 1 & 2 & 1 & 5 \\ 2 & 1 & 4 & 1 & 3 & 1 \end{array} \right\| \left\| \begin{array}{cccccc} 1 & 3 & 1 & 4 & 1 & 2 & 6 \\ 6 & 1 & 2 & 1 & 3 & 1 & 5 \end{array} \right\|$$

$$\left\| \begin{array}{cccccc} 1 & 3 & 1 & 2 & 1 & 5 \\ 2 & 1 & 4 & 1 & 3 & 1 \end{array} \right\| \left\| \begin{array}{cccccccc} 1 & 4 & 1 & 2 & 3 & 1 & 4 & 1 & 6 \\ 2 & 1 & 6 & 1 & 5 & 2 & 1 & 3 & 1 \end{array} \right\|$$

$$\left\| \begin{array}{cccccc} 1 & 3 & 1 & 2 & 1 & 5 \\ 2 & 1 & 4 & 1 & 3 & 1 \end{array} \right\| \left\| \begin{array}{cccccc} 1 & 3 & 1 & 2 & 6 \\ 2 & 1 & 4 & 1 & 5 \end{array} \right\|$$

It remains to consider four cases, namely $n = 7, 8, 9, 14$, which we consider separately.

1. $n = 7$.

We first claim that $\chi_\rho(CL_7) \geq 7$. Note that $\rho_1(CL_7) = 6$, $\rho_2(CL_7) = 3$, $\rho_3(CL_7) = 2$, and $\rho_i(CL_7) = 1$ for every $i \geq 4$. However, if we use six times colour 1, colour 2 can be used at most twice. Hence, at most 13 vertices of CL_7 can be coloured with a colour in $\{1, \dots, 6\}$ and the claim follows.

A packing 7-colouring of CL_7 is given by the following pattern:

$$\begin{array}{cccccc} 1 & 3 & 1 & 2 & 1 & 4 & 5 \\ 2 & 1 & 6 & 1 & 3 & 1 & 7 \end{array}$$

2. $n = 8$.

We first claim that $\chi_\rho(CL_8) \geq 7$. Note that $\rho_1(CL_8) = 8$, $\rho_2(CL_8) = 4$, $\rho_3(CL_8) = \rho_4(CL_8) = 2$, and $\rho_i(CL_8) = 1$ for every $i \geq 5$. However, if we use eight times colour 1, colour 2 can be used at most twice, and then colour 4 at most once. On the other hand, if we use seven times colour 1, colour 2 can be used at most thrice and colour 4 at most once. Hence, at most 15 vertices of CL_8 can be coloured with a colour in $\{1, \dots, 6\}$ and the claim follows.

A packing 7-colouring of CL_8 is given by the following pattern:

$$\begin{array}{cccccc} 1 & 3 & 1 & 2 & 1 & 5 & 1 & 7 \\ 2 & 1 & 4 & 1 & 3 & 1 & 6 & 1 \end{array}$$

3. $n = 9$.

We first claim that $\chi_\rho(CL_9) \geq 7$. Note that $\rho_1(CL_9) = 8$, $\rho_2(CL_9) = 4$, $\rho_3(CL_9) = \rho_4(CL_9) = 2$, and $\rho_i(CL_9) = 1$ for every $i \geq 5$. However, if we use eight times colour 1, colour 2 can be used at most thrice. Hence, at most 17 vertices of CL_9 can be coloured with a colour in $\{1, \dots, 6\}$ and the claim follows.

A packing 7-colouring of CL_9 is given by the following pattern:

$$\begin{array}{cccccc} 1 & 3 & 1 & 2 & 1 & 5 & 1 & 4 & 6 \\ 2 & 1 & 4 & 1 & 3 & 1 & 2 & 1 & 7 \end{array}$$

4. $n = 14$.

We first claim that $\chi_\rho(CL_{14}) \geq 6$. Note that $\rho_1(CL_{14}) = 14$, $\rho_2(CL_{14}) = 6$, $\rho_3(CL_{14}) = 4$, $\rho_4(CL_{14}) = 3$ and $\rho_5(CL_{14}) = 2$. However, if we use 14 times colour 1, colour 2 can be used at most four times. On the other hand, if we use 13 times colour 1, colour 2 can be used at most five times. Hence, at most 27 vertices of CL_{14} can be coloured with a colour in $\{1, \dots, 5\}$ and the claim follows.

A packing 6-colouring of CL_{14} is given by the following pattern:

$$\begin{array}{cccccc} 1 & 3 & 1 & 2 & 1 & 5 & 1 & 2 & 1 & 4 & 1 & 3 & 1 & 6 \\ 2 & 1 & 4 & 1 & 3 & 1 & 6 & 1 & 3 & 1 & 2 & 1 & 5 & 1 \end{array}$$

This completes the proof of Theorem 7. □

4 H-graphs

The *H-graph* $H(r)$, $r \geq 2$, is the 3-regular graph of order $6r$, with vertex set

$$V(H(r)) = \{u_i, v_i, w_i : 0 \leq i \leq 2r - 1\},$$

and edge set (subscripts are taken modulo $2r$)

$$\begin{aligned} E(H(r)) = & \{(u_i, u_{i+1}), (w_i, w_{i+1}), (u_i, v_i), (v_i, w_i) : 0 \leq i \leq 2r - 1\} \\ & \cup \{(v_{2i}, v_{2i+1}) : 0 \leq i \leq r - 1\}. \end{aligned}$$

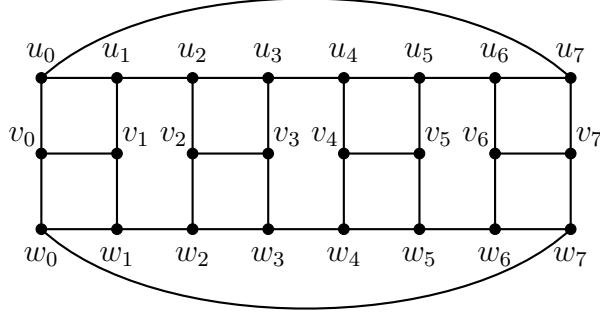


Figure 4: The H-graph $H(4)$.

Figure 4 depicts the H-graph $H(4)$.

William and Roy proved in [31] that $\chi_\rho(H(r)) \leq 5$ for every H-graph $H(r)$ with even $r \geq 4$. We complete their result in Theorem 10 below.

We first prove a technical lemma. For every integers $r \geq 2$ and $0 \leq i \leq r - 1$, we denote by $G_i(r)$ the subgraph of $H(r)$ induced by the set of vertices $\{u_{2i}, u_{2i+1}, v_{2i}, v_{2i+1}, w_{2i}, w_{2i+1}\}$. Observe that for every $r \geq 2$, all the subgraphs $G_i(r)$ are isomorphic to the graph depicted on Figure 5(a). For a given packing 5-colouring π of $H(r)$, we denote by $\pi(G_i(r))$ the set of colours assigned to the vertices of $G_i(r)$.

We then have the following result.

Lemma 8 *For every $r \geq 3$ and every packing 5-colouring π of $H(r)$, the following statements hold for every i , $0 \leq i \leq r - 1$ (subscripts are taken modulo r):*

1. $\pi(G_i(r)) \neq \{1, 2, 3\}$ (that is, $\chi_\rho(G_i(r)) > 3$),
2. $\pi(G_i(r)) \cap \pi(G_{i+1}(r)) = \{1, 2, 3\}$.

Proof. We first prove that every packing 5-colouring of $H(r)$ must use colour 4 or colour 5 on every $G_i(r)$, $0 \leq i \leq r - 1$. Indeed, on every such $G_i(r)$, colour 1 can be used at most thrice, colour 2 at most twice and colour 3 at most once. But we cannot use colour 1 thrice and colour 2 twice in the same colouring, and the result follows. This gives the first item of Lemma 8.

We now prove that if colour 4 (resp. colour 5) is used on $G_i(r)$, then colour 4 (resp. colour 5) cannot be used on $G_{i+1}(r)$. Observe first that every vertex of $G_i(r)$ is at distance at most 5 from every vertex of $G_{i+1}(r)$. Therefore, colour 5 cannot be used on both $G_i(r)$ and $G_{i+1}(r)$. Suppose now that colour 4 is used on both $G_i(r)$ and $G_{i+1}(r)$. Up to symmetries, we necessarily have one of the two following cases.

1. $\pi(u_{2i}) = \pi(w_{2i+3}) = 4$ (see Figure 5(b)).

Since every vertex of G_{i-1} is at distance at most 4 from u_{2i} and two consecutive $G_i(r)$'s cannot both use colour 5, it follows from the first item of Lemma 8 that colour 5 is used neither on $G_i(r)$, nor, by symmetry, on $G_{i+1}(r)$.

Now, on the remaining uncoloured vertices of $G_i(r)$, colour 1 can be used at most thrice, colour 2 at most twice and colour 3 at most once. If colour 1 is used thrice, then we necessarily have $\pi(u_{2i+1}) = \pi(v_{2i}) = \pi(w_{2i+1}) = 1$, so that $\{\pi(v_{2i+1}), \pi(w_{2i})\} = \{2, 3\}$, and no colour is available for w_{2i+2} (recall that colour 5 is not used on $G_{i+1}(r)$). If colour 1 is used twice, then we necessarily have, up to symmetry, $\pi(v_{2i}) = \pi(w_{2i+1}) = 1$, $\pi(u_{2i+1}) = \pi(w_{2i}) = 2$, and $\pi(v_{2i+1}) = 3$, and no colour is available for w_{2i+2} .

