

# Homomorphisms of signed graphs

Éric SOPENA  
LaBRI, Bordeaux University  
France

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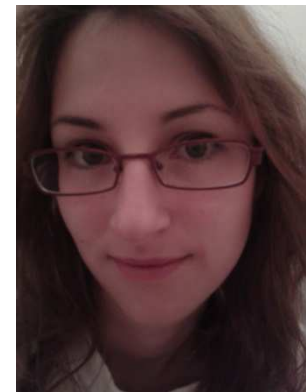
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*based on joint work with  
Reza Naserasr (Orsay, France)  
and Edita Rollová (Plzeň, Czech Republic)*



CID 2013 - September 15-20, 2013, Szklarska Poręba, Poland

# Motivation

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Graph colouring

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## Graph colouring

### Brook's Theorem

*Every connected graph  $G$  which is neither an odd cycle nor a complete graph is  $\Delta(G)$ -colourable.*

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*If  $G$  has no  $K_n$  as a minor,  
then  $G$  is  $(n-1)$ -colourable*

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Hadwiger's  
conjecture

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Signed graphs may help to get "better" results...

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Since the 1950's, several theories have been extended from graphs to signed graphs (minors, nowhere-zero flows, colouring, ...).

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## Our goal

- Investigate relations between minors and homomorphisms of signed graphs,
- Extend colouring (or homomorphism) results from graphs to signed graphs,
- ...with a special focus on minor-closed families of graphs (e.g. planar graphs).

# Outline

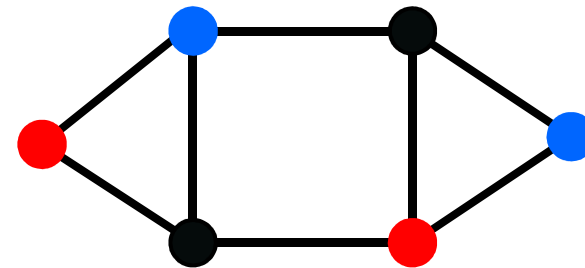
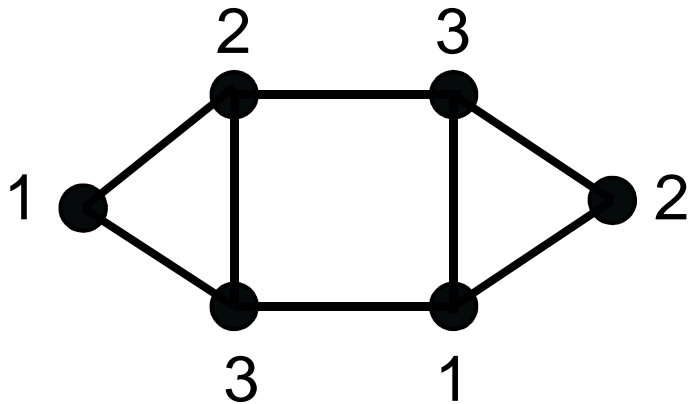
- Preliminaries
- Signed graphs
- Odd signed graphs
- Signed bipartite graphs
- Homomorphisms to signed projective cubes
- Chromatic number of signed graphs

# Preliminaries



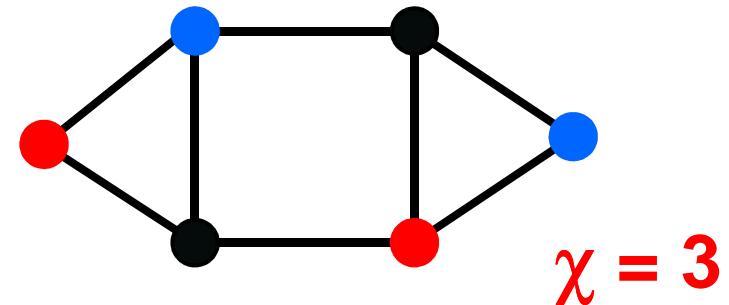
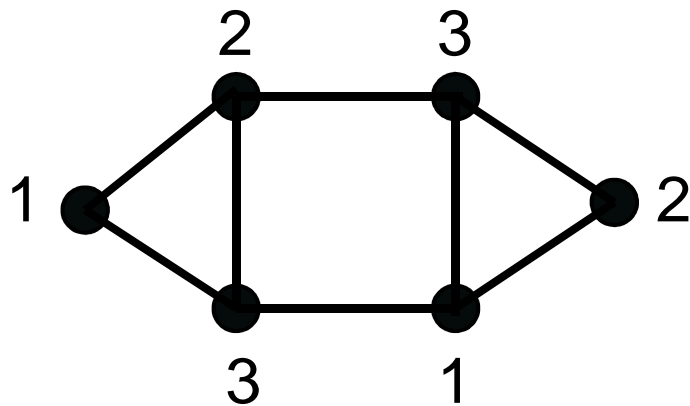
# Vertex Colouring

A (proper) **k-vertex-colouring** of a graph  $G$  is a mapping  $c : V(G) \rightarrow \{1, 2, \dots, k\}$  such that every two *adjacent vertices* are assigned *distinct colours*.



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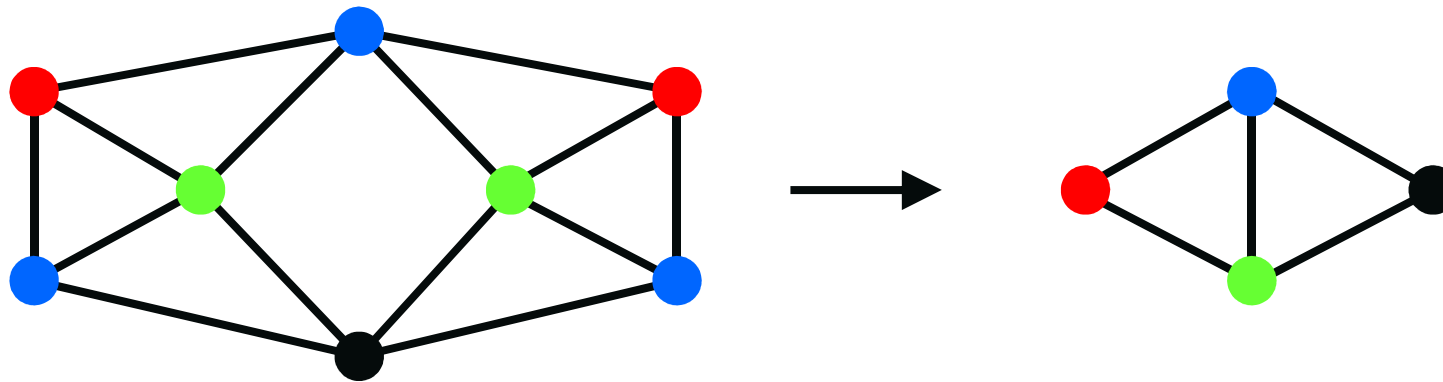


The **chromatic number**  $\chi(G)$  of  $G$  is the smallest  $k$  for which  $G$  has a  $k$ -vertex-colouring.

# Graph homomorphisms

A **homomorphism** from  $G$  to  $H$  is a mapping  $h : V(G) \rightarrow V(H)$  such that :

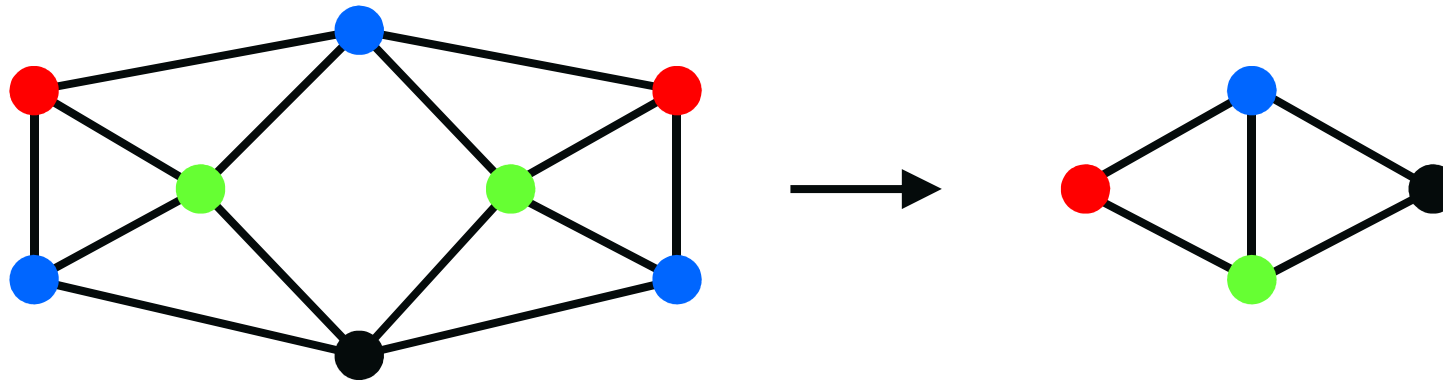
$$xy \in E(G) \Rightarrow h(x)h(y) \in E(H)$$



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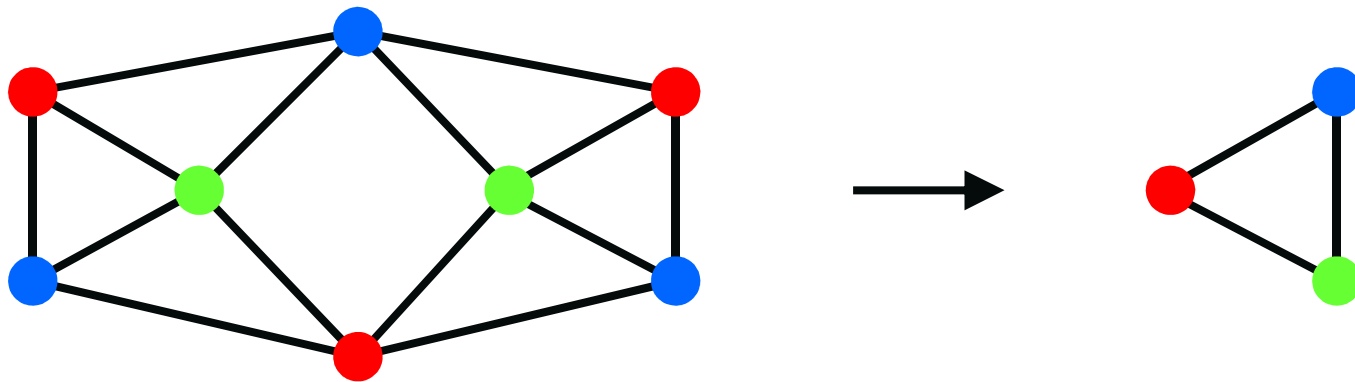


