

Incidence Coloring of Graphs

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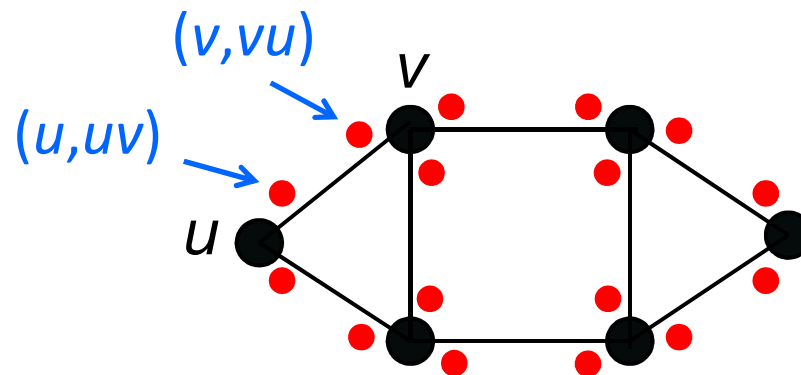
Incidence colorings

Incidences

(1)

Incidences

An **incidence** in a graph G is a pair (u, uv) with $u \in V(G)$ and $uv \in E(G)$.



We denote by $Inc(G)$ the set of incidences of a graph G .

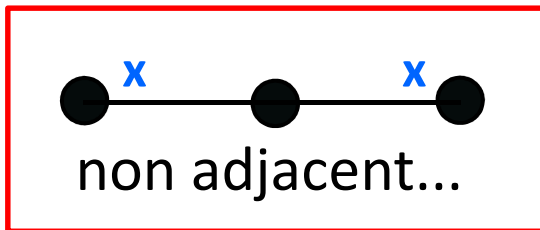
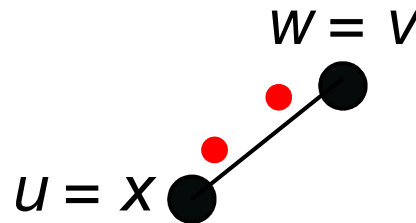
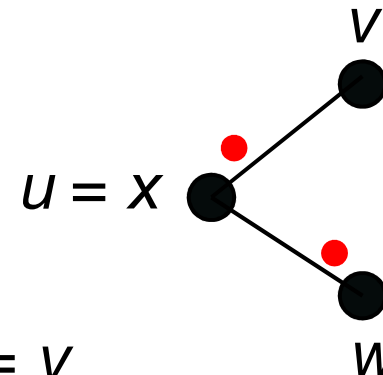
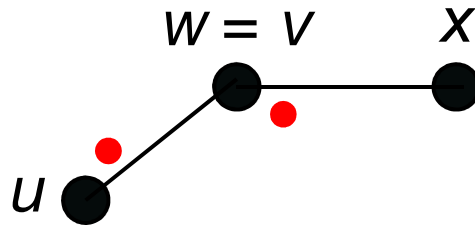
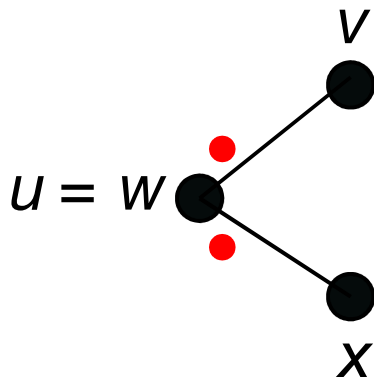
Incidences

(2)

Adjacent incidences

Two distinct incidences (u, uv) and (w, wx) are **adjacent** in G if:

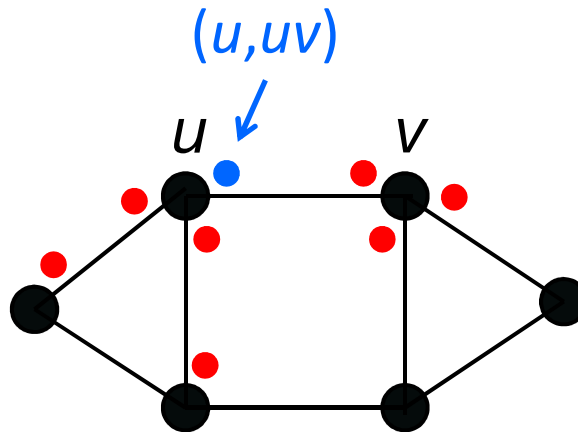
- i. $u = w$ ($v \neq x$), or
- ii. $w = v$, or
- iii. $u = x$.



Adjacent incidences

Two distinct incidences (u, uv) and (w, wx) are **adjacent** in G if:

- i. $u = w$ ($v \neq x$), or
- ii. $w = v$, or
- iii. $u = x$.



Incidence coloring

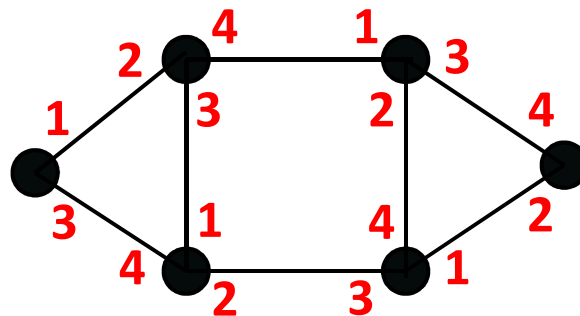
(1)

Incidence coloring [BRUALDI & QUINN MASSEY, 1993]

An **incidence k -coloring** of a graph G is a mapping

$$\lambda: \text{Inc}(G) \rightarrow \{1, \dots, k\}$$

that assigns distinct colors to adjacent incidences.



$$\chi_i = 4$$

*(may be defined
for multigraphs)*

Incidence chromatic number

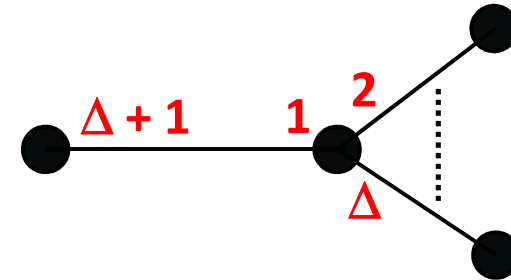
The **incidence chromatic number** $\chi_i(G)$ of a graph G is the smallest k for which G admits an incidence k -coloring.

Incidence coloring

(2)

General lower bounds

For every graph G , $\chi_i(G) \geq \Delta(G) + 1$.



For every graph G , $\chi_i(G) \geq 2|E| / (|V| - \gamma(G))$, where $\gamma(G)$ denotes the domination number of G .

