

Homomorphisms and colourings of oriented graphs: An updated survey

É. Sopena^{a,b}

^a*Univ. Bordeaux, LaBRI, UMR5800, F-33400 Talence*

^b*CNRS, LaBRI, UMR5800, F-33400 Talence*

Abstract

An oriented graph is a loopless digraph with no opposite arcs. An oriented k -colouring of an oriented graph \vec{G} is a partition of its set of vertices into k parts in such a way that no two adjacent vertices belong to the same part, and all the arcs connecting every two parts have the same direction. Hence, such a colouring exists if and only if \vec{G} admits a homomorphism to some oriented graph of order k .

The oriented chromatic number of \vec{G} is then defined as the smallest k for which \vec{G} admits an oriented k -colouring or, equivalently, as the minimum order of an oriented graph to which \vec{G} admits a homomorphism.

In this paper, we survey the main results about oriented colourings and propose a few open problems.

Keywords: Homomorphism, Oriented chromatic number, Oriented colouring, Oriented graph

1. Introduction

A (proper) k -colouring of an undirected graph G is a partition of $V(G)$ into k parts, called colour classes, such that no two adjacent vertices belong to the same colour class. Such a k -colouring can be equivalently regarded as a homomorphism of G to the complete graph K_k on k vertices. Therefore the chromatic number $\chi(G)$ of G , defined as the smallest k such that G admits a k -colouring, corresponds to the smallest k such that G admits a

Email address: `eric.sopena@labri.fr` (É. Sopena)

homomorphism to K_k (for a survey on graph homomorphisms, see e.g. Hahn and Tardif [30] or the monography by Hell and Nešetřil [31]).

By considering homomorphisms of oriented graphs (that is loopless digraphs with no opposite arcs), we get a natural extension of the notion of colouring to oriented graphs, leading in particular to the notion of the oriented chromatic number of a graph. This notion was first considered by Courcelle [19] as a tool for encoding graph orientations by means of vertex labels. Since then, oriented colourings and homomorphisms of oriented graphs have attracted much attention.

Several notions of colourings of oriented graphs, or digraphs, have been considered in the literature, which differ from the notion of oriented colouring discussed here. In particular, Neumann-Lara [45] introduced in 1982 the *dichromatic* number of a digraph, a well-studied parameter, defined as the minimum number of parts into which its vertex set can be partitioned in such a way that each part induces an acyclic subgraph.

The aim of this paper is to survey the main results concerning colourings of oriented graphs and to propose a few open problems. It can thus be considered as an update of the papers [64] and [65] (see also [63]), and is organised as follows. In the next section, we introduce the main definitions and notation. We give general results in Section 3, relating the oriented chromatic number to other graph parameters, and focus on the special classes of planar graphs in Section 4 and of graphs with bounded degree in Section 5. We then consider various other classes of graphs in Section 6. We discuss complexity issues in Section 7 and the game version of oriented colouring in Section 8. Finally, we deal with two weaker versions of oriented colourings, namely simple and 2-dipath colourings, in Section 9 and propose some open problems in the last section.

2. Preliminaries

If G is an undirected graph, we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. If \vec{G} is a digraph, we denote by $V(\vec{G})$ its set of vertices and by $A(\vec{G})$ its set of arcs. The *order* of an undirected graph, or a digraph, is the cardinality of its vertex set. All the graphs we consider are simple and have no loops.

Definition 1 (Homomorphism). Let G and H be two undirected graphs. A *homomorphism* of G to H is a mapping $f : V(G) \rightarrow V(H)$ that preserves

the edges: $f(x)f(y) \in E(H)$ whenever $xy \in E(G)$.

Similarly, if \vec{G} and \vec{H} are two digraphs, a homomorphism of \vec{G} to \vec{H} is a mapping $f : V(\vec{G}) \rightarrow V(\vec{H})$ that preserves the arcs: $\overrightarrow{f(x)f(y)} \in E(\vec{H})$ whenever $\overrightarrow{xy} \in E(\vec{G})$.

An *orientation* of an undirected graph G is a digraph obtained from G by giving to every edge uv of G one of its two possible orientations, namely \overrightarrow{uv} or \overrightarrow{vu} . A digraph \vec{G} is an *oriented graph* if it is an orientation of some undirected graph, called the *underlying graph* of \vec{G} . In the following, an arc \overrightarrow{uv} will be simply denoted uv .

2.1. Oriented colourings

As for undirected graphs, homomorphisms of oriented graphs induce a notion of oriented graph colouring as follows.

Definition 2 (Oriented k -colouring). An *oriented k -colouring* of an oriented graph \vec{G} is a partition of $V(\vec{G})$ into k colour classes such that no two adjacent vertices belong to the same colour class, and all the arcs connecting every two colour classes have the same direction.

Equivalently, an oriented k -colouring of \vec{G} may be regarded as a homomorphism of \vec{G} to some oriented graph \vec{H}_k of order k (the vertices of \vec{H}_k are then used as colours and such a homomorphism is sometimes called an \vec{H}_k -colouring), or as a mapping γ from $V(\vec{G})$ to a set of k colours such that

- (i) $\gamma(u) \neq \gamma(v)$ for every arc uv in $A(\vec{G})$,
- (ii) $\gamma(u) \neq \gamma(x)$ for every two arcs uv and wx with $\gamma(v) = \gamma(w)$.

Definition 3 (Oriented chromatic number). The *oriented chromatic number* $\chi_o(\vec{G})$ of an oriented graph \vec{G} is the smallest k for which \vec{G} admits an oriented k -colouring or, equivalently, the minimum order of an oriented graph \vec{H} such that \vec{G} admits a homomorphism to \vec{H} .

This notion can be extended to undirected graphs as follows: the oriented chromatic number of an undirected graph G is defined as the maximum of the oriented chromatic numbers of its orientations. Similarly, the oriented

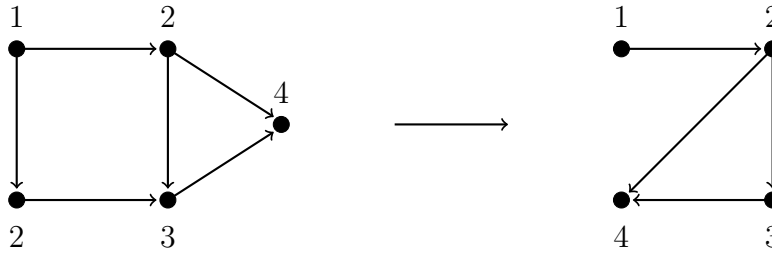


Figure 1: An oriented 4-colouring of an oriented graph \vec{G}

chromatic number of a class of undirected or oriented graphs is defined as the maximum of the oriented chromatic numbers of its members.

Figure 1 shows an oriented 4-colouring of an oriented graph \vec{G} , viewed as a homomorphism to an oriented graph \vec{H} of order 4. It can easily be checked that \vec{G} admits no oriented 3-colouring, so that $\chi_o(\vec{G}) = 4$.

Observe that conditions (i) and (ii) above imply that any two vertices connected by a directed path of length 1 or 2 must be assigned distinct colours in any oriented colouring. We then get for instance that the oriented chromatic number of the directed cycle \vec{C}_5 on five vertices is 5. This property can also be expressed by saying that homomorphisms of oriented graphs “preserve” directed paths of length at most 2 (the image of such a path is still a directed path of the same length). Homomorphisms preserving other types of configurations, namely paths of length k or trees, have been considered in [48].

2.2. Oriented cliques

An oriented graph \vec{G} is an *oriented clique* (or, simply, an *o-clique*) if $\chi_o(\vec{G}) = |V(\vec{G})|$. Therefore, an oriented graph is an o-clique if and only if every two of its vertices are connected (in either direction) by a directed path of length 1 or 2. Every tournament is obviously an o-clique but being a tournament is not a necessary condition for being an o-clique. Figure 2 depicts some sample o-cliques of order up to 7.