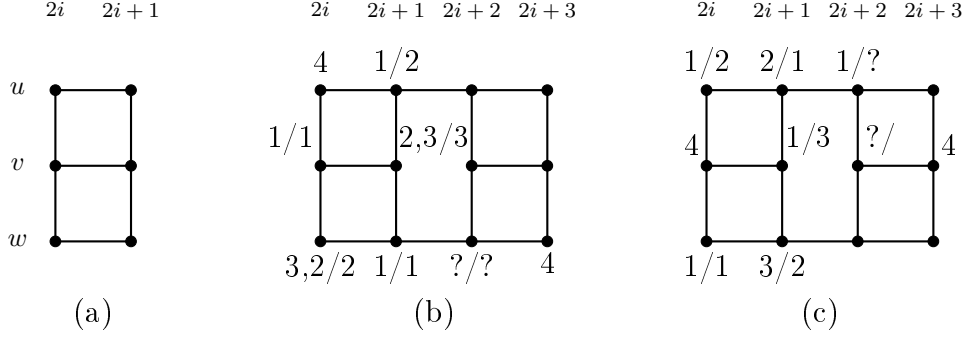


Figure 5: The subgraph $G_i(r)$ and two configurations for the proof of Lemma 8.

2. $\pi(v_{2i}) = \pi(v_{2i+3}) = 4$ (see Figure 5(c)).

Similarly as before, since every vertex of G_{i-1} is at distance at most 4 from v_{2i} and two consecutive $G_i(r)$'s cannot both use colour 5, it follows from the first item of Lemma 8 that colour 5 is used neither on $G_i(r)$, nor, by symmetry, on $G_{i+1}(r)$.

Again, on the remaining uncoloured vertices of $G_i(r)$, colour 1 can be used at most thrice, colour 2 at most twice and colour 3 at most once. If colour 1 is used thrice, then we necessarily have $\pi(u_{2i}) = \pi(v_{2i+1}) = \pi(w_{2i}) = 1$, so that $\{\pi(u_{2i+1}), \pi(w_{2i+1})\} = \{2, 3\}$. Up to symmetry, we may assume $\pi(u_{2i+1}) = 2$ and $\pi(w_{2i+1}) = 3$, which implies $\pi(u_{2i+2}) = 1$, and no colour is available for v_{2i+2} (recall that colour 5 is not used on $G_{i+1}(r)$). If colour 1 is used twice, then we necessarily have, up to symmetry, $\pi(u_{2i+1}) = \pi(w_{2i}) = 1$, $\pi(u_{2i}) = \pi(w_{2i+1}) = 2$, and $\pi(v_{2i+1}) = 3$, and no colour is available for u_{2i+2} .

This completes the proof. \square

From Lemma 8, it follows that every $G_i(r)$ must use colour 4 or 5, and that no two consecutive $G_i(r)$'s can use the same colour from $\{4, 5\}$. Therefore, $H(r)$ does not admit any packing 5-colouring when r is odd.

Corollary 9 *For every odd r , $r \geq 3$, $\chi_\rho(H(r)) > 5$.*

We are now able to prove the main result of this section.

Theorem 10 *For every integer $r \geq 2$, $\chi_\rho(H(r)) = 5$ if r is even, and $6 \leq \chi_\rho(H(r)) \leq 7$ if r is odd.*

Proof. We consider two cases, according to the parity of r .

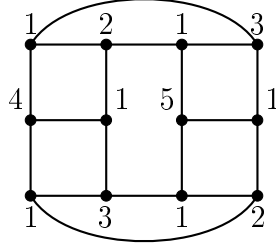
1. r is even.

Since the corona graph $C_{2r} \odot K_1$ is a subgraph of $H(r)$, we get $\chi_\rho(H(r)) \geq 5$ by Proposition 2 and Theorem 3. A packing 5-colouring of $H(r)$ is then obtained by repeating the pattern depicted in Figure 6(a), and thus $\chi_\rho(H(r)) = 5$.

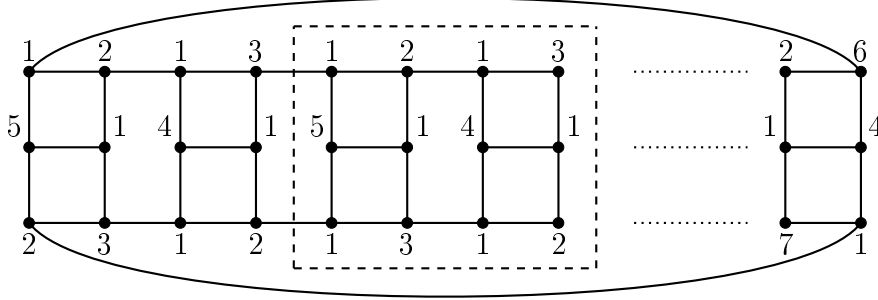
2. r is odd.

From Corollary 9, we get $\chi_\rho(H(r)) \geq 6$. A packing 7-colouring of $H(r)$ is described in Figure 6(b), where the circular pattern (surrounded by the dashed box) is repeated $\frac{r-3}{2}$ times. This gives $\chi_\rho(H(r)) \leq 7$.

This concludes the proof. \square



(a) A packing 5-colouring pattern for $H(r)$, r even



(b) A packing 7-colouring pattern for $H(r)$, r odd ($r \geq 5$)

Figure 6: Packing colouring patterns for H-graphs.

5 Generalised H-graphs

We now consider a natural extension of H-graphs. For every integer $r \geq 2$, the *generalised H-graph* $H^\ell(r)$ with ℓ levels, $\ell \geq 1$, is the 3-regular graph of order $2r(\ell + 2)$, with vertex set

$$V(H^\ell(r)) = \{u_j^i : 0 \leq i \leq \ell + 1, 0 \leq j \leq 2r - 1\}$$

and edge set (subscripts are taken modulo $2r$)

$$\begin{aligned} E(H^\ell(r)) = & \{(u_j^0, u_{j+1}^0), (u_j^{\ell+1}, u_{j+1}^{\ell+1}) : 0 \leq j \leq 2r - 1\} \\ & \cup \{(u_{2j}^i, u_{2j+1}^i) : 1 \leq i \leq \ell, 0 \leq j \leq r - 1\} \\ & \cup \{(u_j^i, u_j^{i+1}) : 0 \leq i \leq \ell, 0 \leq j \leq 2r - 1\}. \end{aligned}$$

Figure 7 depicts the generalised H-graph with three levels $H^3(4)$. Note that generalised H-graphs with one level are precisely H-graphs.

The three following lemmas will be useful for determining the packing chromatic number of generalised H-graphs.

Lemma 11 *Let $H^\ell(r)$, $\ell, r \geq 3$, be a generalised H-graph with $\chi_\rho(H^\ell(r)) \leq 5$, and π be a packing 5-colouring of $H^\ell(r)$. For every edge $u_{2j}^i u_{2j+1}^i$, $1 \leq i \leq \ell$, $0 \leq j \leq r - 1$, with $\pi(u_{2j}^i) \neq 1$ and $\pi(u_{2j+1}^i) \neq 1$, either u_{2j}^i or u_{2j+1}^i has colour 2 and its three neighbours have colours 3, 4 and 5.*

Proof. Observe that every such edge $u_{2j}^i u_{2j+1}^i$ belongs to a subgraph of $H^\ell(r)$ isomorphic to the graph X depicted in Figure 2, in such a way that $u_{2j}^i u_{2j+1}^i$ corresponds to one of the edges $u_3 v_3$, $u_4 v_4$ or $u_5 v_5$ of X . Therefore, the result follows by Lemma 4. \square

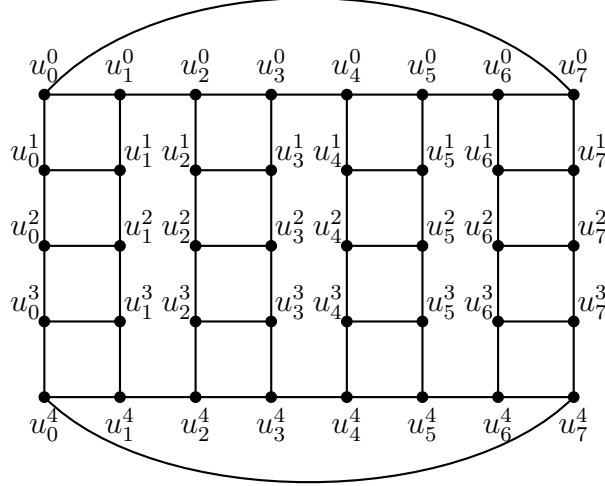


Figure 7: The generalised H-graph $H^3(4)$.

From Lemma 11, it follows that if $\pi(u_{2j}^i) \neq 1$ and $\pi(u_{2j+1}^i) \neq 1$ for some edge $u_{2j}^i u_{2j+1}^i$ of $H^\ell(r)$, $1 \leq i \leq \ell$, $0 \leq j \leq r-1$, then the colour 2 has to be used on this edge and, since the neighbours of the 2-coloured vertex are coloured with 3, 4 and 5, the colour 2 can be replaced by colour 1. Therefore, we get the following corollary.