General upper bound

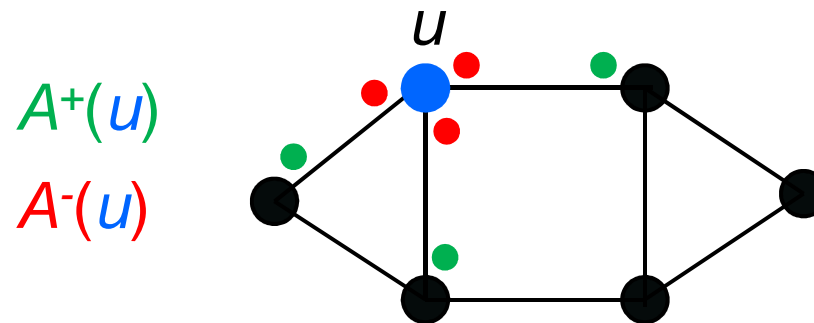
For every graph G , $\chi_i(G) \leq 2\Delta(G)$. [BRUALDI & QUINN MASSEY, 1993]

For every graph G , $\chi_i(G) \leq \Delta(G) + 20 \cdot \log_2 \Delta(G) + 84$. [GUIDULI, 1997]

Incidence (k,p) -coloring

The sets of incidences $A^+(u)$ and $A^-(u)$

For every vertex u in $V(G)$, let $A^+(u)$ denote the set of incidences of the form (v,vu) and $A^-(u)$ the set of incidences of the form (u,uv) .



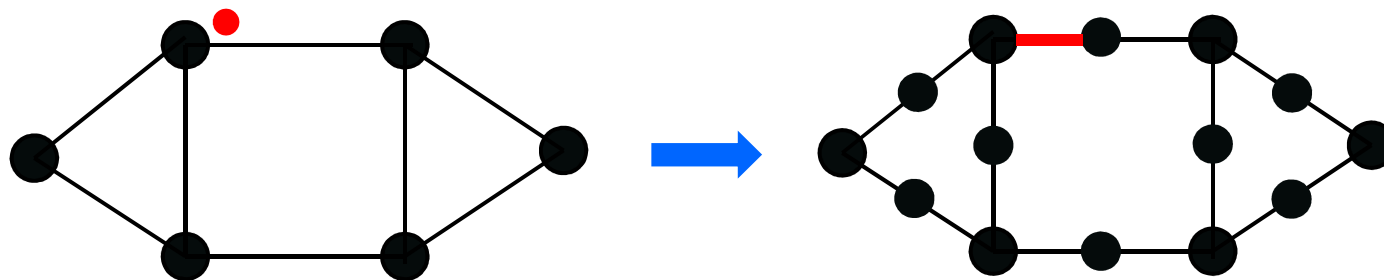
Incidence (k,p) -coloring

An **incidence (k,p) -coloring** of G is an incidence k -coloring of G such that, for every vertex u in G , at most p colors are used on $A^+(u)$.

Subdivided graph

The subdivided graph $S(G)$

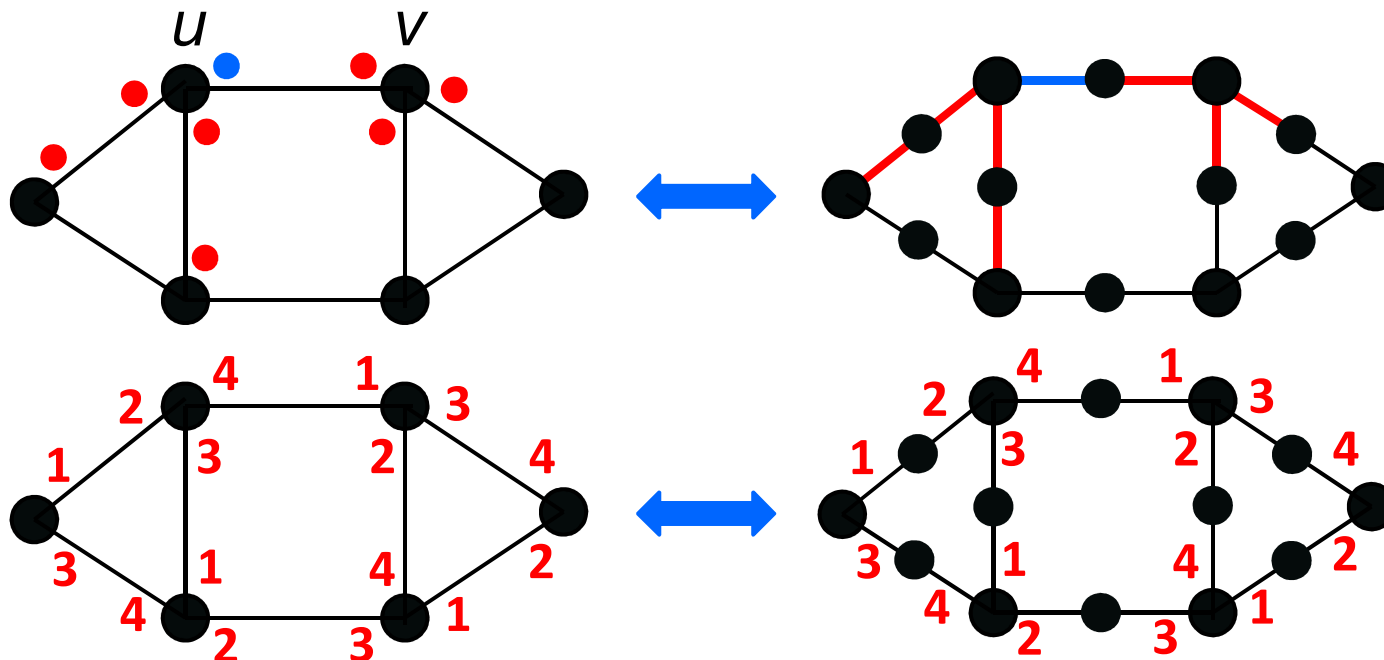
Obtained from G by subdividing once each edge of G :



Strong edge coloring of $S(G)$

Strong edge coloring of $S(G)$

An edge coloring is **strong** if every two edges at distance one or two are assigned distinct colors. An incidence coloring of G is nothing but a strong edge coloring of $S(G)$.



(original motivation of BRUALDI & QUINN MASSEY, 1993)

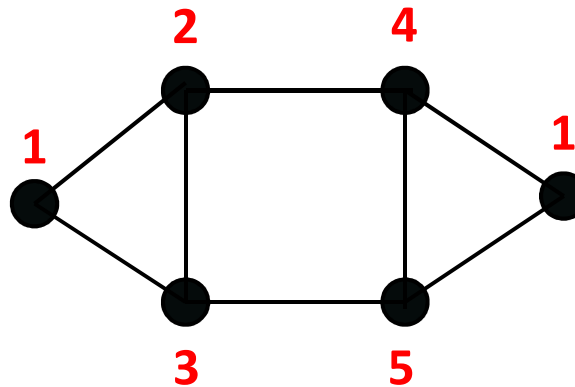
Distance-two (vertex) coloring (1)

Distance-two (vertex) coloring (a.k.a. square coloring)

A **distance-two k -coloring** of a graph G is a mapping

$$\lambda: V(G) \rightarrow \{1, \dots, k\}$$

that assigns distinct colors to vertices at distance one or two from each other.