Let \vec{O}_n be an o-clique of order n . Take now two copies of \vec{O}_n , a new vertex v , and add all possible arcs from vertices of the first copy towards v and all possible arcs from v towards vertices of the second copy. The oriented graph so-obtained is indeed an o-clique of order $2n + 1$. (The o-clique \vec{O}_7

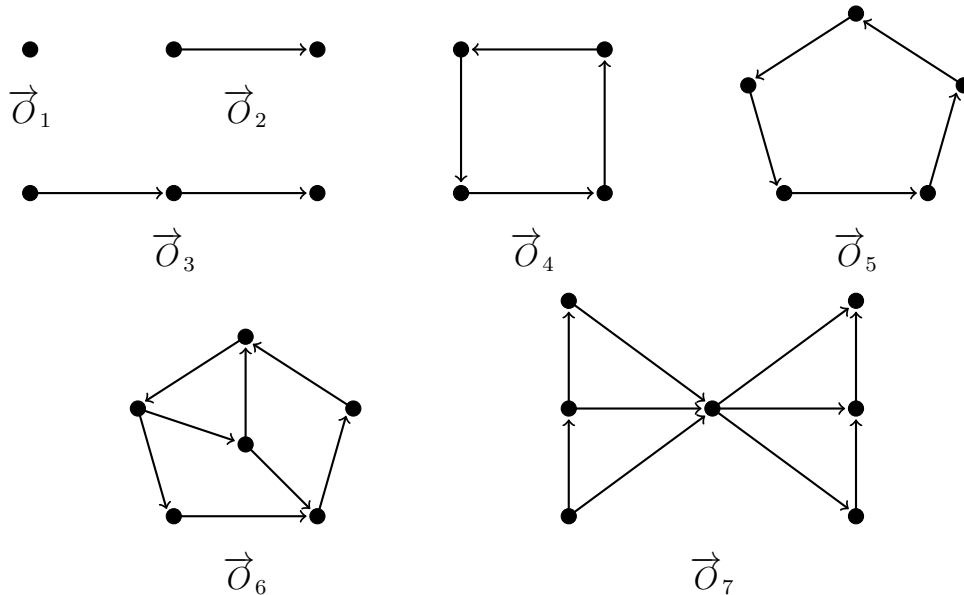


Figure 2: Sample o-cliques of order k , $1 \leq k \leq 7$

given in Figure 2 is obtained in this way from the o-clique \vec{O}_3 , which is itself so-obtained from the o-clique \vec{O}_1 .)

An undirected graph G is a k -tree if either $G = K_k$ or G is obtained from a k -tree H by adding a new vertex connected to a subgraph of H isomorphic to K_k . A subgraph of a k -tree is a *partial k -tree*. Every forest is thus a 1-tree and every outerplanar graph is a partial 2-tree (recall that a planar graph is *outerplanar* if it admits a planar embedding such that every vertex lies on the outerface). In [35], Klostermeyer and MacGillivray proved that the maximum order of an o-clique whose underlying graph is a partial 2-tree is 7 which implies, in particular, that the maximum order of an outerplanar o-clique is 7 (observe that the o-clique \vec{O}_7 of order 7 depicted in Figure 2 is indeed outerplanar). They also proved that every planar o-clique has order at most 36 and conjectured that the maximum order of a planar o-clique is 15. Notice that applying the above-described construction to the outerplanar o-clique \vec{O}_7 produces a planar o-clique of order 15. This conjecture has been recently settled by Sen [59]:

Theorem 4 (Sen, 2012). *The maximum order of a planar o-clique is 15.*

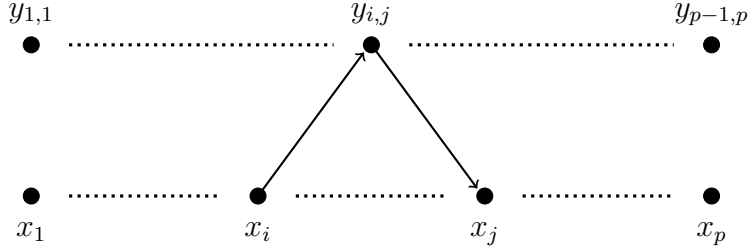


Figure 3: The oriented graph \vec{H}_p

It was proved in [26] and [37] that the minimum number of edges in an o-clique of order n is $(1 + o(1))n \log_2 n$.

2.3. Oriented clique numbers

The clique number $\omega(G)$ of an undirected graph G is the maximum order of a complete subgraph of G and, obviously, a lower bound on the chromatic number of G . We can define the *absolute* oriented clique number of an oriented graph \vec{G} in a similar way:

Definition 5 (Absolute oriented clique number). The *absolute oriented clique number* $\omega_{ao}(\vec{G})$ of an oriented graph \vec{G} is the maximum order of an o-clique subgraph of \vec{G} .

However, it may happen that all vertices of a subset S of $V(\vec{G})$ require distinct colours in every oriented colouring of \vec{G} although the subgraph of \vec{G} induced by S is not itself an o-clique. This leads to the notion of the *relative* oriented clique number:

Definition 6 (Relative oriented clique number). The *relative oriented clique number* $\omega_{ro}(\vec{G})$ of an oriented graph \vec{G} is the maximum cardinality of a subset S of \vec{G} such that every two vertices of S are connected in \vec{G} by a directed path of length 1 or 2.

From these definitions we clearly have, for every oriented graph \vec{G} , the following inequalities:

$$\omega_{ao}(\vec{G}) \leq \omega_{ro}(\vec{G}) \leq \chi_o(\vec{G}).$$

girth	ω_{ao}	ω_{ro}
3	15	$15 \leq \omega_{ro} \leq 80$
4	6	$10 \leq \omega_{ro} \leq 26$
5	5	6
6	3	4
≥ 7	3	3

Table 1: Maximum absolute and relative oriented clique numbers of planar graphs with given girth

The following example (see Figure 3) shows that the difference between the absolute and the relative clique numbers can be arbitrarily large. Let \vec{H}_p , $p \geq 3$, be the bipartite oriented graph given by:

$$V(\vec{H}_p) = \{x_i \mid 1 \leq i \leq p\} \cup \{y_{j,k} \mid 1 \leq j < k \leq p\},$$

$$E(\vec{H}_p) = \{x_i y_{i,j} \mid 1 \leq i < j \leq p\} \cup \{y_{i,j} x_j \mid 1 \leq i < j \leq p\}.$$

Clearly, every two vertices x_i and x_j , $1 \leq i < j \leq p$, are connected by a directed 2-path going through $y_{i,j}$ and, therefore, $\omega_{ro}(\vec{H}_p) \geq p$. On the other hand, it is not difficult to check that there is no o-clique subgraph in \vec{H}_p of order greater than 3, so that $\omega_{ao}(\vec{H}_p) = 3$.

The maximum value of the absolute and relative clique numbers of planar oriented graphs with given girth have been studied in [60] and [44]. The corresponding results are summarized in Table 1.

3. General results

One of the main difficulties when dealing with oriented colourings is that, contrary to the undirected case, we cannot decide “locally” the availability of a colour for a given vertex u : if u has some already coloured neighbour v then, in order to satisfy condition (ii), we need to check the neighbourhood of all other vertices in the oriented graph having the same colour as v . Hence, the (ordinary) chromatic number of an undirected graph G is upper bounded by $\Delta(G) + 1$ (by Brook’s Theorem), while its oriented chromatic number can be exponential in $\Delta(G)$ (see the sentence following Theorem 18). Therefore, it is not surprising that we cannot bound the oriented chromatic number of

an oriented graph by a function of the ordinary chromatic number of its underlying undirected graph. To see this, consider for instance the orientation $\vec{K}_{n,n}$ of the complete bipartite undirected graph $K_{n,n}$ obtained by orienting all the edges from the first part of the bipartition to the other part, and then reversing the orientation of all the edges of any perfect matching of $K_{n,n}$. Doing this, every two vertices are clearly connected by a directed path of length at most 2, so that $\chi_o(\vec{K}_{n,n}) = 2n$ while $\chi(K_{n,n}) = 2$.

In this section, we give some general relations between the oriented chromatic number of an undirected graph and its acyclic chromatic number or its maximum average degree. We then consider oriented graphs that are critical with respect to oriented colourings.