## Notation.

$G \rightarrow H$  : there exists a homomorphism from  $G$  to  $H$

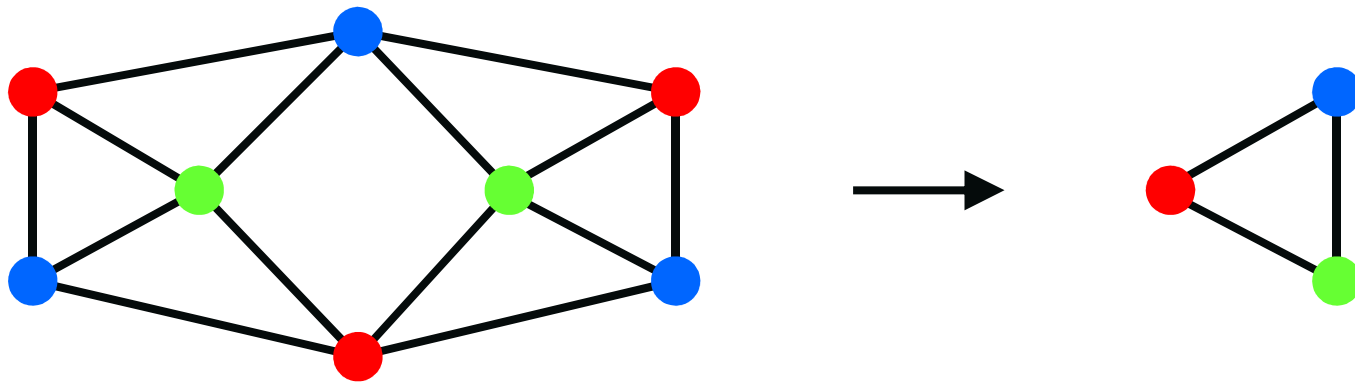
# Homomorphisms and colourings

A  $k$ -vertex-colouring of  $G$  is nothing but a *homomorphism* from  $G$  to  $K_k$ , the *complete graph* on  $k$  vertices.



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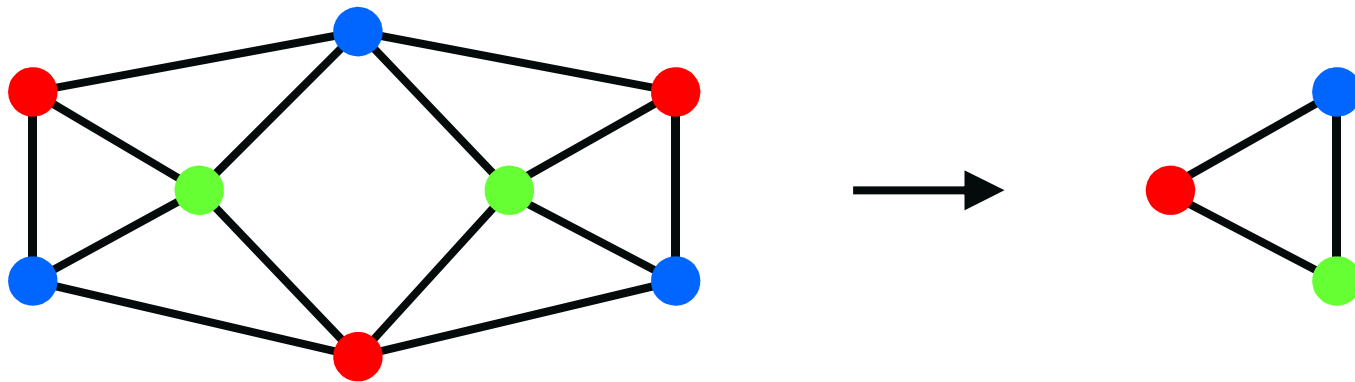


## Remark.

$\chi(G) = k$  if and only if  $G \rightarrow K_k$  and  $G \not\rightarrow K_{k-1}$

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*The chromatic number  $\chi(G)$  of  $G$  can equivalently be defined as the smallest order of a (complete) graph  $H$  such that  $G \rightarrow H$ .*

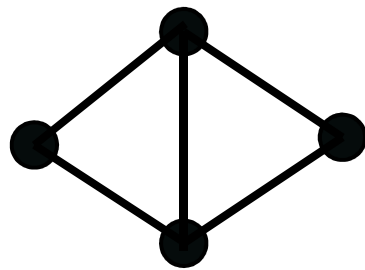
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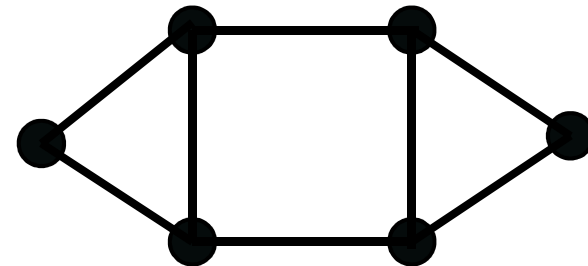


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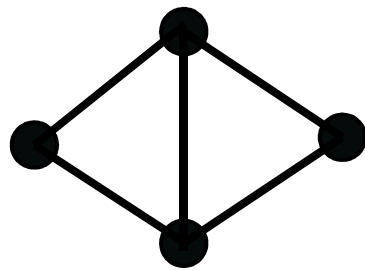


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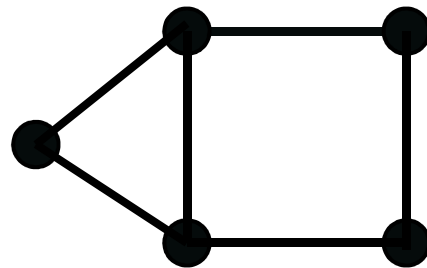
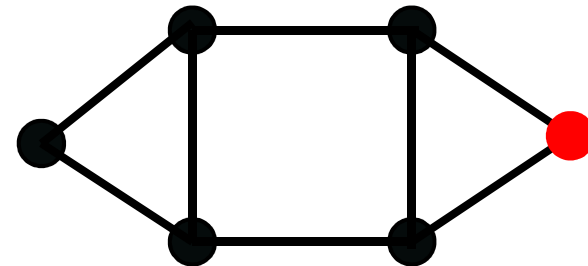


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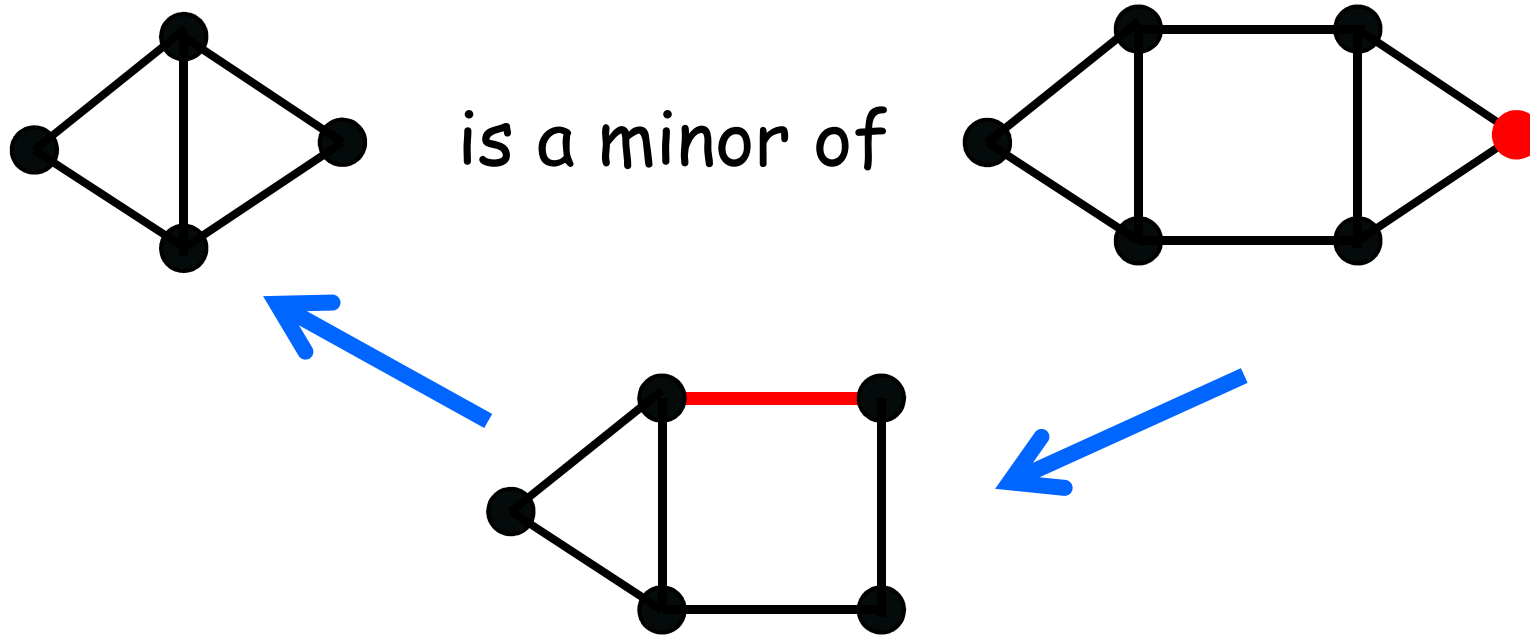


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# Hadwiger's Conjecture

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If a graph  $G$  has no  $K_n$  as a minor, then  $G$  is  $(n-1)$ -colourable.