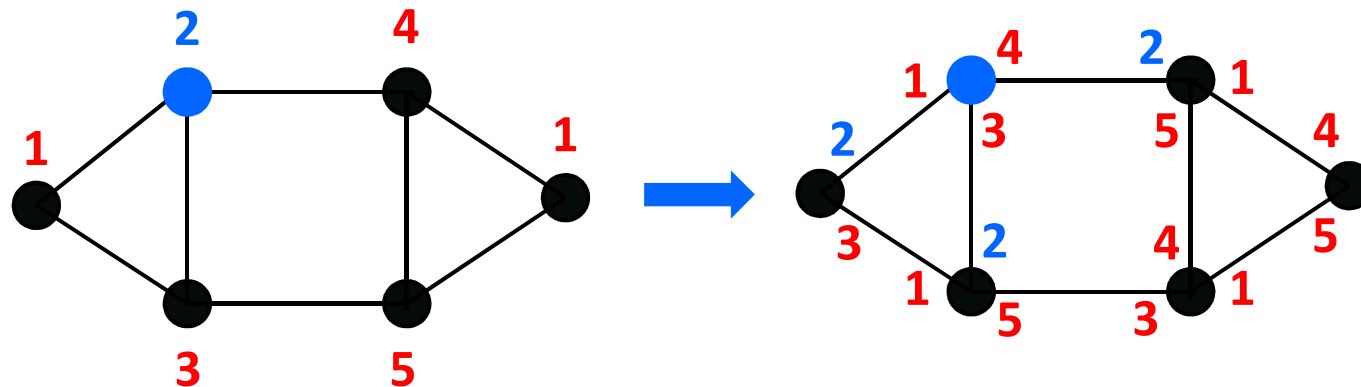
Distance-two chromatic number

The **distance-two chromatic number** $\chi_2(G)$ of a graph G is the smallest k for which G admits a distance-two k -coloring.

Distance-two (vertex) coloring (2)

Distance-two (vertex) coloring (a.k.a. square coloring)

Observation. For every graph G , $\chi_i(G) \leq \chi_2(G)$.



In particular...

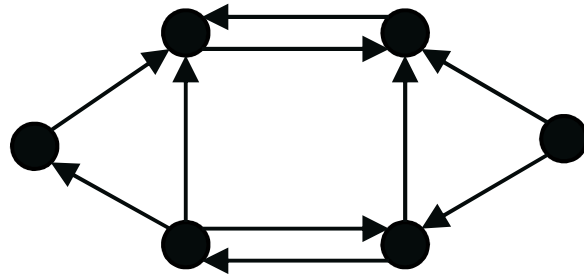
For every graph G , $\chi_2(G) = \Delta(G) + 1$ implies $\chi_i(G) = \Delta(G) + 1$.
(Our example shows that the converse is not true.)

Note that distance-two k -colorings of a graph G are in one-to-one correspondence with $(k,1)$ -incidence colorings of G ...

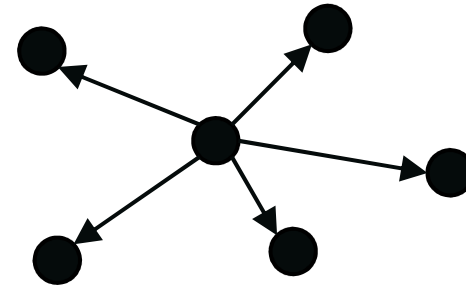
Directed star arboricity

(1)

Digraphs, directed stars



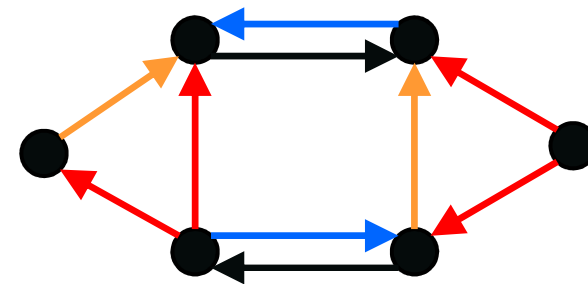
A digraph D



A directed (outgoing) star

Directed star arboricity (ALGOR & ALON, 1989)

The **directed star arboricity** $\text{dst}(D)$ of a digraph D is the least number of forests of directed (outgoing) stars needed for partitioning the arc set of D .



$$\text{dst}(D) = 4$$

Directed star arboricity

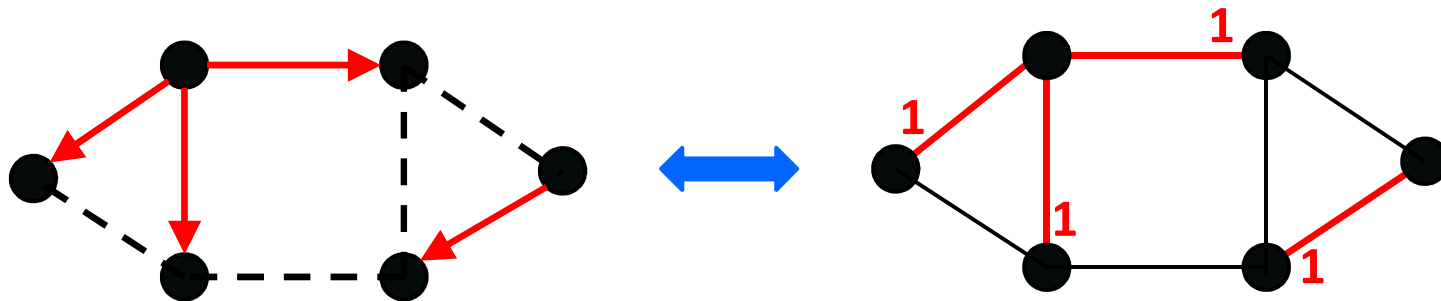
(2)

Symmetric digraph associated with an undirected graph

The **symmetric digraph** $SD(G)$ associated with an undirected graph G is the digraph obtained from G by replacing each edge by a pair of opposite arcs.

Directed star arboricity and incidence coloring

For every undirected graph G , $\chi_i(G) = \text{dst}(SD(G))$.



(first observed by GUIDULI, 1997)

Selected results

Complexity issues

Incidence chromatic number

Theorem [Li & Tu, 2008].

Deciding whether the incidence chromatic number of a graph is at most k , $k \geq 4$, is an NP-complete problem.

Incidence graph

Theorem [HARTKE & HELLELOID, 2012].

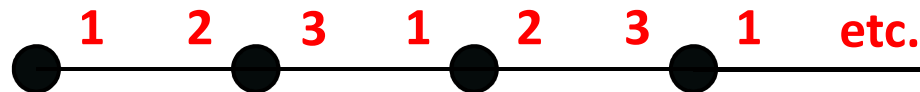
Deciding whether a graph H is the incidence graph of some graph G can be done in polynomial time.

(If the answer is yes, their algorithm outputs the corresponding graph G .)

Paths and cycles

Paths

For every path P_n , $n \geq 3$, $\chi_i(P_n) = 3 = \Delta(P_n) + 1$.



Cycles

For every cycle C_n with $n \geq 3$, $n \equiv 0 \pmod{3}$, $\chi_i(C_n) = 3 = \Delta(C_n) + 1$.

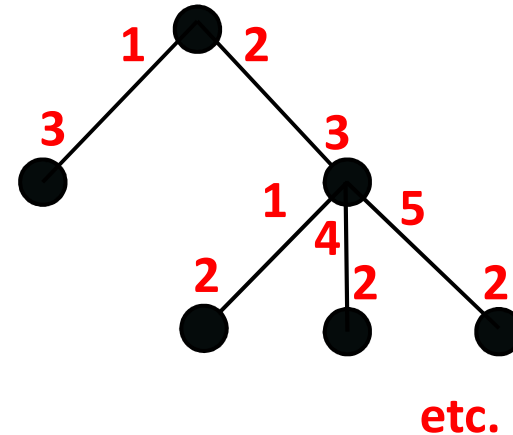
For every cycle C_n with $n \geq 4$, $n \equiv 1, 2 \pmod{3}$, $\chi_i(C_n) = 4 = \Delta(C_n) + 2$.