3.1. Graphs with bounded acyclic chromatic number

A proper vertex colouring of an undirected graph G is *acyclic* if, for every two colours a and b , the subgraph of G induced by the vertices with colour a or b is a forest. In other words, every cycle of G uses at least three colours. The *acyclic chromatic number* $\chi_a(G)$ of G is then the smallest number of colours needed by an acyclic colouring of G .

Raspaud and Sopena [58] proved that every graph with bounded acyclic chromatic number has bounded oriented chromatic number:

Theorem 7 (Raspaud and Sopena, 1994). *If G is an undirected graph with $\chi_a(G) \leq k$, then $\chi_o(G) \leq k \cdot 2^{k-1}$.*

In [50], Ochem proved that this bound is tight for every $k \geq 3$. Kostochka, Sopena and Zhu proved in [38] that, conversely, every class of graphs with bounded oriented chromatic number has bounded acyclic chromatic number:

Theorem 8 (Kostochka, Sopena and Zhu, 1997). *If G is an undirected graph with $\chi_o(G) \leq k$, then $\chi_a(G) \leq k^{\lceil \log_2(\lceil \log_2 k \rceil + k/2) \rceil + 1}$.*

Since every planar graph has acyclic chromatic number at most 5 (a celebrated result of Borodin, see [8]), we get by Theorem 7 that every planar graph has oriented chromatic number at most $5 \cdot 2^4 = 80$. This upper bound is indeed the best known bound for planar graphs up to now (see also Subsections 4.3, 9.1 and Problem 1 in Section 10).

mad	girth	χ_o	Author(s)
$< 10/3$		≤ 19	Borodin, Kostochka, Nešetřil, Raspaud and Sopena [15]
$< 10/3$	≥ 5	≤ 16	Pinlou [56]
< 3		≤ 11	Borodin, Kostochka, Nešetřil, Raspaud and Sopena [15]
$< 11/4$	≥ 5	≤ 7	Borodin, Kostochka, Nešetřil, Raspaud and Sopena [15]
$< 12/5$	≥ 5	≤ 5	Borodin, Ivanova and Kostochka [12]

Table 2: Oriented chromatic number of graphs with bounded maximum average degree

3.2. Graphs with bounded maximum average degree

The *maximum average degree* $\text{mad}(G)$ of an undirected graph G is defined as the maximum of the average degrees of its subgraphs, that is

$$\text{mad}(G) = \max_{H \subseteq G} \left\{ \frac{2|E(H)|}{|V(H)|} \right\}.$$

Intuitively speaking, the maximum average degree of an undirected graph provides a measure of its “local density”. Recall that the *girth* of an undirected graph G is the length of a shortest cycle in G . If G is a planar graph with girth at least g then, using Euler’s formula, it can be proved that G has maximum degree less than $2g/(g-2)$. Therefore, several authors have considered the problem of determining the oriented chromatic number of graphs with bounded maximum average degree (and, sometimes, with high girth), in order to get upper bounds on the oriented chromatic number of planar graphs with high girth [12, 15, 56]. The best known upper bounds are summarized in Table 2, some of them giving the best known upper bounds for planar graphs with high girth (see Subsection 4.3).

3.3. Critical graphs

An oriented graph \vec{G} is *vertex-critical* if, for every vertex u in $V(\vec{G})$, $\chi_o(\vec{G} - u) < \chi_o(\vec{G})$. Similarly, an oriented graph \vec{G} is *arc-critical* if, for every arc uv in $A(\vec{G})$, $\chi_o(\vec{G} - uv) < \chi_o(\vec{G})$.

Let \vec{G} be an oriented graph, γ an oriented colouring of \vec{G} and u and v two non-adjacent vertices of \vec{G} . By giving two new colours to u and v , we clearly get an oriented colouring γ' of $\vec{G} + uv$. Hence, $\chi_o(\vec{G} + uv) - \chi_o(\vec{G}) \leq 2$. On the other hand, Borodin, Fon-der-Flaass, Kostochka, Raspaud and Sopena [9] proved that such a universal bound does not exist when we add new vertices:

Theorem 9 (Borodin *et al.*, 2001). *For every $k > 0$, there exists an oriented graph \vec{G}_k such that $\chi_o(\vec{G}_k) - \chi_o(\vec{G}_k - u) \geq k$ for every vertex u in $V(\vec{G}_k)$ and $\chi_o(\vec{G}_k) - \chi_o(\vec{G}_k - uv) = 2$ for every arc uv in $A(\vec{G}_k)$.*

4. Planar graphs

Due to the famous Four-Colour Theorem for ordinary colourings of undirected planar graphs, it is not surprising that the study of oriented colourings of various classes of planar graphs has attracted much attention in the literature.

Since every oriented forest \vec{F} contains a vertex with degree 1, it clearly admits a homomorphism to \vec{C}_3 , the directed cycle of order 3 (every vertex in \vec{C}_3 has indegree 1 and outdegree 1). Hence, for every undirected forest F , $\chi_o(F) \leq 3$.

As observed in Section 2, the directed cycle \vec{C}_5 on 5 vertices is an o-clique (\vec{O}_5 in Figure 2) and, therefore, has oriented chromatic number 5. It can be proved that every other oriented cycle has oriented chromatic number at most 4. Let \vec{C} be an oriented cycle. We define the *level* $\lambda(\vec{C})$ of C as the number of forward arcs minus the number of backward arcs in \vec{C} . We then have:

Proposition 10. *If \vec{C} is an oriented cycle then:*

- (1) *if the orientation of \vec{C} is alternating then $\chi_o(\vec{C}) = 2$,*
- (2) *if the orientation of \vec{C} is not alternating and either $\lambda(\vec{C}) \equiv 0 \pmod{3}$ or \vec{C} does not contain three consecutive arcs with the same direction then $\chi_o(\vec{C}) = 3$,*
- (3) *otherwise, $\chi_o(\vec{C}) = 4$ if \vec{C} is not the directed cycle \vec{C}_5 on 5 vertices.*

Proof. (1) If the orientation of \vec{C} is alternating then an oriented 2-colouring γ is obtained by setting $\gamma(u) = 1$ for every source vertex u and $\gamma(v) = 2$ for every sink vertex v .

(2) If the orientation of \vec{C} is not alternating, then \vec{C} contains a directed path of length 2, and thus $\chi_o(\vec{C}) \geq 3$.

Suppose that $\lambda(\vec{C}) \equiv 0 \pmod{3}$ and let $\vec{C} = u_0u_1 \dots u_{n-1}u_0$. A homomorphism f of \vec{C} to the directed 3-cycle \vec{C}_3 , with arcs 01, 12 and 20, can be iteratively constructed as follows: $f(u_0) = 0$ and for every i , $1 \leq i \leq n - 1$,

$f(u_i) = f(u_{i-1}) + 1 \pmod{3}$ if $u_{i-1}u_i$ is an arc and $f(u_i) = f(u_{i-1}) - 1 \pmod{3}$ if u_iu_{i-1} is an arc. Since $\lambda(\vec{C}) \equiv 0 \pmod{3}$, we get that $f(u_{n-1}) = 2$ if $u_{n-1}u_0$ is an arc and $f(u_{n-1}) = 1$ if u_0u_{n-1} is an arc. Hence, f is indeed a homomorphism of \vec{C} to \vec{C}_3 .

Suppose now that \vec{C} does not contain three consecutive arcs with the same direction. A homomorphism f of \vec{C} to the transitive tournament \vec{T}_3 of order 3, with arcs 01, 12 and 02, can be constructed as follows: set $f(u) = 0$ for every source vertex u , $f(v) = 2$ for every sink vertex v and $f(w) = 1$ for every vertex w which is neither a source nor a sink. Since \vec{C} does not contain three consecutive arcs with the same direction, the mapping f is well-defined and is indeed a homomorphism of \vec{C} to \vec{T}_3 .