Corollary 12 *If $H^\ell(r)$, $\ell, r \geq 3$, is a generalised H-graph with $\chi_\rho(H^\ell(r)) \leq 5$, then there exists a packing 5-colouring of $H^\ell(r)$ such that, for every integers i and j , $1 \leq i \leq \ell$, $0 \leq j \leq r-1$, the colour 1 is used on the edge $u_{2j}^i u_{2j+1}^i$ of $H^\ell(r)$.*

Lemma 13 *Let $H^\ell(r)$, $\ell, r \geq 3$, be a generalised H-graph with $\chi_\rho(H^\ell(r)) \leq 5$, and π be a packing 5-colouring of $H^\ell(r)$. For every j , $0 \leq j \leq 2r-1$, π must assign colour 1 to one vertex of each of the edges $u_j^0 u_{j+1}^0$ and $u_j^{\ell+1} u_{j+1}^{\ell+1}$ (subscripts are taken modulo $2r$).*

Proof. The proof is done by case analysis and is given in Appendix B. \square

Let $H^\ell(r)$ be a generalised H-graph with $\chi_\rho(H^\ell(r)) \leq 5$. From Corollary 12 and Lemma 13, it follows that one can always produce a packing 5-colouring of $H^\ell(r)$ that uses colour 1 on each edge $u_{2j}^i u_{2j+1}^i$ of $H^\ell(r)$, $0 \leq i \leq \ell+1$, $0 \leq j \leq r-1$. Since adjacent vertices cannot be assigned the same colour and $H^\ell(r)$ is a bipartite graph, we get the following corollary.

Corollary 14 *If $H^\ell(r)$, $\ell, r \geq 3$, is a generalised H-graph with $\chi_\rho(H^\ell(r)) \leq 5$, then there exists a packing 5-colouring of $H^\ell(r)$ such that the colour 1 is used on each edge of $H^\ell(r)$.*

Lemma 15 *If $H^\ell(r)$, $\ell, r \geq 3$, is a generalised H-graph with $\chi_\rho(H^\ell(r)) \leq 5$, then there exists a packing 5-colouring π of $H^\ell(r)$ such that $\pi(u_j^0) \notin \{4, 5\}$ and $\pi(u_j^{\ell+1}) \notin \{4, 5\}$ for every j , $0 \leq j \leq 2r-1$.*

Proof. Let π be a packing 5-colouring of $H^\ell(r)$ such that colour 1 is used on each edge of $H^\ell(r)$ (the existence of such a colouring is ensured by Corollary 14). Thanks to the symmetries of $H^\ell(r)$, it suffices to prove the result for any vertex u_{2j}^0 , $0 \leq j \leq r-1$. Suppose to the contrary that $\pi(u_{2j}^0) \in \{4, 5\}$ for some j , $0 \leq j \leq r-1$. We have two cases to consider.

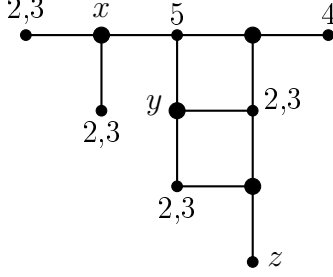


Figure 8: The subgraph Y of $H^\ell(r)$.

1. $\pi(u_{2j}^0) = 4$.

Let Y be the subgraph of $H^\ell(r)$ depicted in Figure 8, where the vertex u_{2j}^0 is the unique vertex with colour 4, and vertices with colour 1 are drawn as “big vertices”. Observe that the three neighbours x , as well as the three neighbours of y , must use colours 2, 3 and 5. Therefore, the common neighbour of x and y must be assigned colour 5. It then follows that no colour is available for z .

2. $\pi(u_{2j}^0) = 5$.

The proof is similar to the proof of the previous case, by switching colours 4 and 5.

This completes the proof. □

Let $H^\ell(r)$ be a generalised H-graph satisfying the hypothesis of Lemma 15, and π be a packing 5-colouring of $H^\ell(r)$. From Lemma 15, it follows that the restriction of π to the $2r$ -cycle induced by the set of vertices $\{u_j^0 \mid 0 \leq j \leq 2r - 1\}$ is a packing 3-colouring. It is not difficult to check (see [23]) that the only packing 3-colouring of a cycle must use the periodic pattern 1213 of length 4. Therefore, we get the following corollary.

Corollary 16 *For every integers $\ell \geq 3$ and odd $r \geq 3$, $\chi_\rho(H^\ell(r)) \geq 6$.*

We are now able to prove the main results of this section. We first consider the case of generalised H-graphs $H^\ell(r)$ with $\ell \notin \{2, 5\}$.

Theorem 17 *For every integers $\ell \geq 3$, $\ell \neq 5$, and $r \geq 2$,*

$$\chi_\rho(H^\ell(r)) = \begin{cases} 5 & \text{if } r \text{ is even,} \\ 6 & \text{otherwise.} \end{cases}$$

Proof. We consider two cases, according to the parity of r .

1. r is even.

Since the corona graph $C_{2r} \odot K_1$ is a subgraph of $H^\ell(r)$ for every $r \geq 2$, we get $\chi_\rho(H^\ell(r)) \geq \chi_\rho(C_{2r} \odot K_1) = 5$ by Theorem 3 and Proposition 2.

We now prove $\chi_\rho(H^\ell(r)) \leq 5$. Figure 9 depicts a packing 5-colouring of $H^4(2)$, together with its corresponding colouring pattern. It can easily be checked that this (6×4) -pattern is periodic, that is, can be repeated, both vertically and horizontally, to produce a packing 5-colouring of any generalised H-graph of the form $H^{6i+4}(2j)$, with $i \geq 0$ and $j \geq 1$.

If $\ell - 4 \not\equiv 0 \pmod{6}$, we use the colouring patterns depicted in Figure 10, depending on the value of ℓ modulo 6. The upper six rows of each colouring pattern, surrounded by double

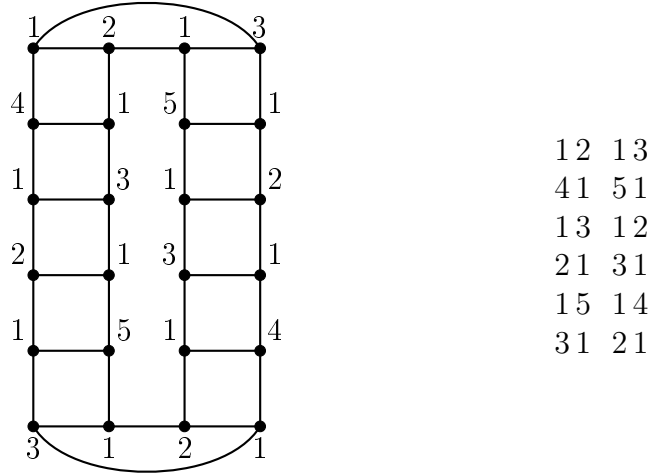


Figure 9: A packing 5-colouring of $H^4(2)$ and its corresponding colouring pattern.

12 13	12 13	12 13	12 13	12 13
41 51	41 51	41 51	41 51	41 51
13 12	13 12	13 12	13 12	13 12
21 31	21 31	21 31	21 31	21 31
15 14	15 14	15 14	15 14	15 14
31 21	31 21	31 21	31 21	31 21
14 15	14 15	12 13	12 13	12 13
21 31	21 31	41 51	41 51	41 51
13 12		13 12	13 12	13 12
51 41			51 41	51 41
12 13			12 13	12 13
41 51			31 21	
13 12			14 15	
			21 31	
			15 14	
			31 21	

$\ell \equiv 5 \pmod{6}$ $\ell \equiv 0 \pmod{6}$ $\ell \equiv 1 \pmod{6}$ $\ell \equiv 2 \pmod{6}$ $\ell \equiv 3 \pmod{6}$

Figure 10: Colouring patterns for $H^\ell(r)$, r even.

13	12	16		12	13		12	15	12	61	23
21	51	41		41	51		41	31	31	14	51
14	23	12		13	12		13	14	14	23	12
31	14	31		51	41		21	21	21	51	41
12	61	25		12	13		15	16	13	12	13

(a)

(b)

Figure 11: Colouring patterns for $H^3(3)$ and for $H^3(r)$, $r \geq 5$, r odd.

12	13	16		12	13		12	13		16		16	13		12	13		12
41	51	21		41	51		41	51		21		41	21		51	41		51
13	12	13		13	12		13	12		13		12	14		13	12		13
21	31	51		21	31		21	31		51		31	51		21	31		21
15	14	12		15	14		15	14		12		15	13		14	15		14
31	21	31		31	21		31	21		31		21	21		31	21		31
				12	15		14	15		14		13	16		12	13		12
				61	31		21	31		21		21	31		51	41		51
												14	12		13	12		13

(a)

(b)

(c)

Figure 12: Colouring patterns for $H^4(r)$, $H^6(r)$ and $H^7(r)$, $r \geq 3$, r odd.

lines, can be repeated as many times as required, or even deleted when $\ell \equiv 1, 2, 3 \pmod{6}$. Therefore, these colouring patterns give us a packing 5-colouring of any generalised H-graph of the form $H^\ell(2)$, for every $\ell \geq 3$, $\ell \neq 5$. It is again easy to check that each of these colouring patterns is “horizontally periodic”, that is, can be horizontally repeated in order to get a packing 5-colouring of any generalised H-graph of the form $H^\ell(r)$, for every $\ell \geq 3$, $\ell \neq 5$, and even r .

2. r is odd.

The inequality $\chi_\rho(H^\ell(r)) \geq 6$ directly follows from Corollary 16. Therefore, we only need to prove the inequality $\chi_\rho(H^\ell(r)) \leq 6$ (recall that $\ell \geq 3$ and $\ell \neq 5$).