(By Sylvester's Theorem, every integer ≥ 6 can be expressed as a sum of 3's and 4's, and we have $2n$ incidences...)

Other simple classes

Trees

For every tree T , $\chi_i(T) = \Delta(T) + 1$.
(Simple induction.)



Complete graphs

For every $n \geq 2$, $\chi_i(K_n) = n = \Delta(K_n) + 1$.

Complete bipartite graphs

For every $m \geq n \geq 2$, $\chi_i(K_{m,n}) = m = \Delta(K_{m,n}) + 2$.

The $(\Delta + 2)$ -Conjecture

Incidence coloring conjecture (ICC)

Conjecture [BRUALDI & QUINN MASSEY, 1993].

For every graph G , $\chi_i(G) \leq \Delta(G) + 2$.

ICC conjecture is satisfied by all the previous examples, and by many other graph classes.

However, the conjecture was disproved by GUIDULI:

Theorem [GUIDULI, 1997].

- There exist graphs with incidence chromatic number at least $\Delta + \Omega(\log \Delta)$ (follows from ALGOR & ALON, 1989).
- For every graph G , $\chi_i(G) \leq \Delta(G) + 20 \cdot \log_2 \Delta(G) + 84$.

Grid graphs

Rectangular grids

Theorem [HUANG, WANG & CHUNG, 2004].

For every $m, n \geq 2$, $\chi_i(G_{m,n}) = 5 = \Delta(G_{m,n}) + 1$.

Toroidal grids

Theorem [Wu & S., 2013].

For every $m, n \geq 3$, $\chi(T_{m,n}) \leq 6 = \Delta(T_{m,n}) + 2$.

Moreover, $\chi(T_{m,n}) = 5$ if and only if $m, n \equiv 0 \pmod{5}$.

k -degenerated graphs

(1)

k -degenerated graphs

Let k be a positive integer. A graph G is k -degenerated if every subgraph of G contains a vertex of degree at most k .

For instance:

- forests are 1-degenerated,
- K_4 -minor-free graphs are 2-degenerated,
- planar graphs are 5-degenerated,
- triangle-free planar graphs are 3-degenerated.

Theorem [HOSSEINI-DOLAMA, S. & ZHU, 2004].

Every k -degenerated graph G admits a $(\Delta(G) + 2k - 1, k)$ -incidence coloring and, therefore, $\chi_i(G) \leq \Delta(G) + 2k - 1$.

k -degenerated graphs

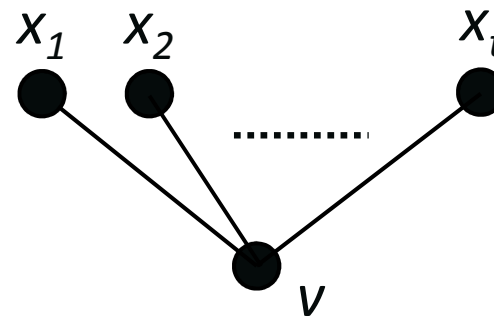
(2)

Theorem. Every k -degenerated graph G admits a $(\Delta(G) + 2k - 1, k)$ -incidence coloring.

Proof.

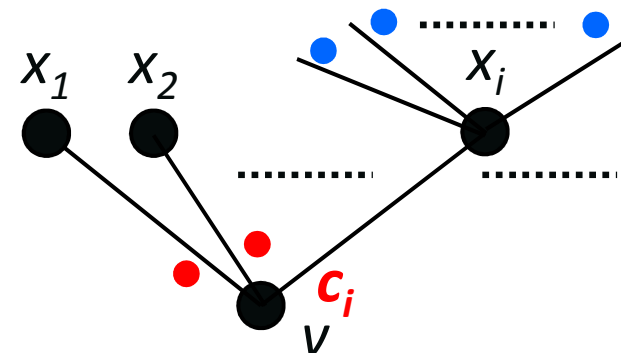
Assume the theorem is false and let G be a minimal counter-example.

Let v be a vertex of degree t , $t \leq k$.



Since G is minimal, $G' = G - v$ admits a $(\Delta(G) + 2k - 1, k)$ -incidence coloring γ' .

Claim. For every i , $1 \leq i \leq t$, there exists a color $c_i \notin \{c_1, \dots, c_{i-1}\}$ that can be assigned to the incidence (v, vx_i) and such that $|\gamma'(A^+(x_i)) \cup \{c_i\}| \leq k$.



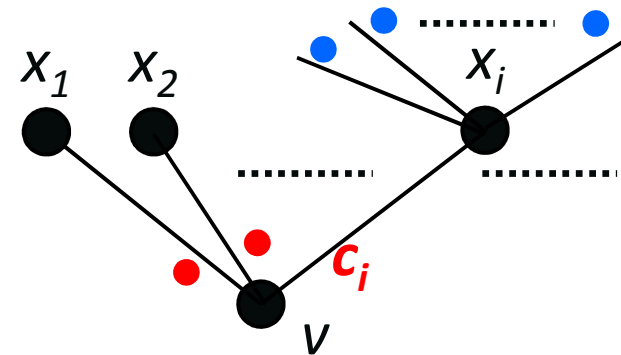
k -degenerated graphs

(3)

Theorem. Every k -degenerated graph G admits a $(\Delta(G) + 2k - 1, k)$ -incidence coloring.

Proof (cont.).

Claim. For every i , $1 \leq i \leq t$, there exists a color $c_i \notin \{c_1, \dots, c_{i-1}\}$ that can be assigned to the incidence (v, vx_i) and such that $|\gamma(A^+(x_i)) \cup \{c_i\}| \leq k$.



By induction:

- $i = 1$: take any color c_1 in $\gamma(A^+(x_1))$, or any color if $\deg_G(x_1) = 1$.
- $i > 1$: if $\gamma(A^+(x_i)) \setminus \{c_1, \dots, c_{i-1}\}$ is not empty, take any color c_i in $\gamma(A^+(x_i)) \setminus \{c_1, \dots, c_{i-1}\}$;
otherwise (which implies $|\gamma(A^+(x_i))| \leq i - 1 \leq k - 1$), take any color not in $\gamma(A^-(x_i)) \cup \{c_1, \dots, c_{i-1}\}$, which is possible since
$$|\gamma(A^-(x_i)) \cup \{c_1, \dots, c_{i-1}\}| \leq \Delta(G) - 1 + i - 1 \leq \Delta(G) + k - 2.$$

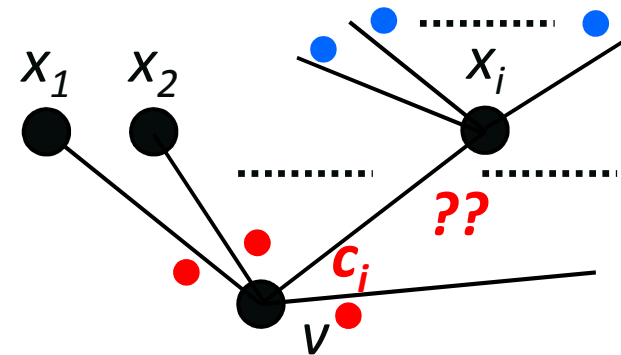
k -degenerated graphs

(4)

Theorem. Every k -degenerated graph G admits a $(\Delta(G) + 2k - 1, k)$ -incidence coloring.