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- $n = 3$ : if  $G$  has no cycle ( $G$  is a forest), then  $G$  is 2-colourable.
- $n = 4$ : if  $G$  is a partial 2-tree, then  $G$  is 3-colourable.

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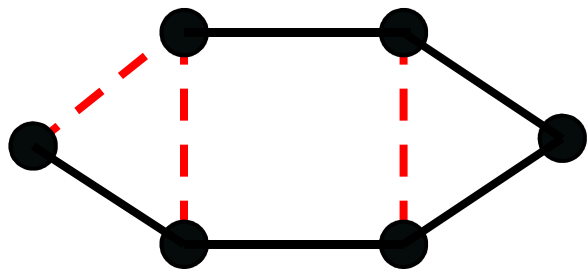


- $n = 5$ : *equivalent to the Four-Colour Theorem (Wagner, 1937).*
- $n = 6$ : Robertson, Seymour & Thomas (1993).
- $n \geq 7$ : open...

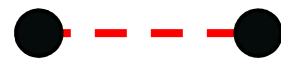
# Signed graphs

# Signified graphs

A **signified graph** is a pair  $(G, \Sigma)$  where  $G$  is a graph and  $\Sigma$  a *signature* on  $G$ , that is a subset of  $E(G)$ . Edges belonging to  $\Sigma$  are *negative* edges, while other edges are *positive* edges.



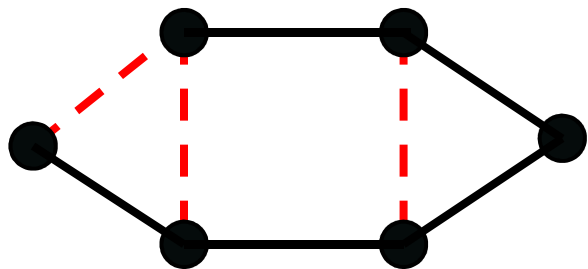
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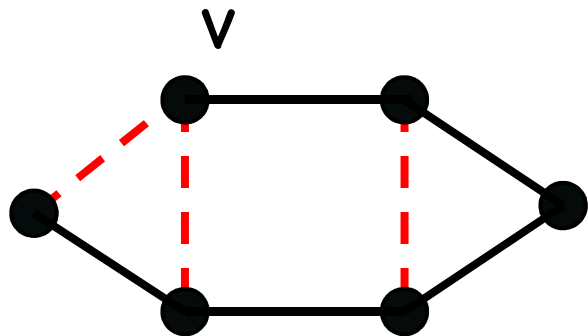
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**Frank Harary.** On the notion of balance of a signed graph.  
*Michigan Math. J.* 2(2):143-146 (1954). (*Social Psychology: "like" and "dislike" relations in social networks.*)



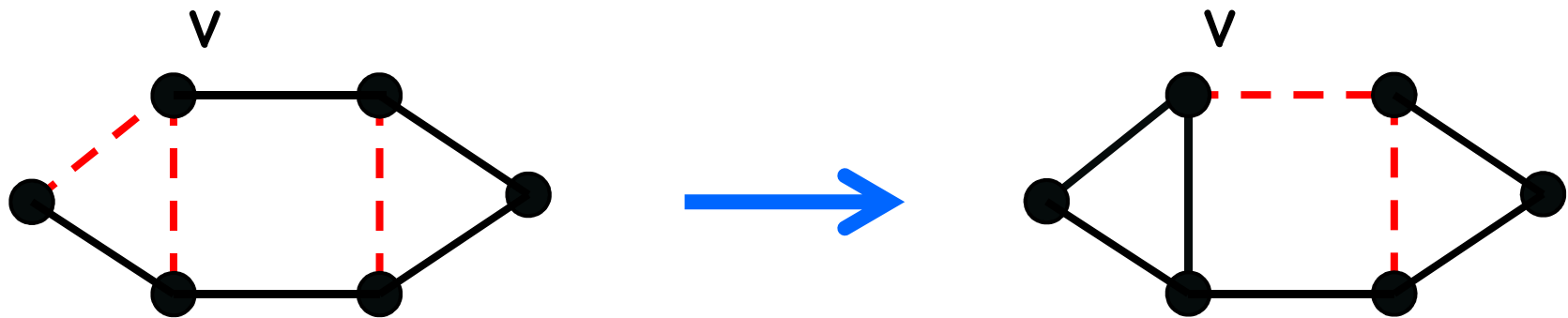
# Resigning

Let  $(G, \Sigma)$  be a signified graph and  $v$  a vertex of  $V(G)$ . By *resigning at  $v$* , we mean changing the sign of every edge incident to  $v$ .



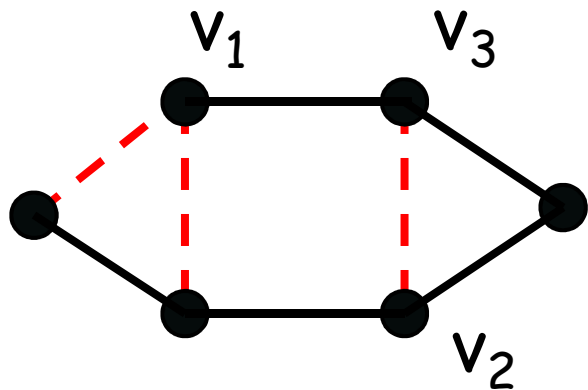
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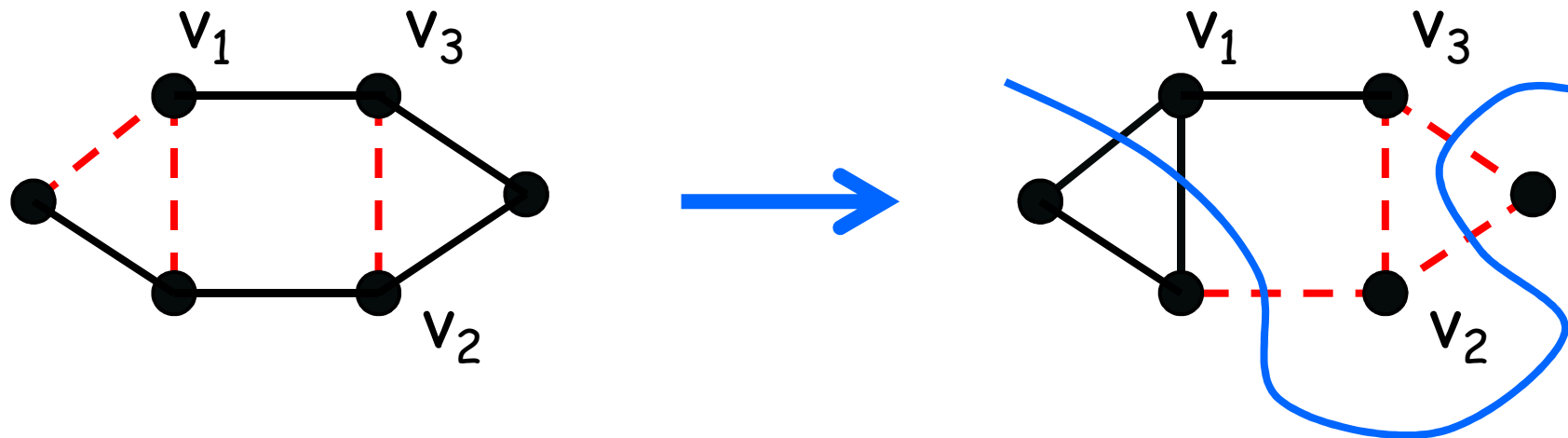
Resigning sequentially at vertices  $v_1, \dots, v_k$ , is equivalent to changing the signs of the *edge-cut*  $(S, E(G) \setminus S)$ , where  $S = \{v_1, \dots, v_k\}$ .





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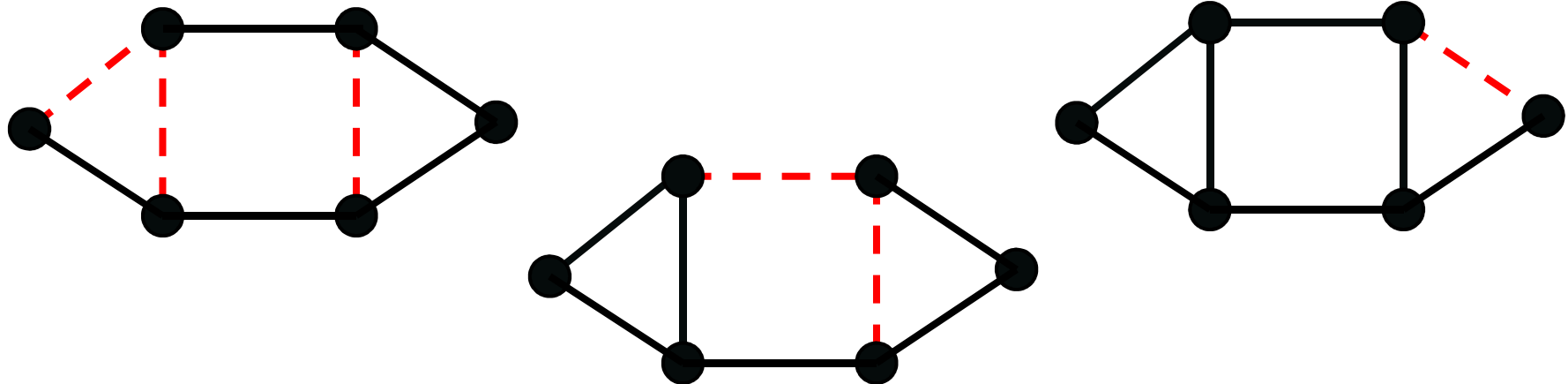
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A **signed graph** is an *equivalence class* of signified graphs. A signed graph is denoted  $(G, \Sigma)$ , where  $(G, \Sigma)$  is any representative of this class.