We first consider a few particular cases. A packing 6-colouring of $H^3(3)$ is depicted in Figure 11(a), and a packing 6-colouring of $H^3(r)$, for every odd $r \geq 5$, is depicted in Figure 11(b) (the first four columns, surrounded by a double line, are repeated $\frac{r-5}{2}$ times, and thus do not appear if $r = 5$). A packing 6-colouring of $H^4(r)$, for every odd $r \geq 3$, is depicted in Figure 12(a) (the first four columns are repeated $\frac{r-3}{2}$ times), and a packing 6-colouring of $H^6(r)$ and of $H^7(r)$, for every odd $r \geq 3$, is depicted in Figure 12(b) and in Figure 12(c), respectively (the four columns surrounded by a double line, are repeated $\frac{r-3}{2}$ times, and thus do not appear if $r = 3$).

In order to produce a packing 6-colouring of $H^\ell(r)$, with $\ell \geq 8$, $r \geq 3$, and r odd, we use the colouring patterns depicted in Figure 13. The six rows surrounded by double lines must be repeated $\frac{\ell-6-(\ell \bmod 6)}{6}$ times (and thus do not appear if $\ell \equiv 2, 3, 4, 5 \pmod{6}$ and $\ell = 6 + (\ell \bmod 6)$). The four columns surrounded by double lines must be repeated $\frac{r-1}{2}$ times if $\ell \equiv 0, 1 \pmod{6}$, or $\frac{r-3}{2}$ times if $\ell \equiv 2, 3, 4, 5 \pmod{6}$ (and thus do not appear if

12	13	12	13	16
41	51	41	51	21
13	12	13	12	13
21	31	21	31	41
15	14	15	14	12
31	21	31	21	31
12	13	12	13	15
41	51	41	51	21
13	12	13	12	13
21	41	51	41	41
15	13	12	13	12
31	21	31	21	61
12	15	14	15	13
41	31	21	31	21

$\ell \equiv 0 \pmod{6}$

12	13	12	13	16
41	51	41	51	21
13	12	13	12	13
21	31	21	31	41
15	14	15	14	12
31	21	31	21	31
12	13	12	13	15
41	51	41	51	21
13	12	13	12	16
21	31	21	31	41
15	14	15	14	13
31	21	31	21	21
12	13	12	13	15
61	51	41	51	31
13	12	13	12	14

$\ell \equiv 1 \pmod{6}$

12	13	16
41	51	21
13	12	13
21	31	41
15	14	12
31	21	31
12	13	15
41	51	21
13	12	16
51	41	31
12	13	14
31	21	21
14	15	13
21	31	51
15	14	12
31	21	61

$\ell \equiv 2 \pmod{6}$

12	13	16
41	51	21
13	12	13
21	31	41
15	14	12
31	21	31
12	13	15
41	51	21
13	12	13
51	41	41
12	13	12
31	21	31
14	15	15
21	31	21
13	12	14
51	41	31
13	12	16

$\ell \equiv 3 \pmod{6}$

12	13	16
41	51	21
13	12	13
21	31	41
15	14	12
31	21	31
12	13	15
41	51	21
13	12	13
21	31	41
15	14	12
31	21	31
12	13	15
41	51	21
13	12	14
21	31	31
15	14	12
31	21	61

$\ell \equiv 4 \pmod{6}$

12	13	16
41	51	21
13	12	13
21	31	41
15	14	12
31	21	31
12	13	15
41	51	21
13	12	13
51	41	61
12	13	12
31	21	51
14	15	13
21	31	41
15	14	12
21	31	31
13	12	15
41	51	21
12	13	16

$\ell \equiv 5 \pmod{6}$

Figure 13: Colouring patterns for $H^\ell(r)$, $\ell \geq 8$, $r \geq 3$, r odd.

13 16	13 16 12 15	13 16 14 12 17 14 15
21 21	21 21 31 31	21 21 21 51 31 21 31
14 17	14 13 14 12	14 13 13 13 12 13 12
31 51	31 51 21 71	31 51 71 21 41 51 61

$r = 2$

$r = 4$

$r = 7$

13 16 13 12 17 13 12 15	13 16 13 12 14 13 16 14 12 17 15
21 21 21 54 31 21 41 31	21 21 21 51 31 21 21 21 31 31 31
14 13 14 21 14 15 13 12	14 13 14 13 12 15 13 13 15 12 12
31 51 71 36 21 31 21 61	31 51 71 21 61 31 41 71 21 41 61

$r = 8$

$r = 11$

Figure 14: Packing 7-colourings of $H^2(r)$, $r \in \{2, 4, 7, 8, 11\}$.

$r = 3$).

This completes the proof. □

The last two theorems of this section deal with the cases not covered by Theorem 17, that is, $\ell = 2$ and $\ell = 5$, respectively.

Theorem 18 *For every integer $r \geq 2$,*

$$\chi_\rho(H^2(r)) = \begin{cases} 7 & \text{if } r \in \{2, 4, 7, 8, 11\}, \\ 6 & \text{otherwise.} \end{cases}$$

Proof. The fact that $H^2(r)$ does not admit a packing 6-colouring for every $r \in \{2, 4, 7, 8, 11\}$ has been checked by a computer program. Packing 7-colourings for each of these graphs are depicted in Figure 14.

Assume now $r \notin \{2, 4, 7, 8, 11\}$. Again, the fact that in that case $H^2(r)$ does not admit a packing 5-colouring has been checked by a computer program. Packing 6-colourings of such generalised H-graphs are depicted in Figure 15, according to the value of r , r modulo 3, or r modulo 6 (periodic patterns, made of 6 or 12 columns, are surrounded by double lines). □

Theorem 19 *For every integer $r \geq 2$, $\chi_\rho(H^5(r)) = 6$.*

Proof. The fact that $H^5(r)$ does not admit a packing 5-colouring for every $r \geq 2$ has been checked by a computer program. Packing 6-colourings of $H^5(r)$, $r \in \{2, 3, 5\}$, are depicted in Figure 16, while packing 6-colourings of $H^5(r)$, $r = 4$ or $r \geq 6$, are depicted in Figure 17 according to the value of r modulo 4 (periodic patterns, made of eight columns, are surrounded by double lines). Note that for $r = 6$ and $r = 7$, the first eight columns have to be deleted. □

$$\begin{array}{ccc}
\begin{array}{c} 13 \ 12 \ 13 \ 12 \ 16 \\ 21 \ 51 \ 41 \ 51 \ 31 \\ 14 \ 13 \ 12 \ 14 \ 15 \\ 31 \ 21 \ 61 \ 31 \ 21 \end{array} & \left\| \begin{array}{c} 13 \ 12 \ 15 \\ 21 \ 41 \ 31 \\ 16 \ 13 \ 14 \\ 31 \ 51 \ 21 \end{array} \right\| & \left\| \begin{array}{c} 13 \ 12 \ 15 \\ 21 \ 41 \ 31 \\ 16 \ 13 \ 14 \\ 31 \ 51 \ 21 \end{array} \right\| \begin{array}{c} 13 \ 14 \ 12 \ 13 \ 14 \ 12 \ 15 \\ 61 \ 21 \ 31 \ 51 \ 21 \ 61 \ 31 \\ 12 \ 13 \ 16 \ 12 \ 15 \ 13 \ 12 \\ 31 \ 51 \ 21 \ 41 \ 31 \ 21 \ 41 \end{array} \\
r = 5 & r \equiv 0 \pmod{3} & r \equiv 1 \pmod{3}, r \geq 10
\end{array}$$

$$\left\| \begin{array}{c} 13 \ 12 \ 14 \ 13 \ 16 \ 12 \\ 21 \ 51 \ 31 \ 21 \ 21 \ 31 \\ 14 \ 13 \ 12 \ 15 \ 13 \ 15 \\ 31 \ 21 \ 61 \ 31 \ 41 \ 21 \end{array} \right\| \begin{array}{c} 13 \ 12 \ 13 \ 16 \ 12 \ 13 \ 12 \ 16 \\ 41 \ 51 \ 41 \ 21 \ 31 \ 41 \ 51 \ 31 \\ 12 \ 13 \ 15 \ 14 \ 15 \ 12 \ 23 \ 15 \\ 31 \ 61 \ 21 \ 31 \ 21 \ 36 \ 14 \ 21 \end{array}$$

$$r \equiv 2 \pmod{6}, r \geq 14$$

$$\left\| \begin{array}{c} 13 \ 12 \ 14 \ 13 \ 16 \ 12 \\ 21 \ 51 \ 31 \ 21 \ 21 \ 31 \\ 14 \ 13 \ 12 \ 15 \ 13 \ 15 \\ 31 \ 21 \ 61 \ 31 \ 41 \ 21 \end{array} \right\| \begin{array}{c} 13 \ 12 \ 14 \ 13 \ 12 \ 41 \ 63 \ 12 \ 13 \ 12 \ 16 \\ 41 \ 51 \ 31 \ 21 \ 51 \ 32 \ 21 \ 51 \ 41 \ 51 \ 31 \\ 12 \ 23 \ 15 \ 16 \ 13 \ 15 \ 14 \ 13 \ 12 \ 14 \ 15 \\ 36 \ 14 \ 21 \ 31 \ 41 \ 21 \ 31 \ 21 \ 61 \ 31 \ 21 \end{array}$$

$$r \equiv 5 \pmod{6}, r \geq 17$$

Figure 15: Colouring patterns for $H^2(r)$, $r \notin \{2, 4, 7, 8, 11\}$.