Proof (cont.).

Claim. For every i , $1 \leq i \leq t$, there exists a color $c_i \notin \{c_1, \dots, c_{i-1}\}$ that can be assigned to the incidence (v, vx_i) and such that $|\gamma(A^+(x_i)) \cup \{c_i\}| \leq k$.



The theorem then easily follows:

- Color the incidences of $A^-(v)$ thanks to the claim,
- Color the incidences of $A^+(v)$ using any available color, that is, color each incidence (v, vx_i) with a color not in $\gamma(A^+(x_i)) \cup \gamma(A^-(x_i)) \cup \{c_1, \dots, c_{i-1}\}$, which is possible since

$$\begin{aligned} |\gamma(A^+(x_i)) \cup \gamma(A^-(x_i)) \cup \{c_1, \dots, c_{i-1}\}| &\leq k + \Delta(G) - 1 + t - 1 \\ &= \Delta(G) + 2k - 2. \end{aligned}$$

k -degenerated graphs

(5)

Theorem. Every k -degenerated graph G admits a $(\Delta(G) + 2k - 1, k)$ -incidence coloring.

Therefore we get the following:

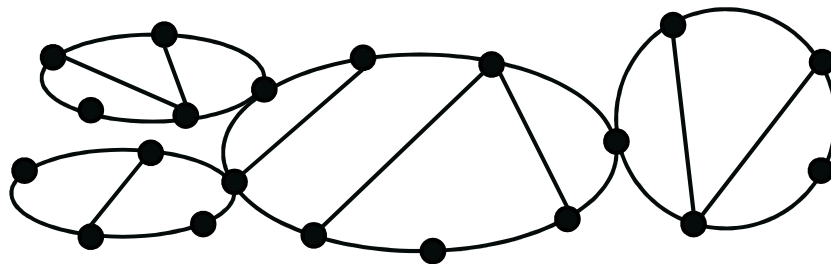
Corollary [HOSSEINI-DOLAMA, S. & ZHU, 2004].

- If G is a K_4 -minor-free graph, then $\chi_i(G) \leq \Delta(G) + 3$.
- If G is a planar graph, then $\chi_i(G) \leq \Delta(G) + 9$.
- If G is a triangle-free planar graph, then $\chi_i(G) \leq \Delta(G) + 5$.

Outerplanar graphs

Outerplanar graphs

A graph is **outerplanar** if it admits a planar drawing such that all its vertices lie on the outer face.



Outerplanar graphs form a subclass of K_4 -minor-free graphs.

Theorem [HOSSEINI-DOLAMA, S. & ZHU, 2004].

Every outerplanar graph G admits a $(\Delta(G) + 2, 2)$ -incidence coloring and, therefore, $\chi_i(G) \leq \Delta(G) + 2$ (tight bound).

(This result holds more generally for K_4 -minor free graphs.)

Planar graphs

Planar graphs

Theorem [HOSSEINI-DOLAMA, S. & ZHU, 2004].

Every planar graph G admits a $(\Delta(G) + 7, 7)$ -incidence coloring and, therefore, $\chi_i(G) \leq \Delta(G) + 7$.

This result has been improved by YANG (best known upper bound up to now):

Theorem [YANG, 2012].

Let G be a planar graph. Then we have:

- $\chi_i(G) \leq \Delta(G) + 5$ if $\Delta(G) \neq 6$,
- $\chi_i(G) \leq \Delta(G) + 6$ if $\Delta(G) = 6$.

Triangle-free planar graphs

Triangle-free planar graphs

Theorem [HOSSEINI-DOLAMA & S., 2005].

Every triangle-free planar graph G admits a $(\Delta(G) + 4, 3)$ -incidence coloring and, therefore, $\chi_i(G) \leq \Delta(G) + 4$.

This result has been improved by BONAMY *et al.* whenever the maximum degree is large enough:

Theorem [BONAMY, HOCQUARD, KERDJOU DJ & RASPAUD, 2015+].

For every triangle-free planar graph G with $\Delta(G) \geq 7$,
 $\chi_i(G) \leq \Delta(G) + 3$.

Planar graphs with high girth (1)

The **girth** $g(G)$ of a graph G is the length of a shortest cycle in G .

Planar graphs with high girth

From results on the incidence chromatic number of graphs with bounded maximum average degree, we get:

Theorem [HOSSEINI-DOLAMA & S., 2005].

Let G be a planar graph. Then we have:

- if $g(G) \geq 6$ then $\chi_i(G) \leq \Delta(G) + 3$,
- if $g(G) \geq 11$, or $g(G) \geq 6$ and $\Delta(G) \geq 5$, then $\chi_i(G) \leq \Delta(G) + 2$,
- if $g(G) \geq 16$ and $\Delta(G) \geq 4$ then $\chi_i(G) \leq \Delta(G) + 1$.

Planar graphs with high girth (2)

The **girth** $g(G)$ of a graph G is the length of a shortest cycle in G .

Planar graphs with high girth

From results on the distance-two chromatic number of planar graphs we get the following improvements:

Theorem [BONAMY, LÉVÊQUE & PINLOU, 2011].

For every planar graph G with girth $g(G) \geq 14$ and $\Delta(G) \geq 4$,
 $(\chi_i(G) \leq) \chi_2(G) \leq \Delta(G) + 1$.

Theorem [BONAMY, CRANSTON & POSTLE, 2015+].

There exists a constant Δ_0 such that every planar graph G with girth $g(G) \geq 5$ and $\Delta(G) \geq \Delta_0$,
 $(\chi_i(G) \leq) \chi_2(G) \leq \Delta(G) + 2$.

Graphs with maximum degree 3

Graphs with maximum degree 3

After some partial results obtained by SHIU, LAM and CHEN (2002) and by WU (2004), MAYDANSKYI finally proved that the ICC conjecture holds for graphs with maximum degree 3:

Theorem [MAYDANSKYI, 2005].

For every graph G with $\Delta(G) = 3$, $\chi_i(G) \leq 5 = \Delta(G) + 2$.

Therefore, the incidence chromatic number of any subcubic graph is either 4 or 5.

However, we have:

every vertex has either degree 1 or degree 3

Theorem [Li & Tu, 2008].

The problem of deciding whether a *semi-cubic* graph has incidence chromatic number 4 or 5 is an NP-complete problem.