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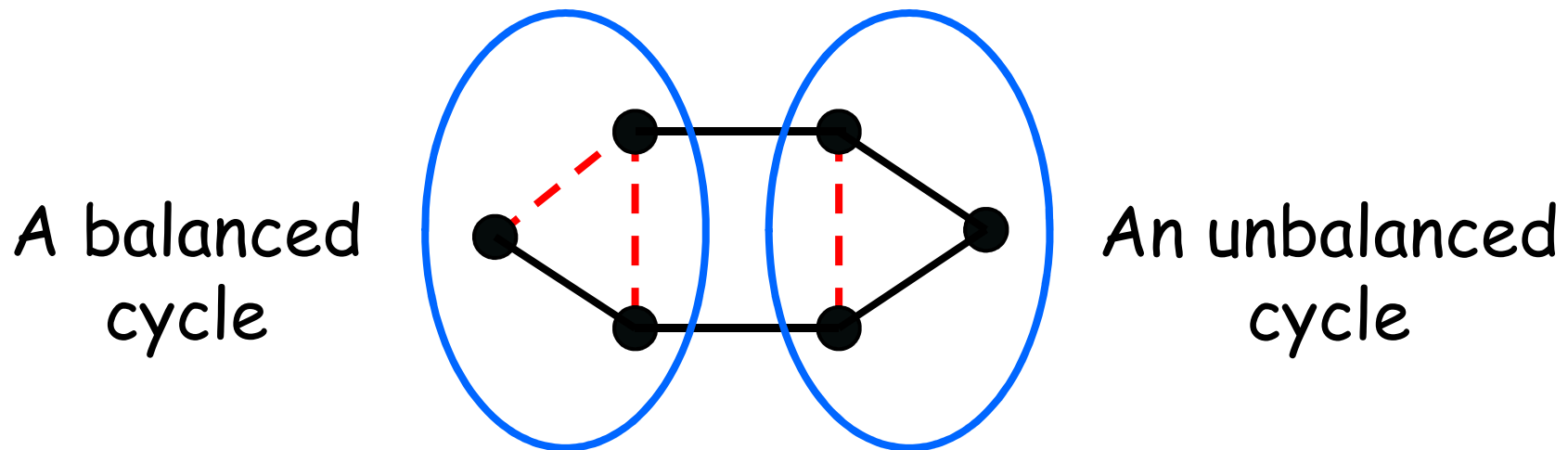


# Cycles in signed graphs

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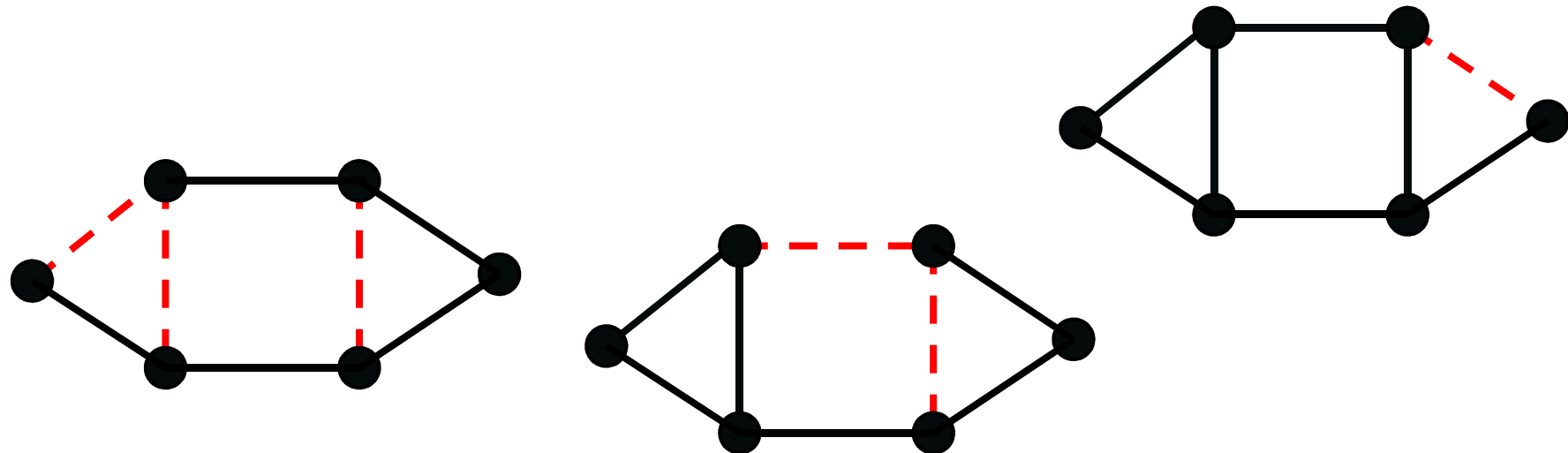
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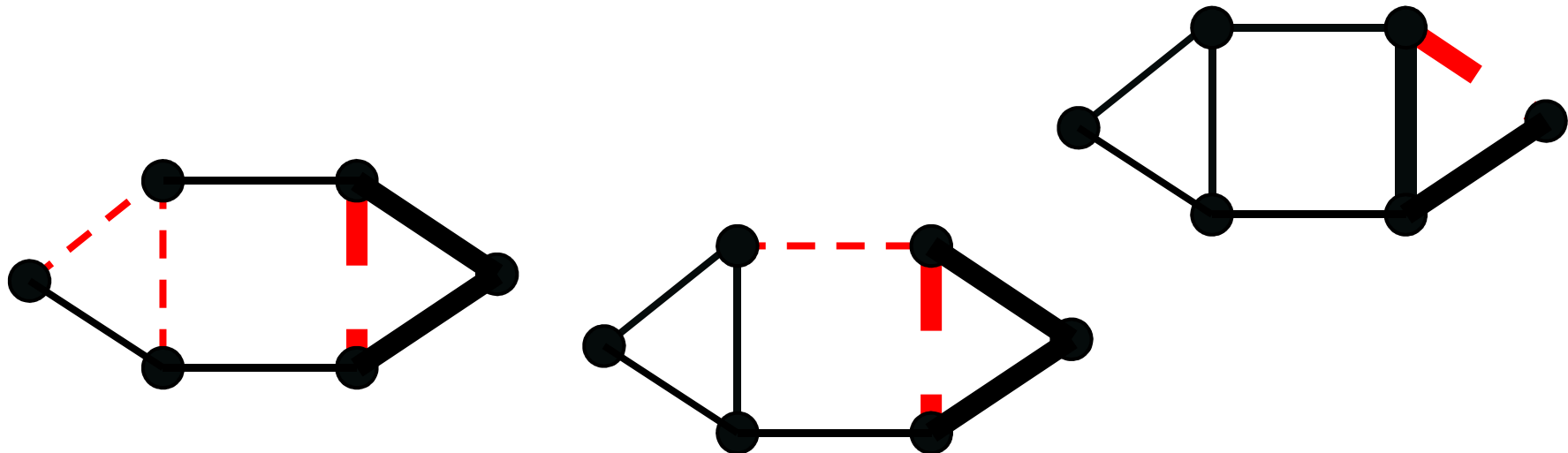