$$\begin{array}{ccc}
\begin{array}{c} 13 \ 12 \\ 41 \ 61 \\ 15 \ 13 \\ 31 \ 21 \\ 12 \ 14 \\ 61 \ 51 \\ 13 \ 12 \end{array} & \begin{array}{c} 13 \ 16 \ 15 \\ 21 \ 21 \ 21 \\ 14 \ 13 \ 14 \\ 31 \ 51 \ 31 \\ 15 \ 12 \ 12 \\ 21 \ 31 \ 51 \\ 16 \ 14 \ 13 \end{array} & \begin{array}{c} 13 \ 15 \ 12 \ 13 \ 14 \\ 21 \ 21 \ 41 \ 61 \ 21 \\ 16 \ 13 \ 13 \ 12 \ 15 \\ 31 \ 41 \ 21 \ 41 \ 31 \\ 14 \ 16 \ 15 \ 15 \ 12 \\ 21 \ 31 \ 31 \ 31 \ 61 \\ 15 \ 12 \ 14 \ 12 \ 13 \end{array} \\
r = 2 & r = 3 & r = 5
\end{array}$$

Figure 16: Packing 7-colourings of $H^5(r)$, $r \in \{2, 3, 5\}$.

$$\begin{array}{c} \left\| \begin{array}{cccc} 13 & 16 & 13 & 15 \\ 21 & 21 & 41 & 21 \\ 14 & 13 & 12 & 14 \\ 31 & 41 & 51 & 31 \\ 15 & 15 & 13 & 12 \\ 21 & 21 & 21 & 51 \\ 16 & 13 & 14 & 13 \end{array} \right\| \end{array} \quad \begin{array}{c} \left\| \begin{array}{cccc} 13 & 16 & 13 & 15 \\ 21 & 21 & 41 & 21 \\ 14 & 13 & 12 & 14 \\ 31 & 41 & 31 & 31 \\ 15 & 15 & 15 & 16 \\ 21 & 21 & 21 & 21 \\ 16 & 13 & 14 & 13 \end{array} \right\| \left\| \begin{array}{cccc} 12 & 61 & 23 & 14 & 15 \\ 31 & 14 & 51 & 21 & 21 \\ 14 & 23 & 12 & 13 & 13 \\ 21 & 51 & 41 & 51 & 61 \\ 13 & 12 & 13 & 12 & 12 \\ 51 & 41 & 61 & 31 & 51 \\ 12 & 13 & 12 & 14 & 13 \end{array} \right\| \end{array}$$

$$r \equiv 0 \pmod{4}, r \geq 4$$

$$r \equiv 1 \pmod{4}, r \geq 9$$

$$\begin{array}{c} \left\| \begin{array}{cccc} 13 & 16 & 13 & 15 \\ 21 & 21 & 41 & 21 \\ 14 & 13 & 12 & 14 \\ 31 & 41 & 51 & 31 \\ 15 & 15 & 13 & 12 \\ 21 & 21 & 21 & 51 \\ 16 & 13 & 14 & 13 \end{array} \right\| \left\| \begin{array}{cccc} 13 & 16 & 13 & 14 & 13 & 15 \\ 21 & 21 & 51 & 21 & 21 & 21 \\ 14 & 13 & 12 & 15 & 16 & 14 \\ 31 & 51 & 31 & 31 & 51 & 31 \\ 15 & 14 & 14 & 42 & 13 & 12 \\ 21 & 21 & 21 & 61 & 21 & 51 \\ 16 & 13 & 15 & 23 & 14 & 13 \end{array} \right\| \end{array}$$

$$r \equiv 2 \pmod{4}, r \geq 6$$

$$\begin{array}{c} \left\| \begin{array}{cccc} 13 & 16 & 13 & 15 \\ 21 & 21 & 41 & 21 \\ 14 & 13 & 12 & 13 \\ 31 & 41 & 51 & 31 \\ 15 & 15 & 13 & 12 \\ 21 & 21 & 21 & 51 \\ 16 & 13 & 14 & 13 \end{array} \right\| \left\| \begin{array}{cccc} 13 & 16 & 13 & 12 & 14 & 13 & 15 \\ 21 & 21 & 41 & 51 & 31 & 21 & 21 \\ 14 & 13 & 12 & 13 & 15 & 16 & 14 \\ 31 & 41 & 51 & 41 & 21 & 51 & 31 \\ 15 & 15 & 13 & 12 & 13 & 12 & 12 \\ 21 & 21 & 21 & 31 & 61 & 31 & 51 \\ 16 & 13 & 14 & 15 & 12 & 14 & 13 \end{array} \right\| \end{array}$$

$$r \equiv 3 \pmod{4}, r \geq 7$$

Figure 17: Colouring patterns for $H^5(r)$, $r = 4$ or $r \geq 6$.

6 Discussion

In this paper, we have studied the packing chromatic number of some classes of cubic graphs, namely ladders, H-graphs and generalised H-graphs. We have determined the exact value of this parameter for every such graph, except for the case of H-graphs $H(r)$ with $r \geq 3$, r odd (see Theorem 10), for which we proved $6 \leq \chi_\rho(H(r)) \leq 7$. We thus propose the following question.

Question 1 *Is it true that $\chi_\rho(H(r)) \leq 6$ for every H-graph $H(r)$ with $r \geq 3$, r odd?*

In [22,23], we have extended the notion of packing colouring to the case of digraphs. If D is a digraph, the (weak) directed distance between two vertices u and v in D is defined as the length of a shortest directed path between u and v , in either direction. Using this notion of distance in digraphs, the packing colouring readily extends to digraphs. Recall that an orientation of an undirected graph G is any antisymmetric digraph obtained from G by giving to each edge of G one of its two possible orientations. It then directly follows from the definition that $\chi_\rho(D) \leq \chi_\rho(G)$ for any orientation D of G . A natural question for oriented graphs, related to this work, is then the following.

Question 2 *Is it true that the packing chromatic number of any oriented graph with maximum degree 3 is bounded by some constant?*

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References

- [1] I. Alegre, M. A. Fiol, and J.L.A. Yebra, Some large graphs with given degree and diameter, *J. Graph Theory* 10 (1986), no. 2, 219–224.
- [2] G. Argiroffo, G. Nasini and P. Torres, Polynomial instances of the packing colouring problem, *Electron. Notes Discrete Math.* 37 (2011), 363–368.
- [3] G. Argiroffo, G. Nasini and P. Torres, The packing colouring problem for $(q, q - 4)$ -graphs, *Lecture Notes in Comput. Sci.* 7422 (2012), 309–319.
- [4] G. Argiroffo, G. Nasini and P. Torres, The packing colouring problem for lobsters and partner limited graphs, *Discrete Appl. Math.* 164 (2014), 373–382.
- [5] J. Balogh, A. Kostochka and X. Liu, Packing chromatic number of subcubic graphs, arXiv:1703.09873 [math.CO] (2017).
- [6] B. Brešar, S. Klavžar and D.F. Rall, On the packing chromatic number of Cartesian products, hexagonal lattice, and trees, *Discrete Appl. Math.* 155 (2007), 2303–2311.
- [7] B. Brešar, S. Klavžar and D.F. Rall, Packing Chromatic Number of Base-3 Sierpiński Graphs, *Graphs Combin.* 32 (2016), 1313–1327.
- [8] B. Brešar, S. Klavžar, D.F. Rall and K. Wash, Packing chromatic number, $(1, 1, 2, 2)$ -colourings, and characterizing the Petersen graph, *Aequat. Math.* 91 (2017), 169–184.
- [9] B. Brešar, S. Klavžar, D.F. Rall and K. Wash, Packing chromatic number under local changes in a graph, *Discrete Math.* 340 (2017), 1110–1115.
- [10] J. Ekstein, J. Fiala, P. Holub and B. Lidický, The packing chromatic number of the square lattice is at least 12, arXiv:1003.2291 [cs.DM] (2010).