Product graphs

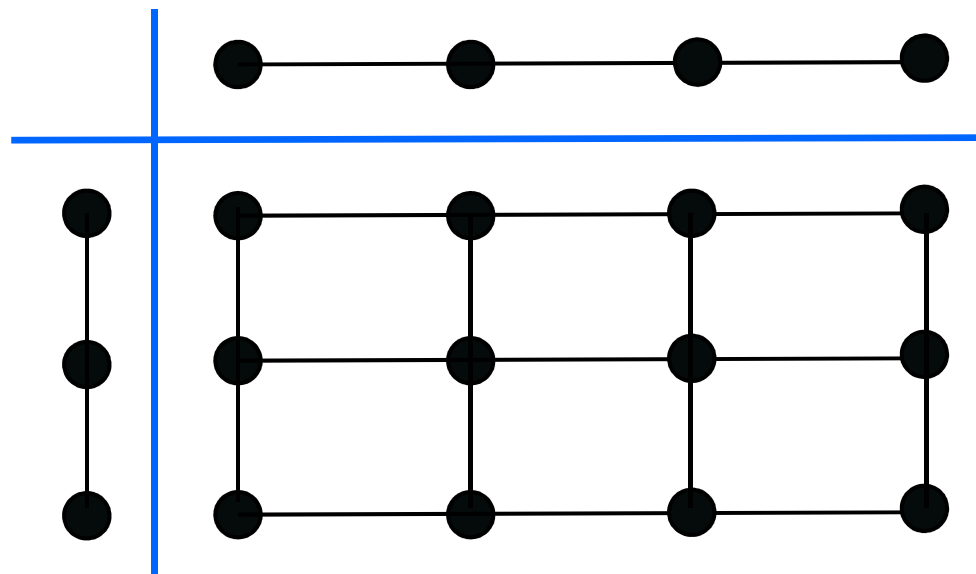
(1)

Cartesian product

The Cartesian product $G \square H$ of two undirected graphs G and H is given by:

- $V(G \square H) = \{ (u,x) \mid u \in V(G), x \in V(H) \}$,
- $E(G \square H) = \{ ((u,x),(v,y)) \mid u = v \text{ and } xy \in E(H), \text{ or } x = y \text{ and } uv \in E(G) \}$.

Example



Incidence chromatic number of Cartesian products

Theorem [SUN & SHIU, 2014].

For every two graphs G_1 and G_2 , $\chi_i(G_1 \square G_2) \leq \chi_i(G_1) + \chi_i(G_2)$.

Let λ be an incidence coloring of G . We denote by $S_\lambda(u)$ the set of colors used on $A^+(u) \cup A^-(u)$. Let $s(\lambda) = \max \{ |S_\lambda(u)|, u \in V(G) \}$.

Theorem [SHIAU, SHIAU & WANG, 2015].

If λ_i is an incidence k_i -coloring of G_i , $i = 1, 2$, then $G_1 \square G_2$ admits a $\max\{k_1 + s(\lambda_2), k_2\}$ -incidence coloring.

Incidence chromatic number of Cartesian products

Theorem [GREGOR & LUŽAR, 2015+].

If $\chi_i(G_1) = \Delta(G_1) + 1$, G_2 is a subgraph of a regular graph H_2 with $\chi_i(H_2) = \Delta(H_2) + 1$ and $\Delta(G_1) + 1 \geq \Delta(H_2) - \Delta(G_2)$ then,

$$\chi_i(G_1 \square G_2) \leq \Delta(G_1 \square G_2) + 2.$$

Incidence chromatic number of Hypercubes

Corollary [SHIAU, SHIAU & WANG, 2015 – GREGOR & LUŽAR, 2015+].

For every $n \geq 1$,

- $\chi_i(Q_n) = n + 1 = \Delta(Q_n) + 1$ if $n = 2^m - 1$ for some $m \geq 1$,
- $\chi_i(Q_n) = n + 2 = \Delta(Q_n) + 2$ otherwise.

(conjectured by PAI, CHANG, YANG & WU, 2014)

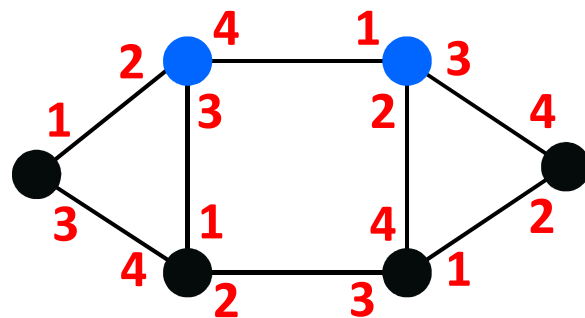
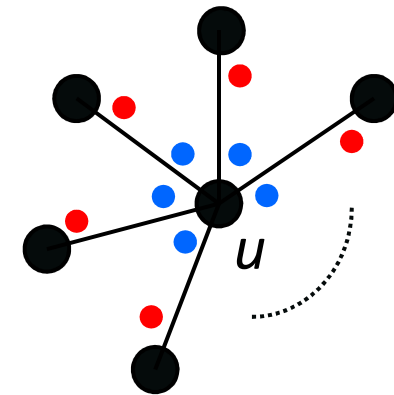
Some variants...

Interval incidence coloring

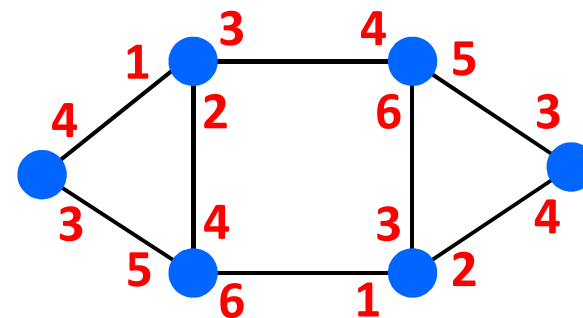
(1)

Interval incidence coloring

An incidence coloring of a graph G is an **interval incidence coloring** if the set of colors used on $A^-(u)$ forms an interval for every vertex u of G .



NO...



YES!

We denote by $\chi_{ii}(G)$ the interval incidence chromatic number of G .

This notion was introduced by JANCZEWSKI, MAŁAFIEJSKA & MAŁAFIEJSKI (2007).

Bipartite graphs

Theorem [JANCZEWSKI, MAŁAFIEJSKA & MAŁAFIEJSKI, 2014].

- For every bipartite graph G , $\Delta(G) + 1 \leq \chi_i(G) \leq \chi_{ii}(G) \leq 2\Delta(G)$.
- If G is a regular bipartite graph, $\chi_{ii}(G) = 2\Delta(G)$.

In particular, if G is a subcubic bipartite graph, $4 \leq \chi_{ii}(G) \leq 6$.

JANCZEWSKI, MAŁAFIEJSKA and MAŁAFIEJSKI characterized those subcubic bipartite graphs G with $\chi_{ii}(G) = 4, 5$ or 6 , respectively. They also characterized those bipartite graphs G with $\Delta(G) = 4$ and $\chi_{ii}(G) = 5$. However, they prove the following:

Theorem [JANCZEWSKI, MAŁAFIEJSKA & MAŁAFIEJSKI, 2014].

Deciding whether $\chi_{ii}(G) \leq 6$ for a bipartite graph G with $\Delta(G) = 4$ is an NP-complete problem.

Some other graph classes

Theorem [JANCZEWSKI, MAŁAFIEJSKA & MAŁAFIEJSKI, 2015].