(3) Assume first that \vec{C} is a directed cycle (of length 3, 4 or at least 6). Since every nonnegative integer, except 1, 2 and 5, can be expressed as a sum of 3's and 4's, we can construct an oriented 4-colouring of \vec{C} by "splitting" the cycle into directed paths of length 3 or 4 and colouring each such path 123 or 1234 (in this way, we obtain a homomorphism of \vec{C} to the oriented graph of order 4 with arcs 12, 23, 34, 31 and 41). Hence, $\chi_o(\vec{C}) \leq 4$. Since $\lambda(\vec{C}) \not\equiv 0 \pmod{3}$, at least one directed 4-path must appear in the above splitting, and thus $\chi_o(\vec{C}) = 4$.

Finally, assume that $\vec{C} = u_0u_1 \dots u_{n-1}u_0$ is not a directed cycle and, without loss of generality, that u_0 is a source vertex. Since $\vec{C} - \{u_0\}$ is a path, it admits a homomorphism to the directed cycle \vec{C}_3 of order 3. Hence, by assigning a fourth colour to the vertex u_0 , we get $\chi_o(\vec{C}) \leq 4$. If $\chi_o(\vec{C}) \leq 3$, then \vec{C} admits a homomorphism to \vec{C}_3 or \vec{T}_3 , the transitive tournament of order 3, which implies that $\lambda(\vec{C}) \equiv 0 \pmod{3}$ or \vec{C} does not contain any directed path of length 3, contradicting the fact that we are not in cases (1) or (2). Hence $\chi_o(\vec{C}) = 4$ and we are done. \square

4.1. Outerplanar graphs

The maximum oriented chromatic number of an outerplanar undirected graph was determined in [64]:

Theorem 11 (Sopena, 1997). *If G is an outerplanar undirected graph, then $\chi_o(G) \leq 7$ and this bound is tight.*

In fact, this result was shown to hold for the larger class of partial 2-trees. The tightness of the bound comes from the outerplanar o-clique \vec{O}_7 depicted in Figure 2.

Pinlou and Sopena [57] later determined the oriented chromatic number of outerplanar graphs (in fact, again more generally, of partial 2-trees) of given girth:

Theorem 12 (Pinlou and Sopena, 2006). *If G is an outerplanar undirected graph with girth g , then*

- (1) $\chi_o(G) \leq 6$ if $g \geq 4$, and this bound is tight,
- (2) $\chi_o(G) \leq 5$ if $g \geq 5$, and this bound is tight.

A graph is 2-outerplanar if it has a planar embedding such that the subgraph obtained by removing the vertices of the outerface is outerplanar. In [24], Esperet and Ochem proved that every 2-outerplanar graph has oriented chromatic number at most 67. This bound was recently decreased to 40 by Ochem and Pinlou [53]:

Theorem 13 (Ochem and Pinlou, 2014). *If G is a 2-outerplanar undirected graph, then $\chi_o(G) \leq 40$.*

4.2. Grid graphs

The *rectangular grid* $G_{m,n}$ is the Cartesian product (see Subsection 6.3) of P_m by P_n , the two paths with m and n vertices, respectively. Fertin, Raspaud and Roychowdhury [25] proved the following:

Theorem 14 (Fertin, Raspaud and Roychowdhury, 2003). *For every integers $m, n \geq 1$, $\chi_o(G_{m,n}) \leq 11$.*

Moreover, they proved that there exists an orientation of the grid $G_{4,5}$ with oriented chromatic number 7. They also conjectured that the oriented chromatic number of every rectangular grid is at most 7, and that every such grid admits a homomorphism to the Paley tournament \vec{T}_7 , given by $V(\vec{T}_7) = \{0, 1, \dots, 6\}$ and $uv \in A(\vec{T}_7)$ if and only if $v \equiv u + x \pmod{7}$ for some $x \in \{1, 2, 4\}$. They proved that this conjecture is true when $m = 2, 3$.

Szepietowski and Targan [69] later proved that this conjecture is also true when $m = 4$ but that the second part of the conjecture fails to be true whenever $m \geq 5$ by providing an orientation of the grid $G_{5,33}$ which admits no

homomorphism to the tournament \overrightarrow{T}_7 . The first part of the conjecture was finally disproved by Dybizbański and Nenca [22] who exhibited an orientation of the grid $G_{7,212}$ with oriented chromatic number at least 8.

Bielak [5] studied hexagonal grids $H_{m,n}$, made of m rows of n hexagons and she proved the following:

Theorem 15 (Bielak, 2006). *For all integers $m, n \geq 1$, $\chi_o(H_{m,n}) \leq 6$. Moreover, $\chi_o(H_{1,n}) = 5$ for every $n \geq 3$.*

4.3. Planar graphs with given girth

In [8], Borodin proved that every undirected planar graph is acyclically 5-colourable. Therefore, using Theorem 7 [58], we get:

Theorem 16 (Raspaud and Sopena, 1994). *If G is a planar undirected graph, then $\chi_o(G) \leq 80$.*

Despite many efforts, this upper bound has not been improved yet. As observed in Section 3, there exists a planar o-clique of order 15, which immediately gives $\chi_o \geq 15$. This lower bound on the oriented chromatic number of the class of planar graphs has successively been improved to 16 by Sopena [66], to 17 by Marshall [40], and to 18, again by Marshall [41].

Several authors have considered the problem of determining the oriented chromatic number of classes of planar graphs with given girth, successively improving various lower or upper bounds (see [58], [46], [14], [15], [49], [13], [10], [11], [24], [12], [43], [52], [56], [53], [41], [42]). The best known lower and upper bounds are summarized in Table 3 (first reference is for the lower bound, second reference is for the upper bound).

Borodin, Kim, Kostochka and West [13] considered other surfaces and proved the following:

Theorem 17 (Borodin, Kim, Kostochka and West, 2004). *If G is an undirected graph with girth at least 13, embeddable on the torus or the Klein bottle, then $\chi_o(G) \leq 5$.*

5. Graphs with bounded degree

It was proved in [64] that every undirected graph with maximum degree Δ has oriented chromatic number at most $(2\Delta - 1) \cdot 2^{2\Delta-2}$. This bound was further decreased as follows [38]:

girth	χ_o	Author(s)
≥ 3	$18 \leq \chi_o \leq 80$	Marshall [41] - Raspaud and Sopena [58]
≥ 4	$11 \leq \chi_o \leq 40$	Ochem [49] - Ochem and Pinlou [53]
≥ 5	$7 \leq \chi_o \leq 16$	Marshall [42] - Pinlou [56]
≥ 6	$7 \leq \chi_o \leq 11$	Marshall [42] - Borodin <i>et al.</i> [15]
≥ 7	$6 \leq \chi_o \leq 7$	Nešetřil <i>et al.</i> [46] - Borodin and Ivanova [10]
≥ 8	$5 \leq \chi_o \leq 7$	Nešetřil <i>et al.</i> [46] - Borodin and Ivanova [10]
≥ 11	$5 \leq \chi_o \leq 6$	Nešetřil <i>et al.</i> [46] - Ochem and Pinlou [52]
≥ 12	$\chi_o = 5$	Nešetřil <i>et al.</i> [46] - Borodin, Ivanova and Kostochka [12]

Table 3: Oriented chromatic number of planar graphs with large girth

Theorem 18 (Kostochka, Sopena and Zhu, 1997). *If G is an undirected graph with maximum degree Δ , then $\chi_o(G) \leq 2 \cdot \Delta^2 \cdot 2^\Delta$.*

These authors also proved that for every $\Delta \geq 2$, there exists an oriented graph with maximum degree Δ and oriented chromatic number at least $2^{\frac{\Delta}{2}}$.

The upper bound of Theorem 18 has been specialised by Aravind and Subramanian [2]. Recall that the *degeneracy* of an undirected graph G is the smallest integer d such that every subgraph of G contains a vertex of degree at most d . Then we have:

Theorem 19 (Aravind and Subramanian, 2009). *If G is an undirected graph with maximum degree Δ and degeneracy d , then $\chi_o(G) \leq 16 \cdot \Delta \cdot d \cdot 2^d$.*

In [71], Wood proved that every oriented graph of order n with maximum degree Δ has oriented chromatic number at most $2 \cdot \Delta \cdot \sqrt{n-1}$ and constructed such graphs with oriented chromatic number at least $\Delta \cdot \sqrt{\frac{n}{8}}$.