- [11] J. Ekstein, P. Holub and B. Lidický, Packing chromatic number of distance graphs, *Discrete Appl. Math.* 160 (2012), 518–524.
- [12] J. Ekstein, P. Holub and O. Togni, The packing colouring of distance graphs $D(k, t)$, *Discrete Appl. Math.* 167 (2014), 100–106.
- [13] J. Fiala and P. A. Golovach, Complexity of the packing colouring problem for trees, *Discrete Appl. Math.* 158 (2010), 771–778.
- [14] J. Fiala, S. Klavžar and B. Lidický, The packing chromatic number of infinite product graphs, *European J. Combin.* 30 (2009), 1101–1113.
- [15] A.S. Finbow and D. F. Rall, On the packing chromatic number of some lattices, *Discrete Appl. Math.* 158 (2010), 1224–1228.
- [16] N. Gastineau. Dichotomies properties on computational complexity of S-packing colouring problems, *Discrete Math.* 338 (2015), no. 6, 1029–1041.
- [17] N. Gastineau, P. Holub and O. Togni, On the packing chromatic number of subcubic outerplanar graphs, arXiv:1703.05023 [cs.DM] (2017).
- [18] N. Gastineau and O. Togni, S-packing colourings of cubic graphs, *Discrete Math.* 339 (2016), 2461–2470.
- [19] W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, J.M. Harris and D.F. Rall, Broadcast chromatic numbers of graphs, The 16th Cumberland Conference on Combinatorics, Graph Theory, and Computing (2003).
- [20] W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, J.M. Harris and D.F. Rall, Broadcast chromatic numbers of graphs, *Ars Combin.* 86 (2008), 33–49.
- [21] D. Korže and A. Vesel, On the packing chromatic number of square and hexagonal lattice, *Ars Math. Contemp.* 7 (2014), 13–22.
- [22] D. Laïche, I. Bouchemakh and É. Sopena. On the Packing colouring of Undirected and Oriented Generalized Theta Graphs. *Australas. J. Combin.* 66 (2016), no. 2, 310–329.
- [23] D. Laïche, I. Bouchemakh and É. Sopena. Packing colouring of some undirected and oriented corone graphs. *Discuss. Math. Graph Theory* 37 (2017), 665–690.
- [24] B. Martin, F. Raimondi, T. Chen, J. Martin, The packing chromatic number of the infinite square lattice is between 13 and 15, *Discrete Appl. Math.* 225 (2017), 136–142.
- [25] D.F. Rall, B. Brešar, A.S. Finbow and S. Klavžar, On the packing chromatic number of trees, Cartesian products and some infinite graphs, *Electron. Notes Discrete Math.* 30 (2008), 57–61.
- [26] C. Sloper, An eccentric colouring of trees, *Australas. J. Combin.* 29 (2004), 309–321.
- [27] R. Soukal and P. Holub, A note on the packing chromatic number of the square lattice, *Electron. J. Combin.* 17 (2010).
- [28] O. Togni, On packing colourings of distance graphs, *Discrete Appl. Math.* 167 (2014), 280–289.
- [29] P. Torres and M. Valencia-Pabon, The packing chromatic number of hypercubes, *Discrete Appl. Math.* 190-191 (2015), 127–140.
- [30] A. William, I. Rajasingh and S. Roy, Packing chromatic number of enhanced hypercubes, *Int. J. Math. Appl.* 2 (2014), no. 3, 1–6.
- [31] A. William and S. Roy, Packing chromatic number of certain graphs, *Int. J. Pure Appl. Math.* 87 (2013), no. 6, 731–739.

A Proof of Lemma 4

The configurations used in the proof correspond to partial colourings of the graph X and are depicted in Figures 18 and 19, with the following drawing convention. If $\{a, b\}$ is the set of colours assigned to two distinct vertices, then the “colour” of both these vertices is denoted “ a, b ”. If the same configuration describes two partial colourings of X and the colours assigned to some vertex by these two colourings are respectively a and b , then the “colour” of this vertex is denoted “ $a|b$ ”. Finally, if a vertex has no available colour, its “colour” is denoted “?”. Moreover, only the edges of X that are necessary for the argument are drawn.

Let $u_i v_i$, $3 \leq i \leq 5$, be an edge of X with $\pi(u_i) \neq 1$ and $\pi(v_i) \neq 1$. We first prove the following claim.

Claim 1 *For every i , $3 \leq i \leq 5$, $2 \in \{\pi(u_i), \pi(v_i)\}$.*

Proof. Assume to the contrary that this is not the case, that is, $\{\pi(u_i), \pi(v_i)\} \subseteq \{3, 4, 5\}$. Thanks to the symmetry exchanging u_i and v_i , we may assume $\pi(u_i) < \pi(v_i)$, without loss of generality. Recall that there is no edge $u_{i-2} v_{i-2}$ (resp. $u_{i+2} v_{i+2}$) in X when $i = 2$ (resp. $i = 4$). We consider the following cases (subscripts are taken modulo n).

1. $\pi(u_i) = 3$ and $\pi(v_i) = 4$.

In that case, we necessarily have $\pi(u_{i+1}) \in \{1, 2, 5\}$.

If $\pi(u_{i+1}) = 1$, then $\{\pi(v_{i+1}), \pi(u_{i+2})\} = \{2, 5\}$. If $\pi(v_{i+1}) = 2$ (and $\pi(u_{i+2}) = 5$), then $\pi(v_{i-1}) = 1$, so that $\pi(u_{i-1}) = 2$ and no colour is available for v_{i-2} (see Figure 18(a)). If $\pi(u_{i+2}) = 2$ (and $\pi(v_{i+1}) = 5$), then $\{\pi(u_{i-1}), \pi(v_{i-1})\} = \{1, 2\}$, and no colour is available for either for u_{i-2} or for v_{i-2} (see Figure 18(b)).

If $\pi(u_{i+1}) = 2$, then $\pi(v_{i+1}) \in \{1, 5\}$. If $\pi(v_{i+1}) = 5$, then $\pi(u_{i-1}) = 1$, so that $\pi(v_{i-1}) = 2$ and no colour is available for u_{i-2} (see Figure 18(c)). If $\pi(v_{i+1}) = 1$, then either $\pi(u_{i-1}) = 5$, so that no colour is available for v_{i+2} (see Figure 18(d)), or $\pi(u_{i-1}) = 1$, which implies $\{\pi(u_{i-2}), \pi(v_{i-1})\} = \{2, 5\}$, so that again no colour is available for v_{i+2} (see Figure 18(e)). Finally, if $\pi(u_{i+1}) = 5$, then $\{\pi(u_{i-1}), \pi(v_{i-1})\} = \{1, 2\}$, and no colour is available either for u_{i-2} or for v_{i-2} (see Figure 18(f)).

2. $\pi(u_i) = 3$ and $\pi(v_i) = 5$.

Observe that the proof is similar to the proof of the previous case, by switching colours 4 and 5, in all cases illustrated in Figure 18(b), (c), (d) and (f). Therefore, it remains only two cases to be considered, which were illustrated in Figure 18(a) and (e), respectively.

- (a) $\pi(u_{i+1}) = 1$ and $\pi(v_{i+2}) = 1$.

In that case, we have $\pi(u_{i+1}) = 4$, which implies $\pi(v_{i-1}) = 1$ and thus $\pi(u_{i-1}) = 2$, so that $\pi(u_{i-2}) = 1$, $\pi(v_{i-2}) = 4$, and no colour is available for u_{i-3} (see Figure 18(g)).

- (b) $\pi(u_{i+1}) = 2$, $\pi(u_{i-1}) = 1$ and $\pi(v_{i+1}) = 1$.

In that case, we necessarily have $\pi(v_{i+2}) = 4$, so that $\pi(u_{i+2}) = 1$, and no colour is available for u_{i+3} (see Figure 18(h)).

3. $\pi(u_i) = 4$ and $\pi(v_i) = 5$.

In that case, we necessarily have $\pi(u_{i+1}) \in \{1, 2, 3\}$. We consider six subcases, depending on the value of $\pi(u_{i+1})$ and i .

- (a) $\pi(u_{i+1}) = 1$ and $i \in \{3, 4\}$.

In that case, we have $\{\pi(u_{i+2}), \pi(v_{i+1})\} = \{2, 3\}$, which implies $\pi(v_{i+2}) = 1$, and no colour is available for v_{i+3} (see Figure 18(i)).

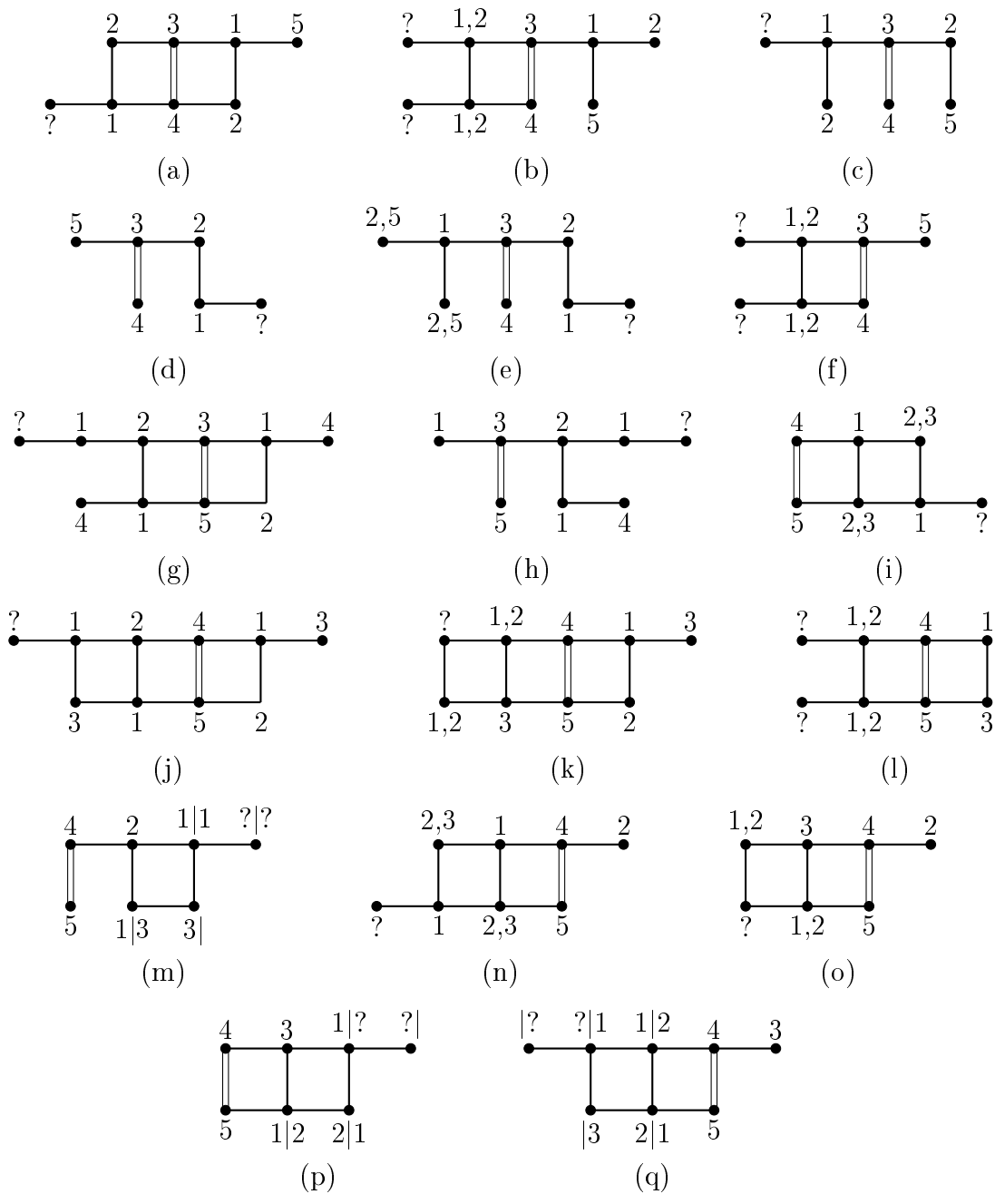


Figure 18: Configurations for the proof of Lemma 4 (the double edge is the edge $u_i v_i$).