- For every graph G , $\Delta(G) + 1 \leq \chi_{ii}(G) \leq \chi(G) \Delta(G)$.
- If G is a path or a cycle, $\chi_{ii}(G) \leq \Delta(G) + 2$, with strict inequality only for P_k , $k \leq 4$ (where k stands for the number of edges).
- For every $n \geq 1$, $\chi_{ii}(K_{1,n}) = \Delta(K_{1,n}) + 1$.
- For every $n \geq 3$, $\chi_{ii}(W_n) = \Delta(W_n) + 2 + (n \bmod 2)$.
- For every $n \geq 1$, $\chi_{ii}(K_n) = 2\Delta(K_n)$.
- For every $n_1 \leq \dots \leq n_k$, $\chi_{ii}(K_{n_1, \dots, n_k}) = 2(n_1 + \dots + n_{k-2}) + n_{k-1} + n_k$.

Subcubic graphs

Theorem [JANCZEWSKI, MAŁAFIEJSKA & MAŁAFIEJSKI, 2015].

- Deciding whether $\chi_{ii}(G) \leq 4$ for a subcubic graph G can be done in linear time.
- Deciding whether $\chi_{ii}(G) \leq 5$ for a subcubic graph G is an NP-complete problem.

Non-hereditary property

Theorem [JANCZEWSKI, MAŁAFIEJSKA & MAŁAFIEJSKI, 2015].

For every $k \geq 5$, there exists a tree T_k with $\Delta(T_k) = k$ such that T_k contains a subtree T with $\chi_{ii}(T) > \chi_{ii}(T_k)$.

The incidence coloring game

(1)

The incidence coloring game (ANDRES, 2009)

Let G be a graph and k a positive integer. The **incidence coloring game with k colors** is played on G as follows:

- The two players, Alice and Bob, play alternately, Alice playing first.
- On his turn, each player assigns a color from $\{1, \dots, k\}$ to an uncolored incidence, respecting the adjacency constraints.
- Alice wins the game if and only if the whole graph G is colored.

The incidence game chromatic number

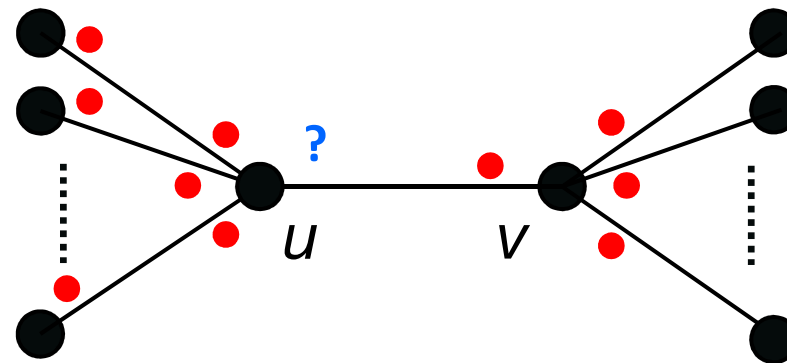
The **incidence game chromatic number** $\chi_{ig}(G)$ of a graph G is the smallest k for which Alice has a winning strategy.

The incidence coloring game

(2)

Upper bound

Consider two adjacent vertices:



Since (u, uv) has at most $3(\Delta(G) - 1) + 1 = 3\Delta(G) - 2$ adjacent incidences, we get $\chi_{ig}(G) \leq 3\Delta(G) - 1$.

Lower bound

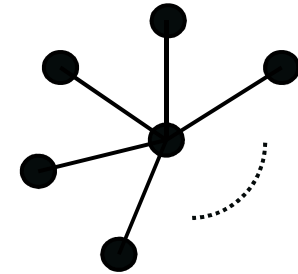
ANDRES proved that for every graph G , $\chi_{ig}(G) > (3\Delta(G) - 1)/2$.

These two bounds are tight...

A sample case: stars

(1)

We denote by $K_{1,n}$ the star on $n + 1$ vertices.



The incidence game chromatic number of stars

Theorem [ANDRES, 2009].

For every $k \geq 1$, we have

- $\chi_{\text{ig}}(K_{1,2k}) = 3k = 3\Delta(K_{1,2k})/2,$
- $\chi_{\text{ig}}(K_{1,2k+1}) = 3k + 2.$

To prove that $\chi_{\text{ig}}(G) = p$ for a given graph G , we need to

1. provide a winning strategy for Alice with p colors,
2. provide a winning strategy for Bob with $p - 1$ colors or fewer.

A sample case: stars

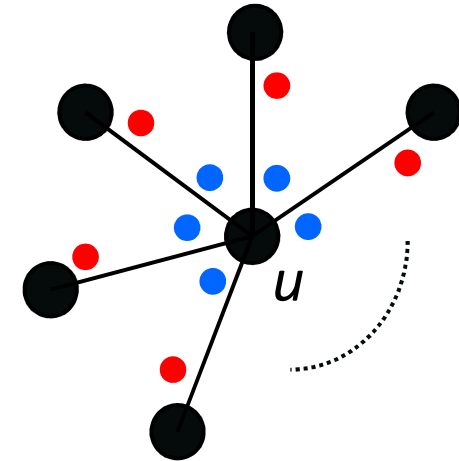
(2)

Theorem. For every $k \geq 1$, $\chi_{\text{ig}}(K_{1,2k}) = 3k$.

Proof of the even case.

(1) Alice's strategy with $3k$ colors is based on the following rules:

- Alice colors preferably incidences from $A^-(u)$.
- If Bob colors an incidence from $A^+(u)$ with color c , Alice colors another incidence from $A^+(u)$, if any, with the same color c .
- When Alice is forced to color an incidence from $A^+(u)$ – which happens when all incidences from $A^-(u)$ are already colored – she uses a color already used in $A^+(u)$ (except when it is the first incidence from $A^+(u)$ to be colored).
 - ⇒ $2k$ colors have to be used for $A^-(u)$,
 - ⇒ at most k colors will be used for $A^+(u)$.



A sample case: stars

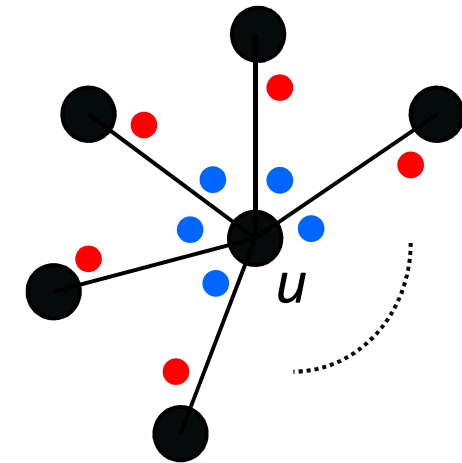
(3)

Theorem. For every $k \geq 1$, $\chi_{ig}(K_{1,2k}) = 3k$.

Proof of the even case.

(2) Bob's strategy with $3k - 1$ colors or fewer is based on the following rule:

- In his k first moves, Bob colors k incidences from $A^+(u)$ with k distinct colors.



\Rightarrow at most $2k - 1$ colors are left for coloring $A^-(u)$, which is not enough.

A few other results

Cycles

Theorem [ANDRES, 2009].

For every $k \geq 7$, $\chi_{\text{ig}}(C_k) = 5 = 3\Delta(C_k) - 1$.

Wheels

Theorem [KIM, 2010].

For every $k \geq 7$, $\chi_{\text{ig}}(W_{2k}) \leq 3k = 3\Delta(W_{2k})/2$.

Paths

Theorem [KIM, 2010].

For every $k \geq 13$, $\chi_{\text{ig}}(P_k) = 5 = 3\Delta(P_k) - 1$.

k -degenerated graphs

(1)

k -degenerated graphs

Theorem [ANDRES, 2009].