It was proved in [64] that every oriented graph with maximum degree 3 has oriented chromatic number at most 16. This upper bound was decreased to 11 by Sopena and Vignal [68] and has not been improved yet.

The following conjecture was proposed in [64]:

Conjecture 20. *Every connected oriented graph with maximum degree 3 has oriented chromatic number at most 7.*

If the conjecture is true then the bound of 7 cannot be improved, as shown by the oriented graph depicted in Figure 4.

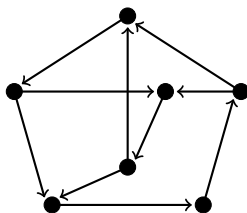


Figure 4: An o-clique of order 7 with maximum degree 3

6. Other graph classes

6.1. Halin graphs

A *Halin graph* is a planar graph which admits a planar embedding such that deleting the edges of its external face gives a tree with at least three vertices. The first result for this class of graphs was proved in [32]:

Theorem 21 (Hosseini Dolama and Sopena, 2006). *If G is a Halin graph, then $\chi_o(G) \leq 9$. Moreover, there exist oriented Halin graphs with oriented chromatic number at least 8.*

The exact bound for the oriented chromatic number of Halin graphs has been recently proved [23]:

Theorem 22 (Dybizbański and Szepietowski, 2014). *If G is a Halin graph, then $\chi_o(G) \leq 8$.*

6.2. Hypercubes

Let Q_d denote the d -dimensional hypercube. Wood [71] proved the following:

Theorem 23 (Wood, 2007). *For every integer $d \geq 1$,*

$$0.8007\dots \cdot \sqrt{2^d} \leq \chi_o(Q_d) \leq 2d \cdot \sqrt{2^d - 1}.$$

6.3. Cartesian products

The *Cartesian product* of two undirected graphs G and H , denoted $G \square H$, is the graph defined by $V(G \square H) = V(G) \times V(H)$ and $(u, v)(u', v') \in E(G \square H)$ if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

In [1], Aravind, Narayanan and Subramanian proved that for every undirected graph G :

- (1) $\chi_o(G \square P_k) \leq (2k - 1) \cdot \chi_o(G)$, for every $k \geq 2$,
- (2) $\chi_o(G \square C_k) \leq 2k \cdot \chi_o(G)$, for every $k \geq 3$,

where P_k and C_k denote the path and the cycle on k vertices, respectively. These results were further improved in [67] as follows:

Theorem 24 (Sopena, 2012). *For every undirected graph G , we have*

- (1) $\chi_o(G \square T) \leq 6 \cdot \chi_o(G)$, for every undirected tree T ,
- (2) $\chi_o(G \square C_k) \leq 12 \cdot \chi_o(G)$, for every $k \geq 3$, $k \neq 5$,
- (3) $\chi_o(G \square C_5) \leq 15 \cdot \chi_o(G)$.

6.4. Fully subdivided graphs

Let G be an undirected graph. We denote by G^s the undirected graph obtained from G by subdividing exactly once every edge of G . Wood [70] proved the following:

Theorem 25 (Wood, 2005). *For every undirected graph G ,*

- (1) $\chi_o(G^s) \leq 7$ if $\chi(G) \leq 7$,
- (2) $\chi_o(G^s) \leq 9$ if $\chi(G) = 8$, and this bound is tight,
- (3) $\chi_o(G^s) = \chi(G)$ if $\chi(G) \geq 9$,

where $\chi(G)$ stands for the ordinary chromatic number of G .

The upper bounds given in this theorem are based on the existence of 2-existentially closed tournaments [6]: a tournament \vec{T} is *2-existentially closed* if, for every two vertices u and v in \vec{T} , there exist four vertices a, b, c and d in \vec{T} such that $au, av, ub, vb, uc, cv, du$ and vd are arcs in T . Bonato and Cameron proved in [7] that 2-existentially closed tournaments of order n exist only for $n = 7$ and $n \geq 9$. Note that this implies that the bound given in Theorem 25(2) is tight, since $\chi_o(K_8^s) = 8$ would imply that the oriented graph \vec{K}_8^s , obtained by connecting every two original vertices of K_8 by a directed 2-path, admits a homomorphism to a 2-existentially closed tournament of order 8, a contradiction.

6.5. Line-digraphs

An edge-colouring of an undirected graph is nothing but a vertex-colouring of its line-graph, so that the *chromatic index* $\chi'(G)$ of G is the chromatic number of the line-graph of G . The same idea applies to oriented graphs.

Let \vec{G} be an oriented graph. The *line-digraph* of \vec{G} , denoted $\text{LD}(\vec{G})$, is the oriented graph whose set of vertices is the set of arcs of \vec{G} , and whose set of arcs is the set of pairs (uv, vw) of “consecutive” arcs in \vec{G} . An *oriented k -arc-colouring* of \vec{G} is then defined as an oriented (vertex) k -colouring of $\text{LD}(\vec{G})$. In other words, an oriented k -arc-colouring of \vec{G} is a mapping c from $A(\vec{G})$ to a set of k colours such that (i) $c(uv) \neq c(vw)$ for every arcs uv and vw in $A(\vec{G})$ and (ii) $c(vw) \neq c(xy)$ for every arcs uv, vw, xy and yz in $A(\vec{G})$ with $c(uv) = c(yz)$.

The *oriented chromatic index* of \vec{G} , denoted $\chi'_o(\vec{G})$, is then the smallest k such that \vec{G} admits an oriented k -arc-colouring.

Let \vec{G} be an oriented graph with oriented chromatic number k and γ be an oriented (vertex) k -colouring of \vec{G} . It is not difficult to check that the mapping c defined by $c(uv) = \gamma(u)$ for every arc uv in $A(\vec{G})$ is an oriented k -arc-colouring of \vec{G} . Therefore, $\chi'_o(\vec{G}) \leq \chi_o(\vec{G})$ for every oriented graph \vec{G} .

Arc-colourings of oriented graphs have been studied by Ochem, Pinlou and Sopena [57, 55, 51, 54]. The main results can be summarized as follows:

Theorem 26.

- (1) *If the underlying undirected graph of \vec{G} has acyclic chromatic number at most k , then $\chi'_o(\vec{G}) \leq 2k(k-1) - \lfloor k \rfloor$. As a corollary, we get that every oriented planar graph has chromatic index at most 38. Moreover, there exist oriented planar graphs with oriented chromatic index at least 10 [54].*
- (2) *If \vec{G} is a planar oriented graph with girth $g \geq 46$ then $\chi'_o(\vec{G}) \leq 4$ [54].*
- (3) *For every partial 2-tree \vec{G} , $\chi'_o(\vec{G}) \leq 7$ and this bound is tight [57].*
- (4) *Every outerplanar graph with girth g has oriented chromatic index at most 7 if $g \geq 3$, at most 6 if $g \geq 4$, at most 5 if $5 \leq g \leq 9$ and at most 4 if $g \geq 10$. Moreover, all these bounds are tight [57].*

- (5) Every partial 2-tree with girth g has oriented chromatic index at most 7 if $g \geq 3$, at most 6 if $g \geq 5$, at most 5 if $7 \leq g \leq 17$ (tight bound), and at most 4 if $g \geq 18$ (tight bound) [51].
- (6) If \vec{G} is an oriented graph with maximum degree Δ , then $\chi'_o(\vec{G}) \leq 2 \cdot (\lfloor \frac{\Delta^2}{2} \rfloor + \Delta)$ [54].
- (7) If \vec{G} is an oriented graph with maximum degree 3, then $\chi'_o(\vec{G}) \leq 7$, and there exist such oriented graphs with oriented chromatic index at least 6 [55].

7. Complexity issues

It is NP-complete to decide whether an undirected graph has ordinary chromatic number at most k , unless $k \leq 2$ [28]. Unsurprisingly, the situation is quite similar in the oriented case. Consider the following decision problem:

PROBLEM: OCN_k (Oriented chromatic number at most k , $k \geq 1$)

Instance: An oriented graph \vec{G} .