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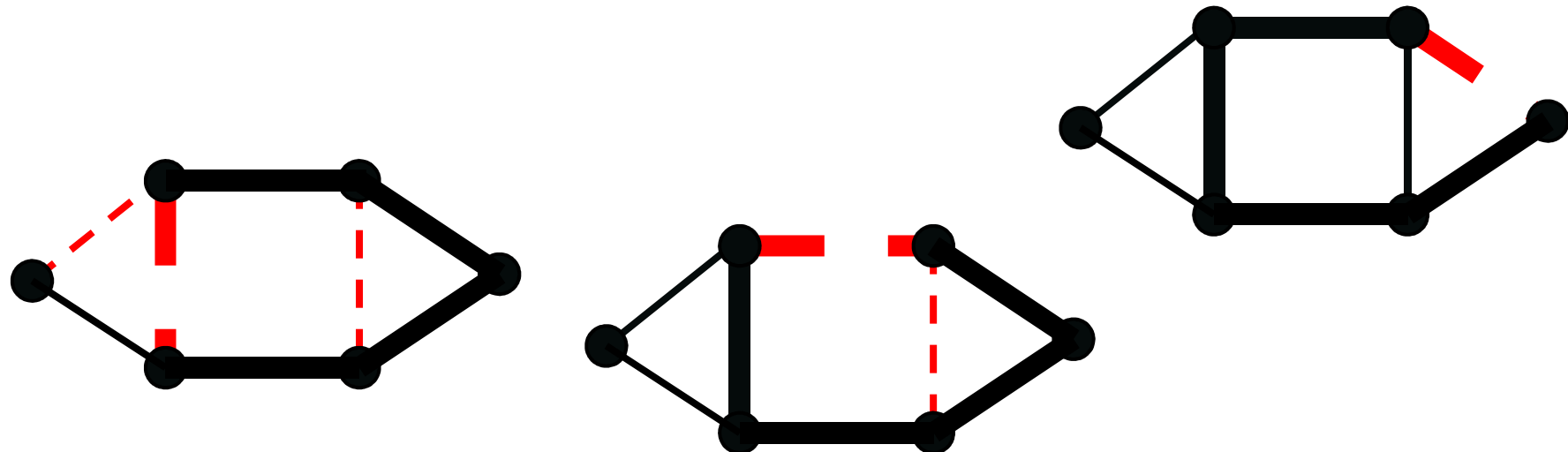
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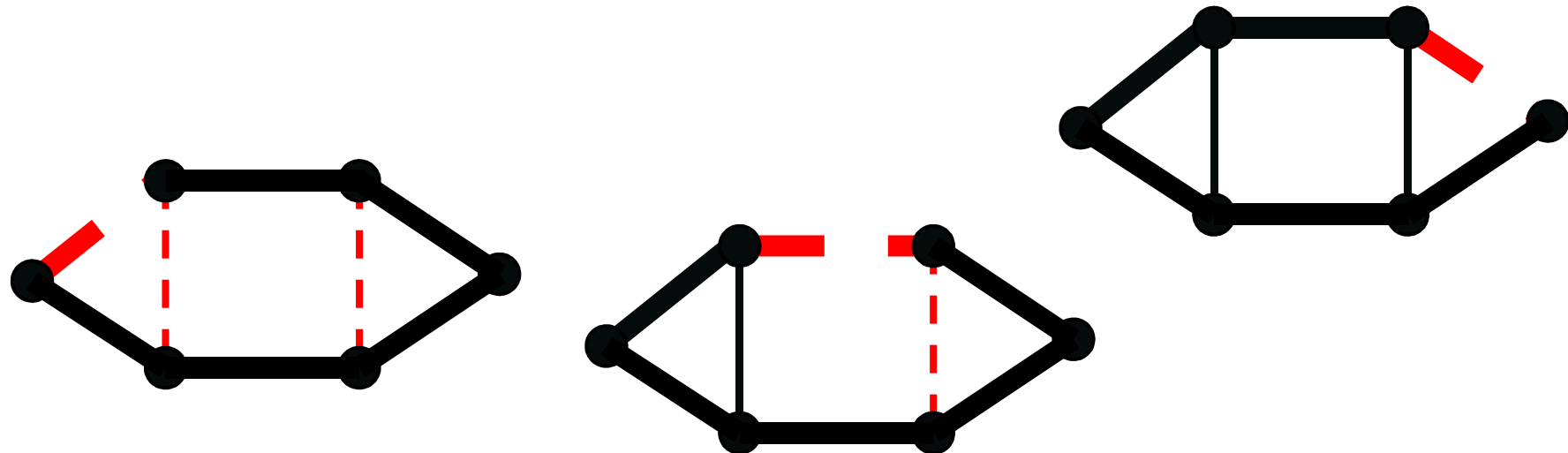
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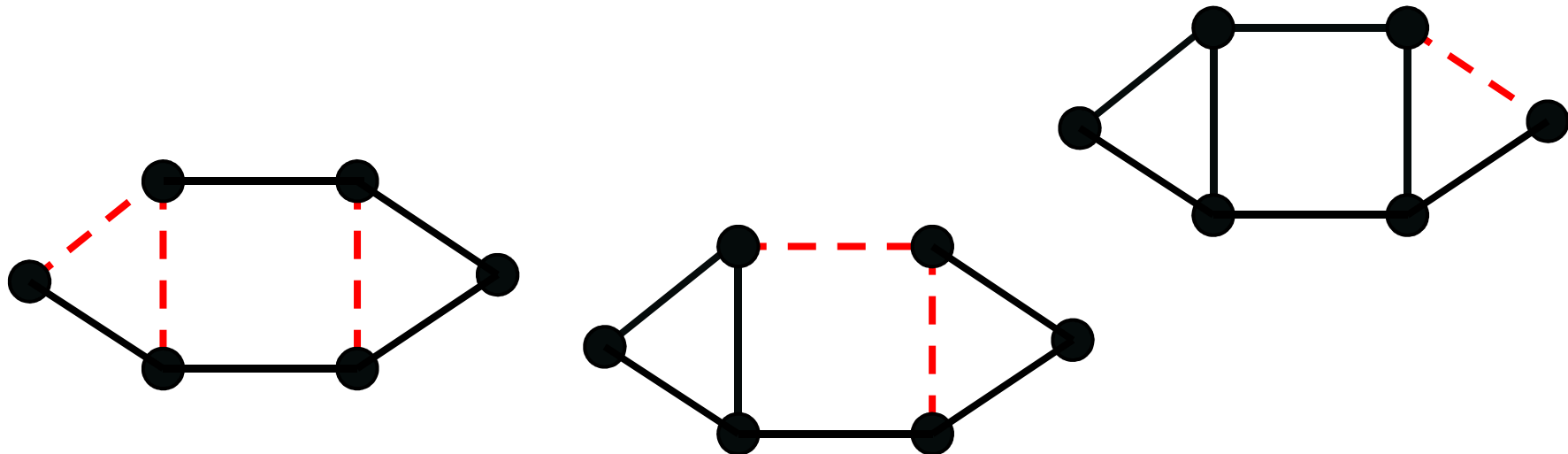
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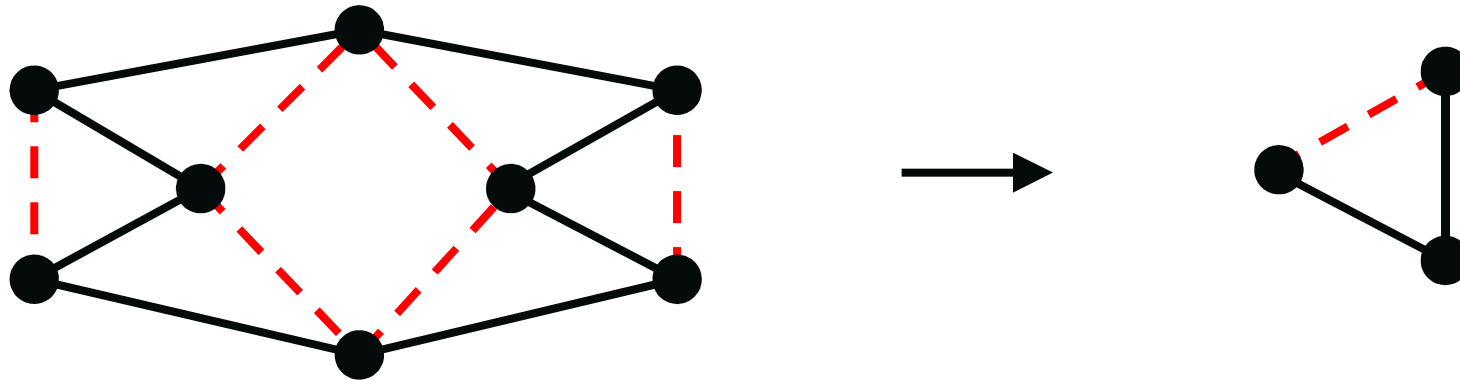


# Homomorphisms of signed graphs

A **homomorphism** of  $(G, \Sigma)$  to  $(H, T)$  is a *sign-preserving* homomorphism from  $(G, \Sigma')$  to  $(H, T)$ , where  $(G, \Sigma')$  and  $(G, \Sigma)$  are equivalent.