For every k -degenerated graph G , $k \geq 1$, we have

- $\chi_{\text{ig}}(G) \leq 2\Delta(G) + 4k - 2$,
- $\chi_{\text{ig}}(G) \leq 2\Delta(G) + 3k - 1$ if $\Delta(G) \geq 5k - 1$,
- $\chi_{\text{ig}}(G) \leq \Delta(G) + 8k - 2$ if $\Delta(G) \leq 5k - 1$.

In particular, this gives:

- For every forest F , $\chi_{\text{ig}}(G) \leq 2\Delta(F) + 2$,
- For every outerplanar graph G , $\chi_{\text{ig}}(G) \leq 2\Delta(G) + 6$,
- For every planar graph G , $\chi_{\text{ig}}(G) \leq 2\Delta(G) + 18$.

Improvement of ANDRES's result

The **arboricity** $a(G)$ of a graph G is the minimum number of forests needed to partition the edge set of G .

Theorem [CHARPENTIER & S., 2013].

For every graph G , $\chi_{ig}(G) \leq \lfloor (3\Delta(G) - a(G))/2 \rfloor + 8a(G) - 1$.

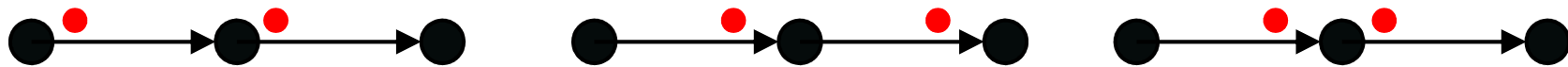
In particular, this gives:

- For every k -degenerated graph G , $\chi_{ig}(G) \leq \lfloor (3\Delta(G) - k)/2 \rfloor + 8k - 1$,
- For every forest F , $\chi_{ig}(G) \leq \lceil 3\Delta(G)/2 \rceil + 6$,
- For every outerplanar graph G , $\chi_{ig}(G) \leq \lfloor 3\Delta(G)/2 \rfloor + 14$,
- For every planar graph G , $\chi_{ig}(G) \leq \lceil 3\Delta(G)/2 \rceil + 21$.

Incidence coloring of digraphs (1)

Adjacent incidences

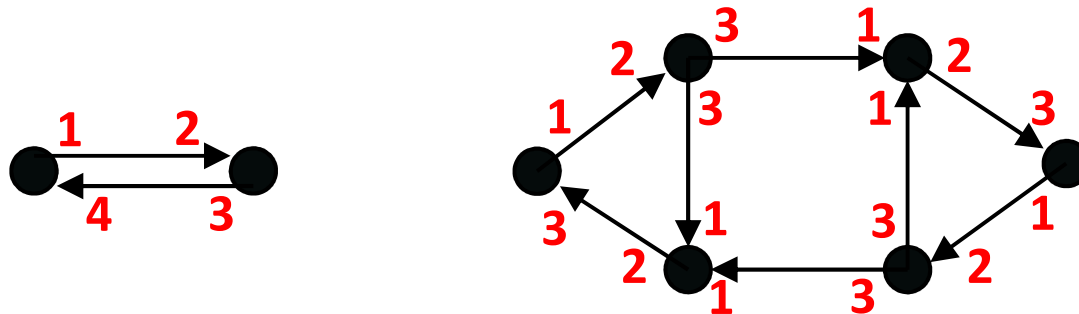
Two incidences are adjacent if and only if they are of one of the following three types:



Incidence coloring and incidence chromatic number

Incidence coloring and incidence chromatic number of digraphs are defined in a natural way (DUFFY, MACGILLIVRAY and RASPAUD, 2015).

Examples.



Incidence coloring of digraphs (2)

Let us denote by K_n^s the complete symmetric digraph of order n , and by $\text{Und}(D)$ the underlying undirected graph of a digraph D .

Then we have:

Theorem [DUFFY, MACGILLIVRAY & RASPAUD, 2015+].

For every digraph D , $\chi_i(D) \leq \chi_i(K_p^s)$, where $p = \chi(\text{Und}(D))$.

The first values of $\chi_i(K_n^s)$ are as follows:

n	1	2	3	4	5	6	7
$\chi_i(K_n^s)$	0	4	4	5	5	6	6

Theorem [DUFFY, MACGILLIVRAY & RASPAUD, 2015+].

For every $p \geq 1$, $\log_2 n \leq \chi_i(K_p^s) \leq (2 + o(1)) \cdot \log_2 n$.

Incidence coloring of digraphs (3)

A digraph with no pair of opposite arcs is an **oriented graph**. Then we have:

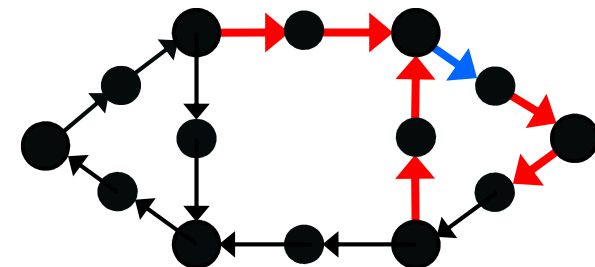
Observation. If D is an oriented graph, then $\chi_i(D) \leq \chi_i(\text{Und}(D))$.

Theorem [DUFFY, MACGILLIVRAY & RASPAUD, 2015+].

- For every oriented tree T , $\chi_i(T) \leq 3$.
- For every oriented cycle C , $\chi_i(C) \leq 4$.
- For every oriented bipartite graph G , $\chi_i(G) \leq 4$.

Do no longer depend on Δ ...

As for the undirected case, an incidence coloring of a digraph D is nothing but a strong arc coloring (using directed distance) of $S(D)$.



Some open questions...

Some open questions...

(1)

Open Question A. *By the result of Maydanskyi, we know that graphs with maximum degree 3 satisfy the ICC Conjecture. What about graphs with maximum degree 4?*

Open Question B. *Is there any characterization of those graphs G with $\chi_i(G) = \Delta(G) + 1$? (Done for regular graphs – SUN, 2012)*

Open Question C. *What is the maximum value of $\chi_i(G)$ when G is a planar graph?*

Open Question D. *What is the smallest k for which every planar graph G with girth at least k and $\Delta(G) \geq 3$ has incidence chromatic number $\Delta(G) + 1$? (The best known value is 14, provided $\Delta(G) \geq 4$.)*

Some open questions...

(2)

Open Question E. What is the smallest k for which every planar graph G with girth at least k has incidence chromatic number at most $\Delta(G) + 2$? (The best known value is 11. This is also true for girth 6, provided $\Delta(G)$ is large enough...)

Open Question F [Gregor & Lužar]. Is it true that $\chi_i(G_1) = \Delta(G_1) + 1$ and $\chi_i(G_2) \leq \Delta(G_2) + 2$ implies $\chi_i(G_1 \square G_2) \leq \Delta(G_1 \square G_2) + 2$?

Open Question G [Janczewski, Małafiejska & Małafiejski]. Is it true that for every graph G , $\chi_{ii}(G) \leq 2\Delta(G)$?

Open Question H. What is the maximum value of $\chi_i(D)$ when D is an oriented planar graph?