Question: Does \vec{G} admit an oriented k -colouring?

Klostermeyer and MacGillivray [36] proved the following:

Theorem 27 (Klostermeyer and MacGillivray, 2004). *The problem OCN_k is polynomial if $k \leq 3$ and NP-complete if $k \geq 4$, even if the input graph is connected.*

Culus and Demange [20] proved that OCN_4 is NP-complete even if the underlying graph of the input graph is bipartite with bounded degree, or acyclic with bounded degree.

Ganian and Hliněný [27] proved that OCN_4 is also NP-complete when restricted to graphs with K-width 1 and DAG-depth 3 ¹.

¹A digraph G has K-width k if k is the smallest integer such that, for any pair of vertices $u, v \in V(G)$, the number of distinct directed paths from u to v is at most k . For a digraph G and any $v \in V(G)$, let G_v denote the subdigraph of G induced by the vertices reachable from v . The maximal elements of the poset $\{G_v : v \in V(G)\}$ in the digraph-inclusion order are then called reachable fragments of G , referred to as $\text{RF}(G)$. The DAG-depth $\text{ddp}(G)$ of a digraph G is then inductively defined as follows: If $|V(G)| = 1$, then $\text{ddp}(G) = 1$. If G has a single reachable fragment, then $\text{ddp}(G) = 1 + \min\{\text{ddp}(G_v) : v \in V(G)\}$. Otherwise, $\text{ddp}(G) = \max\{\text{ddp}(F) : F \in \text{RF}(G)\}$.

Coelho, Faria, Gravier and Klein [18] proved that OCN_4 is still NP-complete when restricted to connected, planar, bipartite, acyclic oriented graphs with maximum degree 3. This gives a dichotomy result for OCN_k with respect to the maximum degree since OCN_k is polynomial for graphs with maximum degree at most 2.

The smallest known class of graphs for which the problem OCN_4 has been proved to be NP-complete is the following [29]:

Theorem 28 (Guegan and Ochem, 2014). *For any fixed $g > 3$, deciding whether an oriented graph \vec{G} has oriented chromatic number at most 4 is NP-complete, even if \vec{G} is restricted to be a planar graph with girth g , bipartite, subcubic, with DAG-depth 3, with maximum outdegree 2 and maximum indegree 2, and such that every vertex with degree 3 is adjacent to at most one vertex of degree 3.*

Recall that the oriented chromatic index of an oriented graph \vec{G} is defined as the oriented chromatic number of its line-digraph $\text{LD}(\vec{G})$ (see Subsection 6.5). The associated decision problem is the following:

PROBLEM: OCI_k (Oriented chromatic index at most k , $k \geq 1$)

Instance: An oriented graph \vec{G} .

Question: Does $\text{LD}(\vec{G})$ admit an oriented k -colouring?

Ochem, Pinlou and Sopena [54] proved the following:

Theorem 29 (Ochem, Pinlou and Sopena, 2008). *The problem OCI_k is polynomial if $k \leq 3$ and NP-complete if $k \geq 4$.*

Let u and v be any two distinct vertices of an oriented graph \vec{G} . The *weak distance* between u and v is the shortest length of a directed path connecting either u to v or v to u . The *weak diameter* of \vec{G} is then the largest weak distance between pairs of vertices of \vec{G} .

In [4], Bensmail, Duvignau and Kirgizov considered the following decision problem:

PROBLEM: OWD_k (Orientation with weak diameter k , $k \geq 1$)

Instance: An undirected graph G .

Question: Does G admit an orientation with weak diameter k ?

They proved the following:

Theorem 30 (Bensmail, Duvignau and Kirgizov, 2013). *The problem OWD_k is polynomial if $k = 1$ and NP-complete if $k \geq 2$.*

Note that an o-clique is nothing but an oriented graph with weak diameter at most 2. Clearly, only tournaments have weak diameter 1. Hence we get:

Corollary 31. *Deciding whether a non-complete undirected graph admits an orientation which is an o-clique is NP-complete.*

8. The oriented colouring game

Nešetřil and Sopena introduced in [47] the oriented version of the graph colouring game (see the survey by Bartnicki, Grytczuk, Kierstead and Zhu [3]). Let \vec{H} be any fixed oriented graph. The *oriented \vec{H} -colouring game* is played on an oriented graph \vec{G} by two players, Alice and Bob, who alternate in turn, Alice having the first move. Each move consists in choosing an uncoloured vertex u in \vec{G} and assigning it a colour α from $V(\vec{H})$ such that:

- (i) if uv (resp. vu) is an arc in $A(\vec{G})$ and v has colour β , then $\alpha\beta$ (resp. $\beta\alpha$) is an arc in $A(\vec{H})$,
- (ii) if there exists a directed 2-path from u to some vertex w with colour γ , then $\alpha \neq \gamma$.

Alice's goal is to eventually produce a homomorphism of \vec{G} to \vec{H} while Bob tries to prevent her from doing so. The *oriented game chromatic number* of \vec{G} , denoted $\chi_o^g(\vec{G})$, is defined as the smallest k for which there exists an oriented graph \vec{H} of order k such that Alice has a winning strategy when playing the oriented \vec{H} -colouring game on \vec{G} .

The authors proved the following:

Theorem 32 (Nešetřil and Sopena, 2001). *For every oriented path \vec{P} , $\chi_o^g(\vec{P}) \leq 7$ and this bound is tight. For every oriented tree \vec{T} , $\chi_o^g(\vec{T}) \leq 19$.*

Moreover, they proved that the oriented game chromatic number of every oriented outerplanar graph is finite and ask whether this is also the case for planar graphs. This question was answered positively by Kierstead and Trotter [33] who proved the following more general result:

Theorem 33 (Kierstead and Trotter, 2001). *For every k , there exists an integer t and a tournament \vec{T}_t of order t such that, if Γ is a topologically closed class of graphs (that is, every subgraph or subdivision of a member of Γ belongs to Γ) and Γ does not contain the complete graph K_k , then Alice wins the oriented \vec{T}_t -colouring game on every orientation of any member of Γ .*

In [34], Kierstead and Tuza proved that, for every k , the oriented game chromatic number of the class of oriented partial k -trees is also finite:

Theorem 34 (Kierstead and Tuza, 2003). *For every oriented partial k -tree \vec{T}_k , $\chi_o^g(\vec{T}_k) \leq 6k - 2$.*

9. Weaker versions of oriented colouring

9.1. Simple colourings

Recall that an oriented k -colouring of an oriented graph \vec{G} is a partition of $V(\vec{G})$ into k independent sets such that all the arcs connecting any two parts have the same direction. If we drop the requirement that every part must be an independent set, and ask to have at least two non-empty parts (in order to avoid monochromatic colourings), we get the notion of a *simple k -colouring*, introduced by Smolíková in 2000 [61]. The *simple chromatic number* of \vec{G} is then defined as the smallest k such that \vec{G} admits a simple k -colouring.

Note that simple colourings can equivalently be defined by means of *simple homomorphisms*, that is, homomorphisms to oriented graphs having a loop at each vertex with the additional requirement that the homomorphic image cannot be reduced to a single vertex.

In [61], Smolíková proved the following quite surprising result (see also [62]):

Theorem 35 (Smolíková, 2000). *The maximum oriented chromatic number and the maximum simple chromatic number of a planar graph (resp. a graph with acyclic chromatic number χ_a , a graph with treewidth k) coincide.*

In particular, this result shows that in order to improve the upper bound of 80 on the oriented chromatic number of planar graphs (see Theorem 16), one could “simply” consider simple colourings of planar graphs.

9.2. 2-dipath colouring

In [17], Chen and Wang introduced and studied *2-dipath colourings* of oriented graphs, that is colourings such that every two vertices connected by a directed path of length 1 or 2 are assigned distinct colours. Every oriented colouring is clearly a 2-dipath colouring. On the other hand, there is no restriction on the orientation of arcs connecting any two colour classes in a 2-dipath colouring. Chen and Wang [17] proved the following:

Theorem 36 (Chen and Wang, 2006). *Every orientation of a Halin graph admits a 2-dipath colouring using 7 colours, and this bound is tight.*

Notice that the optimal upper bound on the oriented chromatic number of Halin graphs is 8 (Theorem 22).