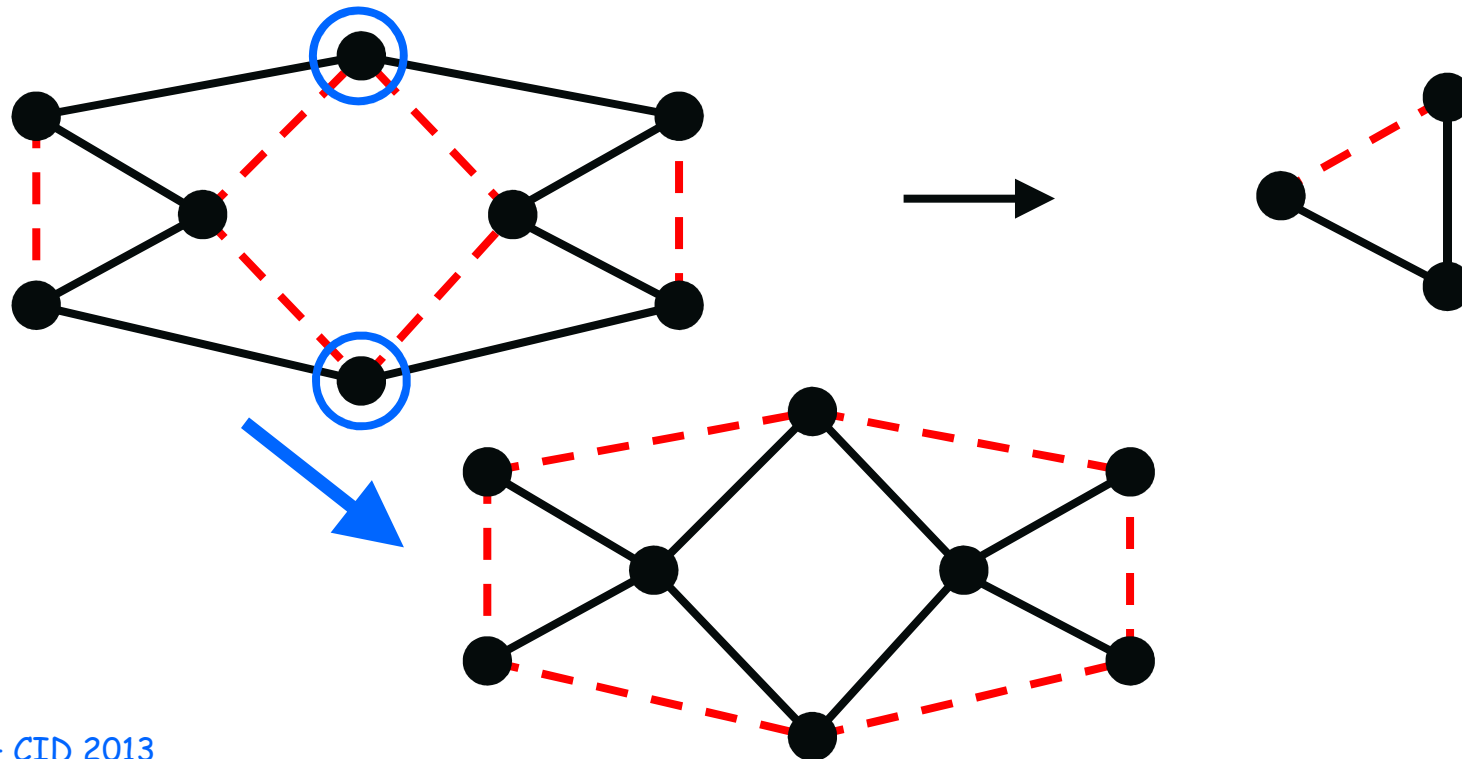
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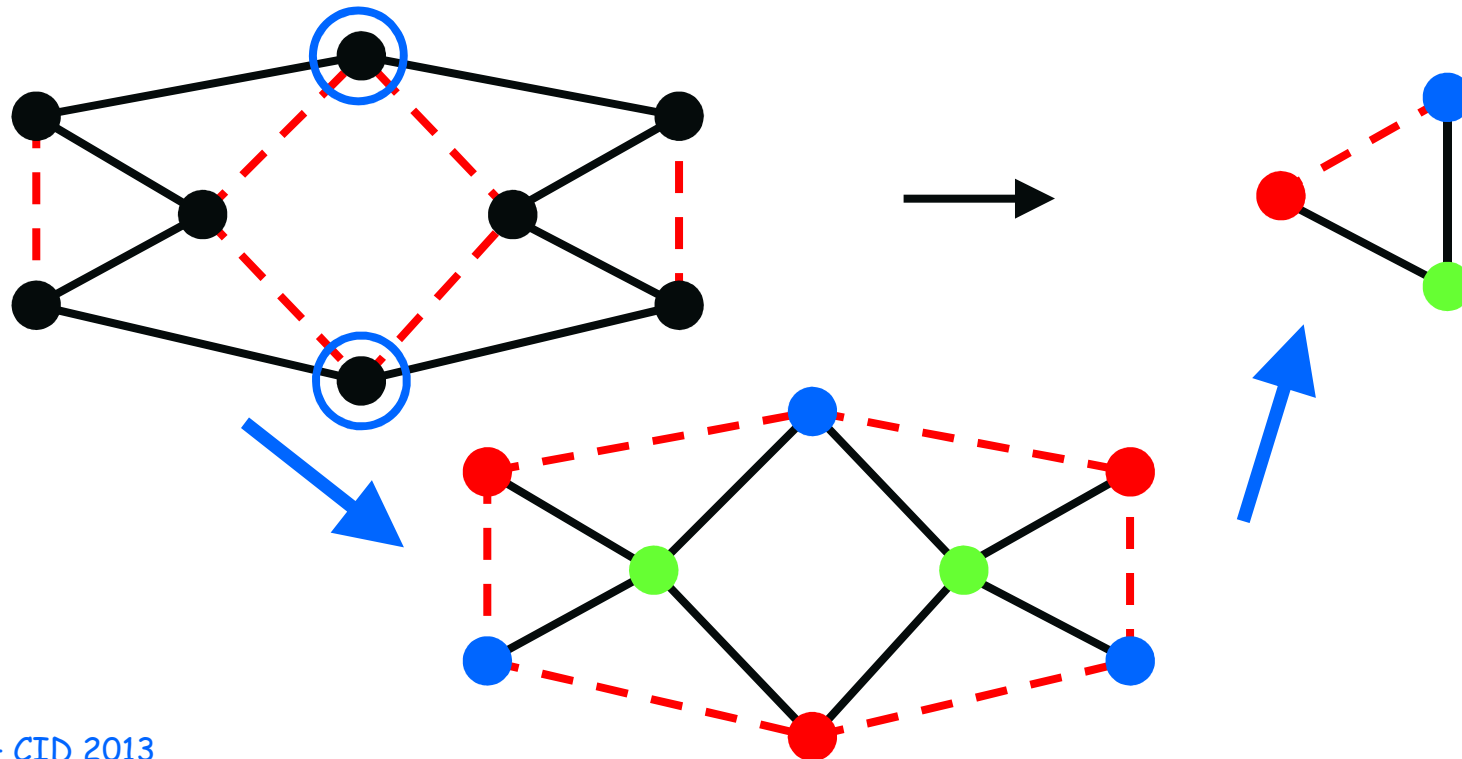
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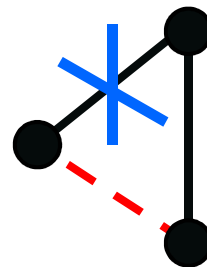




# Minors of signed graphs

A signed graph  $(H, T)$  is a **minor** of  $(G, \Sigma)$  if  $(H, T)$  can be obtained from  $(G, \Sigma)$  by

- *resigning*,
- deleting vertices,
- deleting edges, and
- contracting *positive* edges, *not belonging to an unbalanced 3-cycle*.



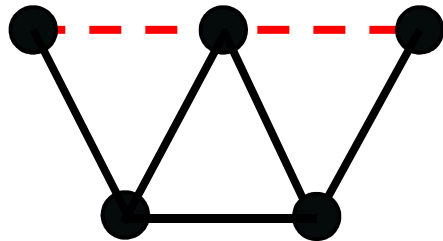
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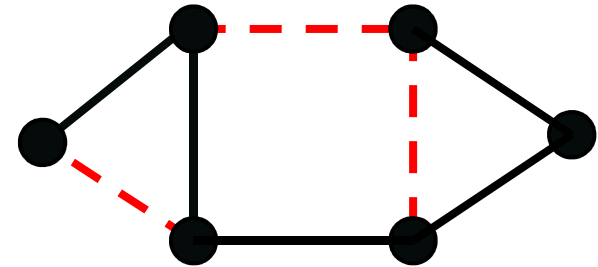
- *resigning*,
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*Note that none of these operations can create an unbalanced cycle, in contrast to graph minor operations that may produce odd cycles...*

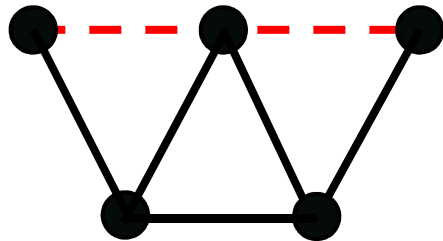
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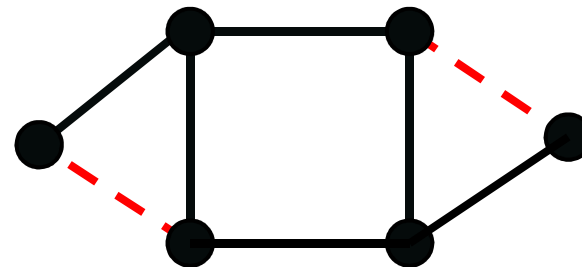
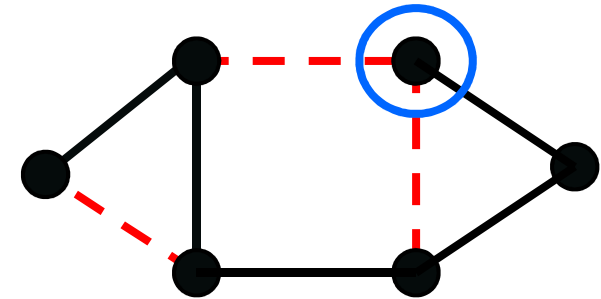
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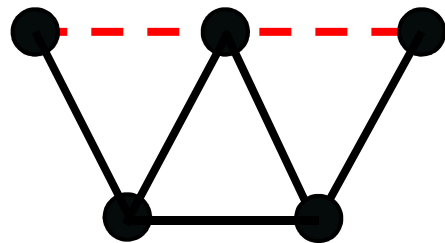
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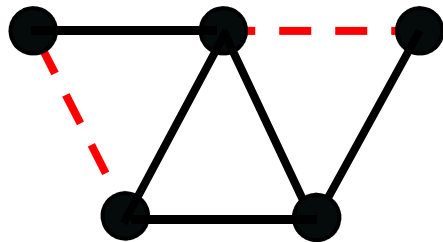
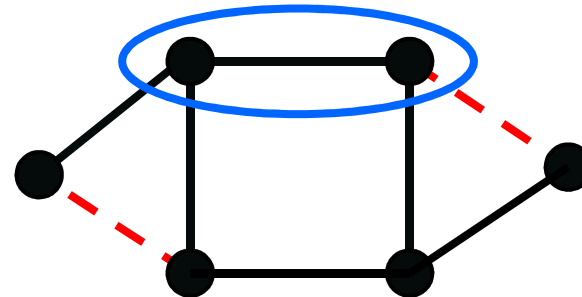
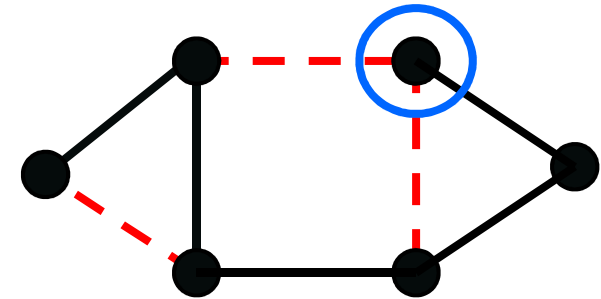
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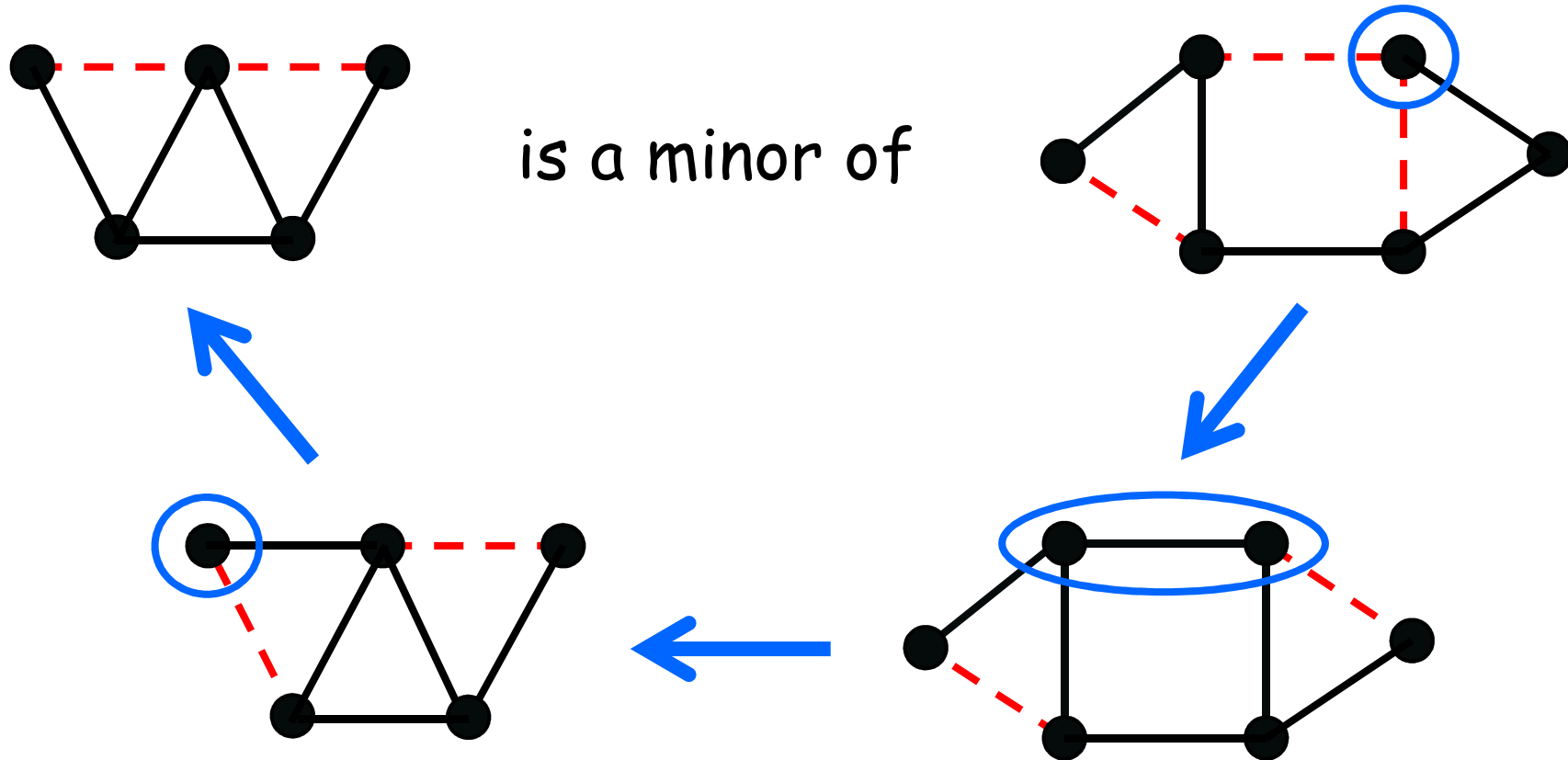
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# Minors of signed graphs