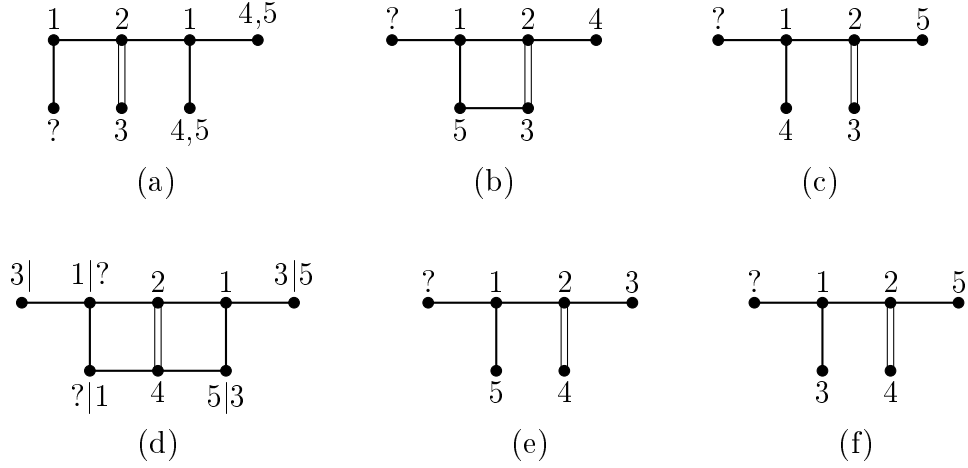


Figure 19: Configurations for the proof of Lemma 4 (cont.).

(b) $\pi(u_{i+1}) = 1$ and $i = 5$.

In that case, we have $\pi(v_6) \in \{3, 4\}$. If $\pi(v_6) = 2$, then we necessarily have $\pi(u_7) = 3$, and thus $\pi(v_4) \in \{1, 3\}$. If $\pi(v_4) = 1$, we get successively $\pi(u_4) = 2$, $\pi(v_3) = 3$, $\pi(u_3) = 1$, and no colour is available for u_2 (see Figure 18(j)). If $\pi(v_4) = 3$, then $\{\pi(u_4), \pi(v_3)\} = \{1, 2\}$ and no colour is available for u_3 (see Figure 18(k)).

If $\pi(v_6) = 3$, then $\{\pi(u_4), \pi(v_4)\} = \{1, 2\}$, and no colour is available either for u_3 or for v_3 (see Figure 18(l)).

(c) $\pi(u_{i+1}) = 2$ and $i \in \{3, 4\}$.

In that case, we necessarily have $\pi(v_{i+1}) \in \{1, 3\}$. If $\pi(v_{i+1}) = 1$, then $\pi(v_{i+2}) = 3$, which implies $\pi(u_{i+2}) = 1$, and no colour is available for u_{i+3} . If $\pi(v_{i+1}) = 3$, then $\pi(u_{i+2}) = 1$, and no colour is available for u_{i+2} (see Figure 18(m)).

(d) $\pi(u_{i+1}) = 2$ and $i = 5$.

In that case, we necessarily have $\pi(u_4) \in \{1, 3\}$. If $\pi(u_4) = 1$, then $\{\pi(u_3), \pi(v_4)\} = \{2, 3\}$, so that $\pi(v_3) = 1$, and no colour is available for v_2 (see Figure 18(n)). If $\pi(u_4) = 3$, then $\{\pi(u_3), \pi(v_4)\} = \{1, 2\}$, and no colour is available for v_3 (see Figure 18(o)).

(e) $\pi(u_{i+1}) = 3$ and $i \in \{3, 4\}$.

In that case, either $\pi(v_{i+1}) = 1$, so that $\pi(v_{i+2}) = 2$, $\pi(u_{i+2}) = 1$, and no colour is available for u_{i+3} , or $\pi(v_{i+1}) = 2$, so that $\pi(v_{i+2}) = 1$ and no colour is available for u_{i+2} (see Figure 18(p)).

(f) $\pi(u_{i+1}) = 3$ and $i = 5$.

In that case, $\pi(u_4) \in \{1, 2\}$. If $\pi(u_4) = 1$, then $\pi(v_4) = 2$ and no colour is available for u_3 . If $\pi(u_4) = 2$, then $\pi(v_4) = 1$, so that $\pi(u_3) = 1$ and $\pi(v_3) = 3$, and no colour is available for u_2 (see Figure 18(q)).

This completes the proof of Claim 1. \square

By Claim 1, we can thus assume $\pi(u_i) = 2$, without loss of generality (again, thanks to the symmetry exchanging u_i and v_i), so that $\pi(v_i) \in \{3, 4, 5\}$. To finish the proof of Lemma 4, we need to prove that $\{\pi(u_{i-1}), \pi(u_{i+1})\} = \{3, 4, 5\} \setminus \{\pi(v_i)\}$. Suppose that this is not the case. We consider the following cases, according to the value of $\pi(v_i)$.

1. $\pi(v_i) = 3$.

In that case, we necessarily have $\pi(u_{i+1}) \in \{1, 4, 5\}$.

If $\pi(u_{i+1}) = 1$, then $\{\pi(u_{i+2}), \pi(v_{i+1})\} = \{4, 5\}$, so that $\pi(u_{i-1}) = 1$, and no colour is available for v_{i-1} (see Figure 19(a)).

If $\pi(u_{i+1}) = 4$, then either $\pi(u_{i-1}) = 1$, so that $\pi(v_{i-1}) = 5$, and no colour is available for u_{i-2} (see Figure 19(b)), or $\pi(u_{i-1}) = 5$, which contradicts our assumption since it would imply $\{\pi(u_{i-1}), \pi(u_{i+1})\} = \{3, 4, 5\} \setminus \{\pi(v_i)\}$.

Similarly, if $\pi(u_{i+1}) = 5$, then either $\pi(u_{i-1}) = 1$, so that $\pi(v_{i-1}) = 4$, and no colour is available for u_{i-2} (see Figure 19(c)), or $\pi(u_{i-1}) = 4$, which again contradicts our assumption.

2. $\pi(v_i) = 4$ (the case $\pi(v_i) = 5$ is similar, by switching colours 4 and 5).

In that case, we necessarily have $\pi(u_{i+1}) \in \{1, 3, 5\}$.

If $\pi(u_{i+1}) = 1$, then $\{\pi(u_{i+2}), \pi(v_{i+1})\} = \{3, 5\}$. If $\pi(u_{i+2}) = 3$ and $\pi(v_{i+1}) = 5$, then $\pi(u_{i-1}) = 1$, so that $\pi(u_{i-2}) = 3$, and no colour is available for v_{i-1} . If $\pi(u_{i+2}) = 5$ and $\pi(v_{i+1}) = 3$, then $\pi(v_{i-1}) = 1$, and no colour is available for u_{i-1} (see Figure 19(d)).

If $\pi(u_{i+1}) = 3$, then either $\pi(u_{i-1}) = 1$, so that $\pi(v_{i-1}) = 5$, and no colour is available for u_{i-2} , or $\pi(u_{i-1}) = 5$, which contradicts our assumption (see Figure 19(e)).

Finally, if $\pi(u_{i+1}) = 5$, then either $\pi(u_{i-1}) = 1$, so that $\pi(v_{i-1}) = 3$, and no colour is available for u_{i-2} , or $\pi(u_{i-1}) = 3$, which contradicts our assumption (see Figure 19(f)).

This completes the proof of Lemma 5.

B Proof of Lemma 13

We first prove the following claim.

Claim 2 For every integer j , $0 \leq j \leq r$, either $\pi(u_{2j}^0) = 1$ or $\pi(u_{2j+1}^0) = 1$.

Proof. Thanks to the symmetries of $H^\ell(r)$, it is enough to prove the claim for the edge $u_2^0 u_3^0$. Suppose to the contrary that $\pi(u_2^0) \neq 1$ and $\pi(u_3^0) \neq 1$. Thanks to the symmetries of $H^\ell(r)$, we can assume $\pi(u_2^0) < \pi(u_3^0)$, without loss of generality.

We consider four cases. The corresponding configurations are depicted in Figure 20, using the same drawing convention as for the proof of Lemma 4 (see Appendix A).

1. $\pi(u_2^0) = 2$ and $\pi(u_3^0) = 3$.