Such 2-dipath colourings thus correspond to the so-called *L(1,1)-labellings* of oriented graphs (see [16] for a recent survey on *L(h,k)-labellings*).

MacGillivray and Sherk [39] recently introduced the improper version of 2-dipath colourings, in which only vertices connected by a directed path of length 2 are required to get distinct colours. In both the proper and the improper case, they describe a homomorphism model², prove a dichotomy theorem for the complexity of the problem of deciding whether there exists such a colouring with a fixed number of colours, and give a polynomial time algorithm for determining the minimum number of colours needed to colour a given multipartite tournament.

10. Open problems

In this section, we propose some open problems concerning the oriented chromatic number or the oriented chromatic index of special classes of graphs.

The most challenging one is certainly the following:

Problem 1. *Determine the largest possible oriented chromatic number of a planar graph.*

²More precisely, for every integer $k \geq 1$, they construct an oriented graph \vec{H}_k (resp. \vec{H}'_k) such that for every oriented graph \vec{G} , \vec{G} admits a homomorphism to \vec{H}_k (resp. \vec{H}'_k) if and only if \vec{G} admits a proper (resp. improper) 2-dipath colouring using at most k colours.

We know from Theorem 16 and [41] that the answer to Problem 1 lies between 18 and 80. Recall also that thanks to Theorem 35 one can consider simple colourings instead of oriented colourings.

In relation with the above problem, it would also be interesting to improve the bounds given in Table 1:

Problem 2. *Determine the largest possible value of the relative oriented clique number of planar graphs.*

Problem 3. *Determine the largest possible value of the relative oriented clique number of triangle-free planar graphs.*

The situation for cubic graphs is also still unknown although the gap between the best known lower and upper bounds is small. We thus propose the following:

Problem 4. *Determine the largest possible oriented chromatic number of a cubic graph. Determine the largest possible oriented chromatic number of a connected cubic graph.*

We know from [68] and Figure 4 that the answer to the first part of Problem 4 lies between 7 and 11. Conjecture 20 claims that the answer to the second part is 7. Recently, Duffy [21] announced that he decreased the upper bound of 11 to 9.

The case of rectangular grids is also quite intriguing:

Problem 5. *Determine the largest possible oriented chromatic number of a rectangular grid.*

We know from Theorem 14 and [22] that the answer to Problem 5 lies between 8 and 11.

The following question was proposed by Raspaud and first results on this problem were obtained by Wood [71] (see Theorem 23):

Problem 6. *Determine the oriented chromatic number of the d -dimensional hypercube.*

We finally propose two questions concerning the oriented chromatic index of graphs:

Problem 7. *Determine the largest possible oriented chromatic index of a planar graph.*

From Theorem 26(1), we know that the answer to Problem 7 lies between 10 and 38.

Problem 8. *Determine the largest possible oriented chromatic index of a cubic graph.*

From Theorem 26(7), we know that the answer to Problem 8 lies between 6 and 7.

Acknowledgement

We would like to thank Janusz Dybizbański and Andrzej Szepietowski for their careful reading of a preliminary version of this survey.

Part of this work has been supported by the Cluster of Excellence CPU from IdEx Bordeaux.

References

- [1] N. R. Aravind, N. Narayanan, and C. R. Subramanian. Oriented colouring of some graph products. *Discuss. Math. Graph Theory*, 31(4):675–686, 2011.
- [2] N. R. Aravind and C. R. Subramanian. Forbidden subgraph colourings and the oriented chromatic number. In *Proc. 20th International Workshop on Combinatorial Algorithms, IWOCA'09*. Lecture Notes Comput. Sci. (LNCS), 5874:60–71, 2009.
- [3] T. Bartnicki, J. Grytczuk, H. A. Kierstead, and X. Zhu. The map colouring game. *Amer. Math. Monthly*, 114:793–803, 2007.
- [4] J. Bensmail, R. Duval, and S. Kirgizov. The complexity of deciding whether a graph admits an orientation with fixed weak diameter. Available at <http://hal.archives-ouvertes.fr/docs/01/06/52/16/PDF/weak-diam.pdf>, 2013.
- [5] H. Bielak. The oriented chromatic number of some grids. *Annales UMCS Informatica AI*, 5:5–17, 2006.

- [6] A. Bonato. The search for n-e.c. graphs. *Contrib. to Discrete Math.*, 4(1):40–53, 2009.
- [7] A. Bonato and K. Cameron. On an adjacency property of almost all graphs. *Discrete Math.*, 231:102–119, 2001.
- [8] O. V. Borodin. On acyclic colourings of planar graphs. *Discrete Math.*, 25:211–236, 1979.
- [9] O. V. Borodin, D. Fon der Flaass, A. V. Kostochka, A. Raspaud, and É. Sopena. On deeply critical oriented graphs. *J. Combin. Theory, Ser. B*, 81:150–155, 2001.
- [10] O. V. Borodin and A. O. Ivanova. An oriented 7-colouring of planar graphs with girth at least 7. *Sib. Electron. Math. Reports*, 2:222–229, 2005.
- [11] O. V. Borodin and A. O. Ivanova. An oriented colouring of planar graphs with girth at least 4. *Sib. Electron. Math. Reports*, 2:239–249, 2005.
- [12] O. V. Borodin, A. O. Ivanova, and A. V. Kostochka. Oriented 5-colouring of sparse plane graphs. *J. Applied and Industrial Math.*, 1(1):9–17, 2007.
- [13] O. V. Borodin, S. J. Kim, A. V. Kostochka, and D. B. West. Homomorphisms from sparse graphs with large girth. *J. Combin. Theory, Ser. B*, 90:147–159, 2004.
- [14] O. V. Borodin, A. V. Kostochka, A. Raspaud J. Nešetřil, and É. Sopena. On universal graphs for planar oriented graphs of a given girth. *Discrete Math.*, 188:73–85, 1998.
- [15] O. V. Borodin, A. V. Kostochka, A. Raspaud J. Nešetřil, and É. Sopena. On the maximum average degree and the oriented chromatic number of a graph. *Discrete Math.*, 206:77–89, 1999.
- [16] T. Calamoneri. The $L(h, k)$ -labelling problem: An updated survey and annotated bibliography. *Computer J.*, 54(8):1344–1371, 2014.
- [17] M. Chen and W. Wang. The 2-dipath chromatic number of Halin graphs. *Inform. Process. Lett.*, 99:47–53, 2006.

- [18] H. Coelho, L. Faria, S. Gravier, and S. Klein. Oriented coloring in planar, bipartite, bounded degree 3 acyclic oriented graphs. *Electron. Notes Discrete Math.*, 44:195–200, 2013.
- [19] B. Courcelle. The monadic second order logic of graphs VI: On several representations of graphs by relational structures. *Discrete Appl. Math.*, 54:117–149, 1994. Erratum in *Discrete Appl. Math.* 63:199-200, 1995.
- [20] J.-F. Culus and M. Demange. Oriented colouring: Complexity and approximation. In *Proc. 32nd International Conference on Current Trends in Theory and Practice of Computer Science, SOFSEM'2006*. Lecture Notes in Computer Science (LNCS), 3831:226–236, 2006.
- [21] C. Duffy. Oriented colouring of bounded degree graphs. In *ICGT'14, 9th International Colloquium on Graph Theory and Combinatorics, June 30–July 4, Grenoble, France, 2014*.
- [22] J. Dybizbański and A. Nenca. Oriented chromatic number of grids is greater than 7. *Inform. Process. Lett.*, 112:113–117, 2012.
- [23] J. Dybizbański and A. Szepietowski. The oriented chromatic number of Halin graphs. *Inform. Process. Lett.*, 114(1–2):45–49, 2014.
- [24] L. Esperet and P. Ochem. Oriented colourings of 2-outerplanar graphs. *Inform. Process. Lett.*, 101:215–219, 2007.
- [25] G. Fertin, A. Raspaud, and A. Roychowdhury. On the oriented chromatic number of grids. *Inform. Process. Lett.*, 85:261–266, 2003.
- [26] Z. Füredi, P. Horak, C.M. Pareek, and X. Zhu. Minimal oriented graphs of diameter 2. *Graphs Combin.*, 14:345–350, 1998.
- [27] R. Galian and P. Hliněný. New results on the complexity of oriented colouring on restricted digraph classes. In *Proc. 36th International Conference on Current Trends in Theory and Practice of Computer Science, SOFSEM'2010*. Lecture Notes in Computer Science (LNCS), 5901:428–439, 2010.
- [28] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman & Co Ltd, 1979.