# Consistent signed graphs

A signed graph is **consistent** if

- the length of every *balanced* cycle is *even*,
- the lengths of all *unbalanced* cycles have the *same parity*.

# Consistent signed graphs

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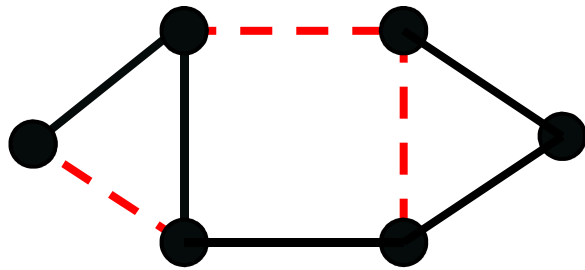
- the length of every *balanced* cycle is *even*,
- the lengths of all *unbalanced* cycles have the *same parity*.

We thus have two classes of consistent signed graphs:

- **odd signed graphs:** all unbalanced cycles are of *odd length*,
- **signed bipartite graphs:** all unbalanced cycles are of *even length*.

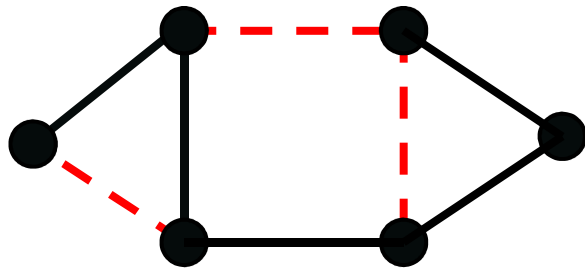


# Consistent signed graphs

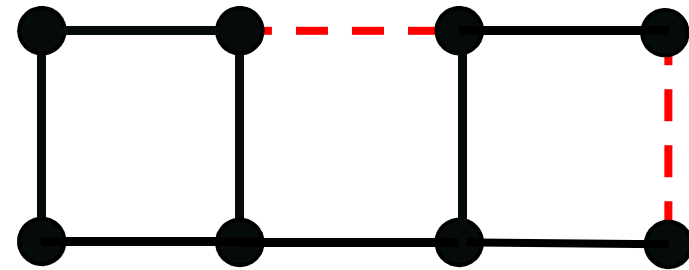


An odd signed graph  
balanced  $\rightarrow$  even length  
unbalanced  $\rightarrow$  odd length

# Consistent signed graphs

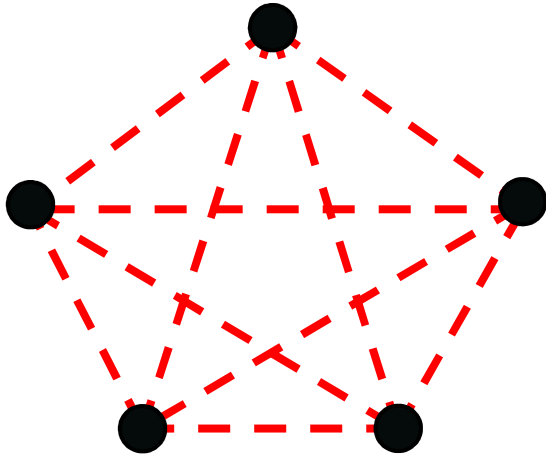


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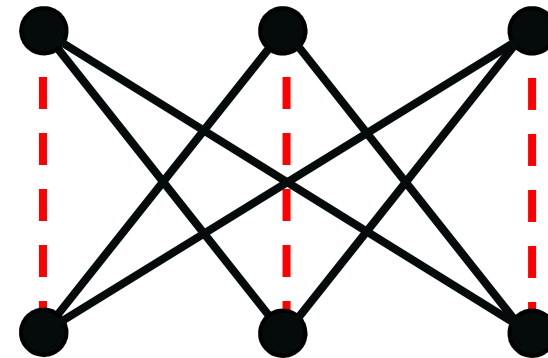


A signed bipartite graph  
balanced  $\rightarrow$  even length  
unbalanced  $\rightarrow$  even length

# Consistent signed graphs



The **odd  $K_5$**   
balanced  $\rightarrow$  even length  
unbalanced  $\rightarrow$  odd length



The signed bipartite  
graph  **$(K_{3,3}, M)$**   
balanced  $\rightarrow$  even length  
unbalanced  $\rightarrow$  even length

Odd signed graphs

# Odd signed graphs

balanced  $\rightarrow$  even length, unbalanced  $\rightarrow$  odd length

**Theorem (Zaslavsky, 1982).**

Two signified graphs are equivalent if and only if they have the same set of unbalanced cycles.

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Every odd signed graph  $(G, \Sigma)$  can be represented by  $(G, E(G))$ . Thus, for  $(G_1, \Sigma_1), (G_2, \Sigma_2)$  two odd signed graphs,

$(G_1, \Sigma_1) \rightarrow (G_2, \Sigma_2)$  if and only if  $G_1 \rightarrow G_2$ .

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*Homomorphism questions can be translated in terms of homomorphisms of odd signed graphs...*



# Odd Hadwiger's conjecture

**Odd Hadwiger's Conjecture (Gerards & Seymour).**

If  $(G, E(G))$  has no  $(K_n, E(K_n))$  as a minor, then  $G$  is  $(n-1)$ -colourable.



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- Odd Hadwiger's Conjecture says that every *bipartite graph* is 2-colourable.

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- $n = 4$ : proved by Catlin (1979),
- $n = 5$ : announced by Guenin (2005, unpublished), would imply the Four-Colour Theorem.

# Signed bipartite graphs

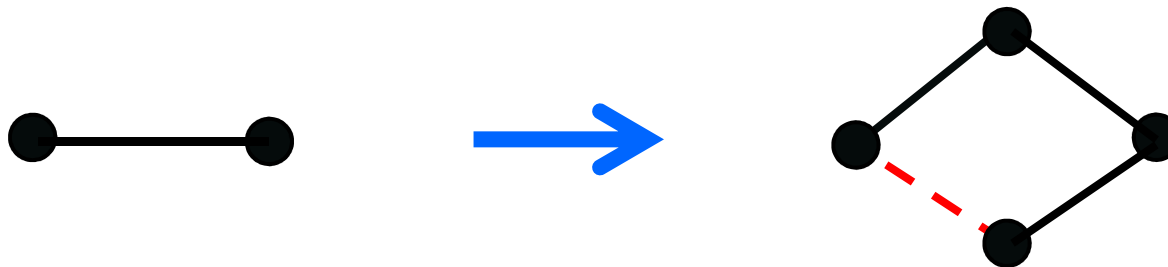
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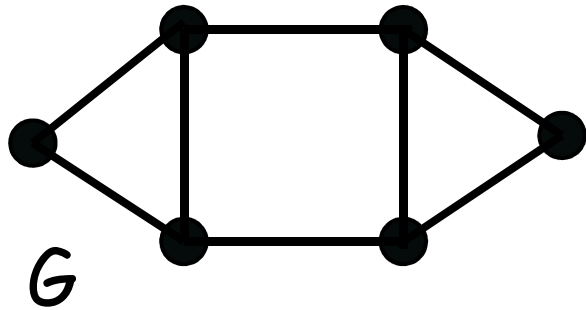
# Signed bipartite graphs

balanced  $\rightarrow$  even length, unbalanced  $\rightarrow$  even length

With every graph  $G$ , one can associate a signed bipartite graph  $S(G)$  by replacing each edge of  $G$  by an *unbalanced 4-cycle*.