In that case, $\pi(u_2^1) \in \{1, 4, 5\}$. If $\pi(u_2^1) = 1$, then $\{\pi(u_2^2), \pi(u_3^1)\} = \{4, 5\}$, which implies $\pi(u_1^0) = 1$, and no colour is available for u_0^0 (see Figure 20(a)). If $\pi(u_2^1) = 4$, then either $\pi(u_1^0) = 1$, which implies $\pi(u_0^0) = 5$, and no colour is available for u_1^1 , or $\pi(u_1^0) = 5$, which implies $\pi(u_4^0) = 1$, $\pi(u_5^0) = 2$, and no colour is available for u_4^1 (see Figure 20(b)). The case $\pi(u_2^1) = 5$ is similar, by switching colours 4 and 5.

2. $\pi(u_2^0) = 2$ and $\pi(u_3^0) = 4$ (the case $\pi(u_2^0) = 2$ and $\pi(u_3^0) = 5$ is similar, by switching colours 4 and 5).

In that case, $\pi(u_2^1) \in \{1, 3, 5\}$. If $\pi(u_2^1) = 1$, then $\{\pi(u_2^2), \pi(u_3^1)\} = \{3, 5\}$, which implies $\pi(u_1^0) = 1$, $\pi(u_0^0) = 3$, and no colour is available for u_1^1 (see Figure 20(c)). If $\pi(u_2^1) = 3$, then either $\pi(u_1^0) = 1$, which implies $\pi(u_0^0) = 5$, and no colour is available for u_1^1 , or $\pi(u_1^0) = 5$, which implies $\pi(u_2^2) = \pi(u_3^1) = 1$, so that $\pi(u_3^2) = 2$, and no colour is available for u_2^3 (see Figure 20(d)). Finally, if $\pi(u_2^1) = 5$, then either $\pi(u_1^0) = 1$, which implies $\pi(u_0^0) = 3$, and no colour is available for u_1^1 , or $\pi(u_1^0) = 3$, which implies $\pi(u_1^1) = 1$, $\pi(u_2^1) = 2$, and no colour is available for u_0^1 (see Figure 20(e)).

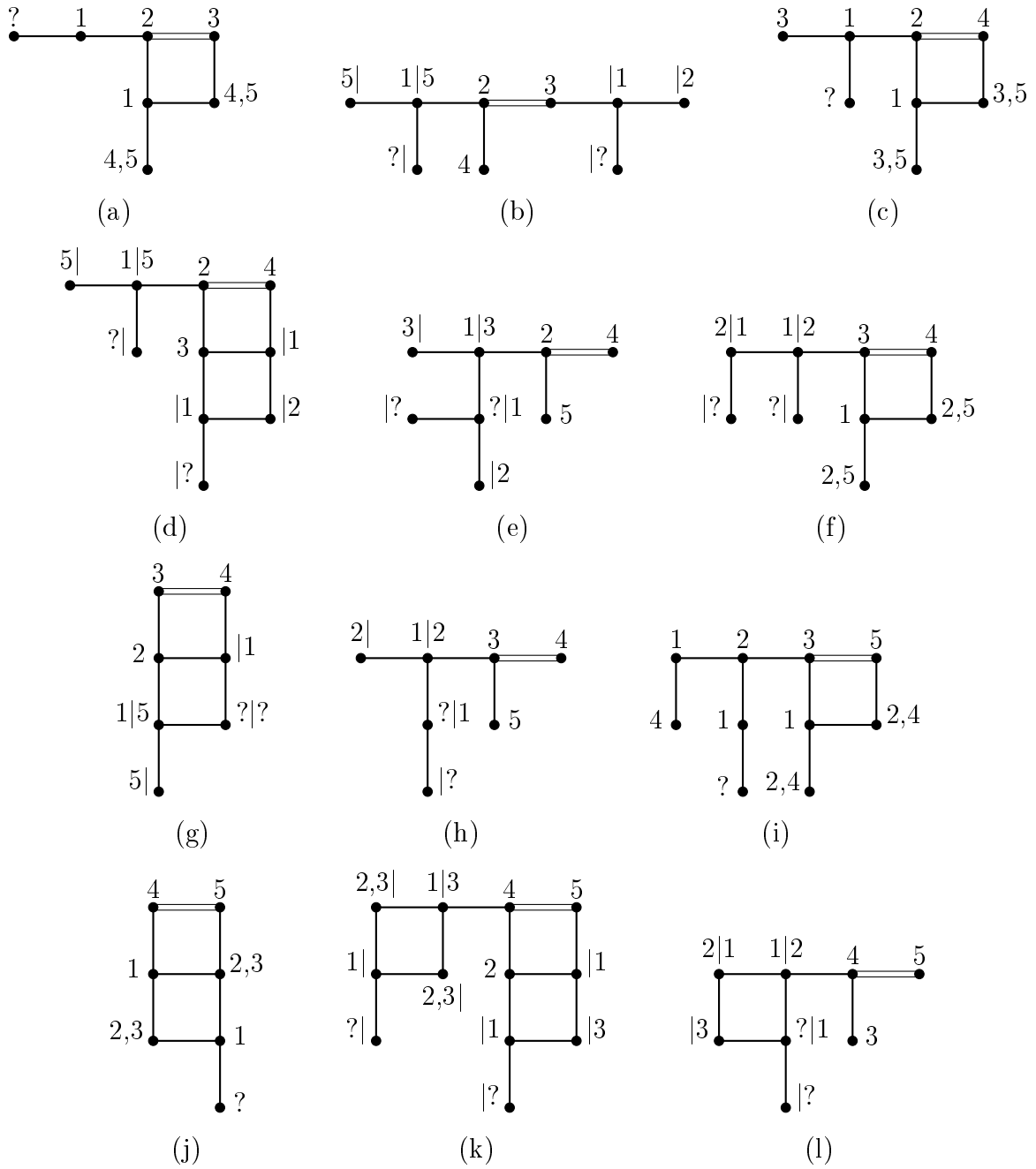


Figure 20: Configurations for the proof of Claim 2 (the double edge is the edge $u_2^0 u_3^0$).

3. $\pi(u_2^0) = 3$ and $\pi(u_3^0) = 4$.

In that case, $\pi(u_2^1) \in \{1, 2, 5\}$. If $\pi(u_2^1) = 1$, then $\{\pi(u_2^2), \pi(u_3^1)\} = \{2, 5\}$, and thus either $\pi(u_1^0) = 1$, so that $\pi(u_0^0) = 2$, and no colour is available for u_1^1 , or $\pi(u_1^0) = 2$, so that $\pi(u_0^0) = 1$, and no colour is available for u_0^1 (see Figure 20(f)). If $\pi(u_2^1) = 2$, then either $\pi(u_2^2) = 1$, which implies $\pi(u_3^2) = 5$, and no colour is available for u_3^2 , or $\pi(u_2^2) = 5$, which implies $\pi(u_3^1) = 1$, and no colour is available for u_3^2 (see Figure 20(g)). Finally, if $\pi(u_2^1) = 5$, then either $\pi(u_1^0) = 1$, which implies $\pi(u_0^0) = 2$, and no colour is available for u_1^1 , or $\pi(u_1^0) = 2$, which implies $\pi(u_1^1) = 1$, and no colour is available for u_1^2 (see Figure 20(h)).

4. $\pi(u_2^0) = 3$ and $\pi(u_3^0) = 5$.

This case is similar to the previous one, by switching colours 4 and 5, except when $\pi(u_2^1) = 1$ (which implies $\{\pi(u_2^2), \pi(u_3^1)\} = \{2, 4\}$) and $\pi(u_1^0) = 2$. In that case, we necessarily have $\pi(u_0^0) = \pi(u_1^1) = 1$, which implies $\pi(u_0^1) = 4$, and no colour is available for u_1^2 (see Figure 20(i)).

5. $\pi(u_2^0) = 4$ and $\pi(u_3^0) = 5$.

In that case, $\pi(u_2^1) \in \{1, 2, 3\}$. If $\pi(u_2^1) = 1$, then $\{\pi(u_2^2), \pi(u_3^1)\} = \{2, 3\}$, which implies $\pi(u_3^2) = 1$, and no colour is available for u_3^3 (see Figure 20(j)). If $\pi(u_2^1) = 2$, then either $\pi(u_1^0) = 1$, which implies $\{\pi(u_0^0), \pi(u_1^1)\} = \{2, 3\}$, so that $\pi(u_0^1) = 1$, and no colour is available for u_0^2 , or $\pi(u_1^0) = 3$, which implies $\pi(u_2^2) = \pi(u_3^1) = 1$, so that $\pi(u_4^1) = 3$, and no colour is available for u_2^3 (see Figure 20(k)). Finally, if $\pi(u_2^1) = 3$, then either $\pi(u_1^0) = 1$, which implies $\pi(u_0^0) = 2$, and no colour is available for u_1^1 , or $\pi(u_1^0) = 2$, which implies $\pi(u_0^0) = \pi(u_1^1) = 1$, so that $\pi(u_0^1) = 3$, and no colour is available for u_1^2 (see Figure 20(l)).

This completes the proof of Claim 2. □

Since the cycle induced by the set of vertices $\{u_0^0, u_1^0, \dots, u_{2r-1}^0\}$ has even length, and adjacent vertices cannot be assigned the same colour, it follows from Claim 2 that colour 1 must be used on each edge $u_j^0 u_{j+1}^0$, $0 \leq j \leq 2r - 1$. This concludes the proof of Lemma 13.