- [29] G. Guégan and P. Ochem. Oriented coloring is NP-complete on a small class of graphs. Available at <http://www.lirmm.fr/~ochem/ocn4npc-els.pdf>, 2014.
- [30] G. Hahn and C. Tardif. Graph homomorphisms: Structure and symmetry. *Graph symmetry (Montreal, PQ, 1996)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 497:107–166, 1997.
- [31] P. Hell and J. Nešetřil. *Graphs and Homomorphisms*, volume 28 of *Oxford Lecture Series in Mathematics and Its Applications*. 2004.
- [32] M. Hosseini Dolama and É. Sopena. On the oriented chromatic number of Halin graphs. *Inform. Process. Lett.*, 98(6):247–252, 2006.
- [33] H.A. Kierstead and W.T. Trotter. Competitive colorings of oriented graphs. *Electron. J. Combin.*, 8(2), 2001. Paper #12.
- [34] H.A. Kierstead and Zs. Tuza. Marking games and the oriented game chromatic number of partial k-trees. *Graphs Combin.*, 19(1):121–129, 2003.
- [35] W. F. Klostermeyer and G. MacGillivray. Analogues of cliques for oriented colouring. *Discuss. Math. Graph Theory*, 24:373–387, 2004.
- [36] W. F. Klostermeyer and G. MacGillivray. Homomorphisms and oriented colourings of equivalence classes of oriented graphs. *Discrete Math.*, 274:161–172, 2004.
- [37] A. V. Kostochka, T. Łuczak, G. Simonyi, and É. Sopena. On the minimum number of edges giving maximum oriented chromatic number. In *Proc. Dimacs/Dimatia conference “Contemporary Trends in Discrete Mathematics”, Štířín, Czech Rep., May 1997*. Dimacs Series in Discrete Math. and Theoret. Comput. Sci., 49:179–182, 1999.
- [38] A. V. Kostochka, É. Sopena, and X. Zhu. Acyclic and oriented chromatic numbers of graphs. *J. Graph Theory*, 24(4):331–340, 1997.
- [39] G. MacGillivray and K.M. Sherk. A theory of 2-dipath colourings. *Australasian J. Combin.*, 60(1):11–26, 2014.
- [40] T. H. Marshall. Homomorphism bounds for oriented planar graphs. *J. Graph Theory*, 55:175–190, 2007.

- [41] T. H. Marshall. On oriented graphs with certain extension properties. *Ars Combinatoria*, to appear, 2012.
- [42] T. H. Marshall. Homomorphism bounds for oriented planar graphs of given minimum girth. *Graphs Combin.*, 29:1489–1499, 2013.
- [43] M. Montassier, P. Ochem, and A. Pinlou. Strong oriented chromatic number of planar graphs without short cycles. *Discrete Math. Theoret. Comput. Sci.*, 10(1):1–24, 2008.
- [44] A. Nandy, S. Sen, and É. Sopena. Outerplanar and planar oriented cliques. Available at <http://www.labri.fr/perso/sopena/Publications>, 2014.
- [45] V. Neumann-Lara. The dichromatic number of a digraph. *J. Combin. Theory, Ser. B*, 33:265–270, 1982.
- [46] J. Nešetřil, A. Raspaud, and É. Sopena. Colourings and girth of oriented planar graphs. *Discrete Math.*, 165–166:519–530, 1997.
- [47] J. Nešetřil and É. Sopena. On the oriented game chromatic number. *Electron. J. Combin.*, 8(2), 2001. Paper #14.
- [48] J. Nešetřil, É. Sopena, and L. Vignal. T-preserving homomorphisms of oriented graphs. *Comment. Math. Univ. Carolinae*, 38(1):125–136, 1997.
- [49] P. Ochem. Oriented colourings of triangle-free planar graphs. *Inform. Process. Lett.*, 92:71–76, 2004.
- [50] P. Ochem. Negative results on acyclic improper colourings. *Discrete Math. Theoret. Comput. Sci. Proc. AE*, pages 357–362, 2005.
- [51] P. Ochem and A. Pinlou. Oriented vertex and arc colourings of partial 2-trees. *Electron. Notes Discrete Math.*, 29:195–199, 2007.
- [52] P. Ochem and A. Pinlou. Oriented colourings of partial 2-trees. *Inform. Process. Lett.*, 108(2):82–86, 2008.
- [53] P. Ochem and A. Pinlou. Oriented colouring of triangle-free planar graphs and 2-outerplanar graphs. *Graphs Combin.*, 30(2):439–453, 2014.

- [54] P. Ochem, A. Pinlou, and É. Sopena. On the oriented chromatic index of oriented graphs. *J. Graph Theory*, 57(4):313–332, 2008.
- [55] A. Pinlou. On oriented arc-colouring of subcubic graphs. *Electron. J. Combin.*, 13:R69, 2006.
- [56] A. Pinlou. An oriented colouring of planar graphs with girth at least five. *Discrete Math.*, 309:2108–2118, 2009.
- [57] A. Pinlou and É. Sopena. Oriented vertex and arc colourings of outerplanar graphs. *Inform. Process. Lett.*, 100(3):97–104, 2006.
- [58] A. Raspaud and É. Sopena. Good and semi-strong colourings of oriented planar graphs. *Inform. Process. Lett.*, 51:171–174, 1994.
- [59] S. Sen. The maximum order of a planar o-clique is 15. In *Proc. 23rd International Workshop on Combinatorial Algorithms, IWOCA '12*. Lecture Notes in Computer Science (LNCS), 7643:130–142, 2012.
- [60] S. Sen. *A contribution to the theory of graph homomorphisms and colorings*. PhD thesis, Bordeaux University, 2014.
- [61] P. Smolíková. The simple chromatic number of oriented graphs. *Electron. Notes Discrete Math.*, 5:281–283, 2000.
- [62] P. Smolíková. *Simple colorings and simple homomorphisms*. PhD thesis, Charles University, Prague, 2002.
- [63] É. Sopena. The oriented colouring page. An online survey available at <http://www.labri.fr/perso/sopena/TheOrientedcolouringPage>.
- [64] É. Sopena. The chromatic number of oriented graphs. *J. Graph Theory*, 25:191–205, 1997.
- [65] É. Sopena. Oriented graph colouring. *Discrete Math.*, 229(1–3):359–369, 2001.
- [66] É. Sopena. There exist oriented planar graphs with oriented chromatic number at least sixteen. *Inform. Process. Lett.*, 81(6):309–312, 2002.
- [67] É. Sopena. Upper oriented chromatic number of undirected graphs and oriented colourings of product graphs. *Discuss. Math. Graph Theory*, 32(3):317–333, 2012.

- [68] É. Sopena and L. Vignal. A note on the oriented chromatic number of graphs with maximum degree three. Technical report, Univ. Bordeaux, LaBRI, 1996. available at <http://www.labri.fr/perso/sopena/pmwiki/uploads/Publications/sv96.pdf>.
- [69] A. Szepietowski and M. Targan. A note on the oriented chromatic number of grids. *Inform. Process. Lett.*, 92:65–70, 2004.
- [70] D. R. Wood. Acyclic, star and oriented colourings of graph subdivisions. *Discrete Math. Theoret. Comput. Sci.*, 7:37–50, 2005.
- [71] D. R. Wood. On the oriented chromatic number of dense graphs. *Contrib. Discrete Math.*, 2(2):145–152, 2007.