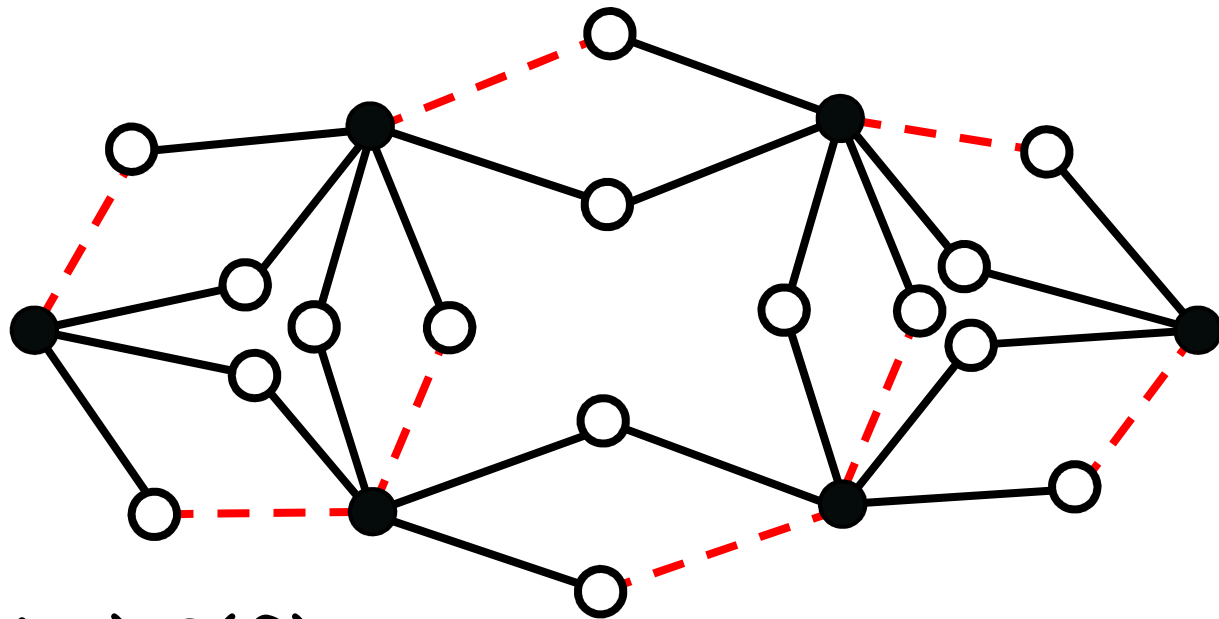
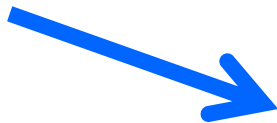
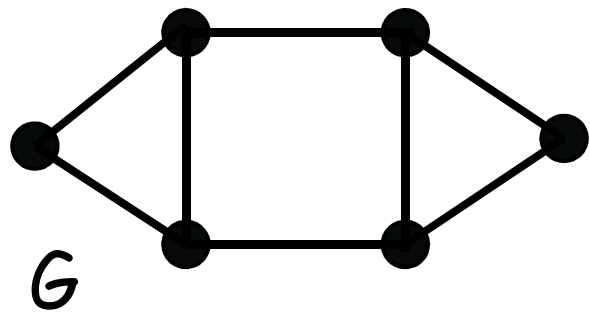


# Signed bipartite graphs





# Signed bipartite graphs



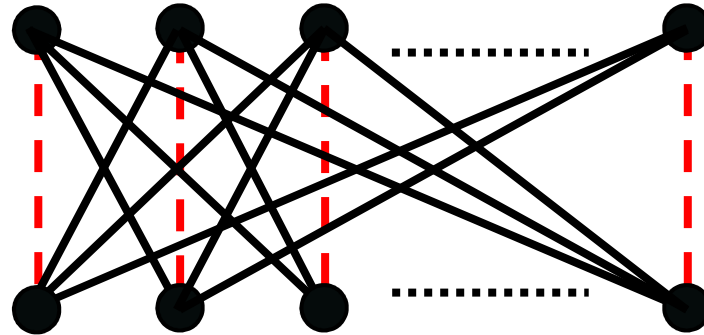
(bipartite)  $S(G)$

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**Theorem (Naserasr, Rollová, ES, 2012+).**

For every graph  $G$ , and every  $k \geq 3$ ,

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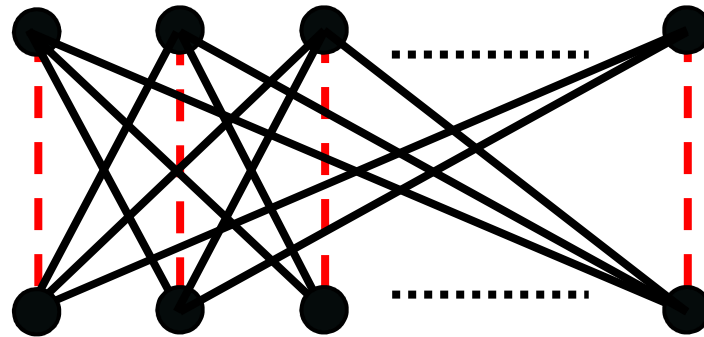


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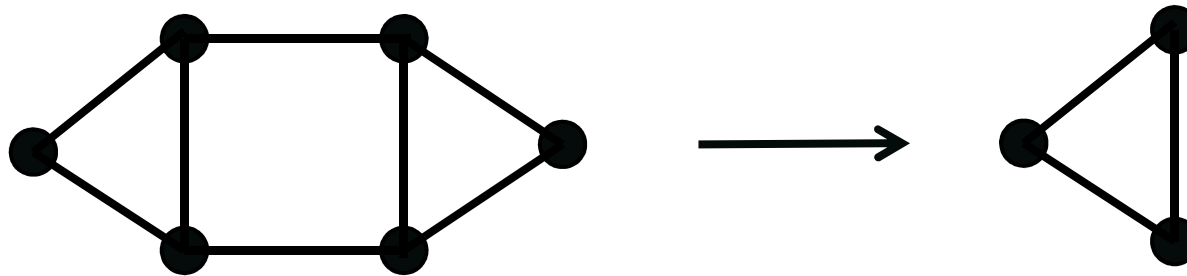
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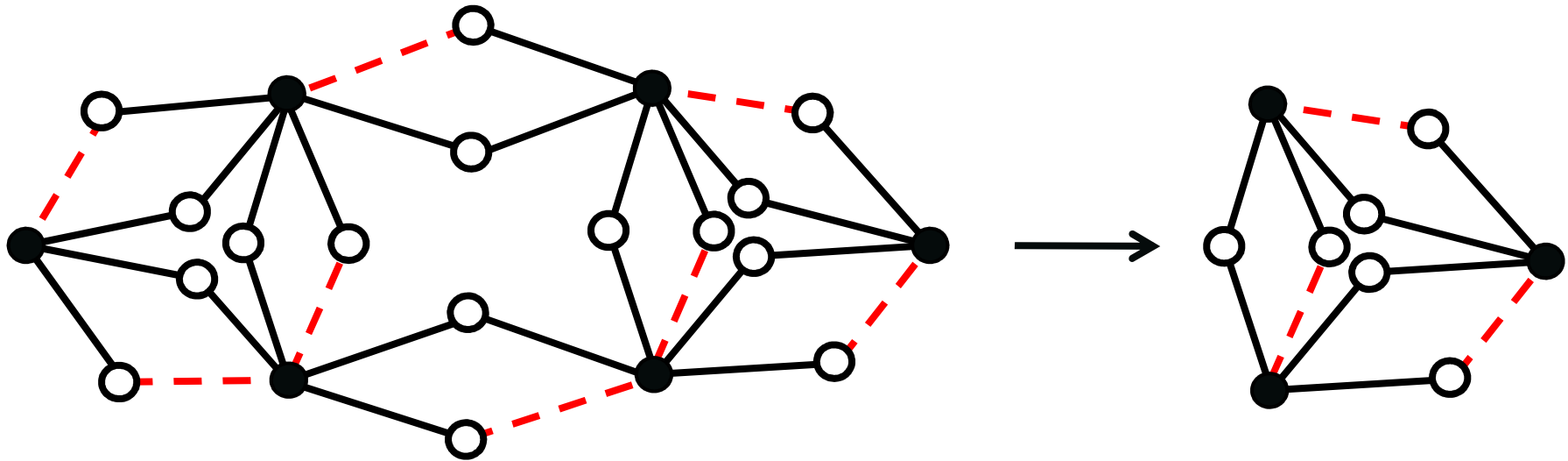


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**Theorem (Naserasr, Rollová, ES, 2012+).**

For every integer  $n$  and every graph  $G$ ,  $K_n$  is a minor of  $G$  if and only if  $(K_n, \Sigma)$  is a minor of  $S(G)$  for some (any)  $\Sigma$ .



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*Hadwiger's Conjecture can thus be restated as:*

**Hadwiger's Conjecture (1943).**

If a graph  $G$  has no  $K_n$  as a minor, then

$$S(G) \rightarrow (K_{n-1, n-1}, M)$$

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For  $n = 4$ , we prove the following generalization:

**Theorem (Naserasr, Rollová, ES, 2012+).**

If  $G$  is bipartite and has no  $K_4$ -minor then, for every signature  $\Sigma$  on  $G$ ,  $(G, \Sigma) \rightarrow (K_{3,3}, M)$ .

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We propose the following stronger statement:

**Conjecture.** If  $G$  is bipartite and  $(G, \Sigma)$  has no  $(K_4, E(K_4))$  as a minor, then  $(G, \Sigma) \rightarrow (K_{3,3}, M)$ .

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Guenin proposed the following:

**Conjecture (Guenin, 2005).**

If  $G$  is bipartite and  $(G, \Sigma)$  has no  $(K_5, E(K_5))$  as a minor, then  $(G, \Sigma) \rightarrow (K_{4,4}, M)$ .



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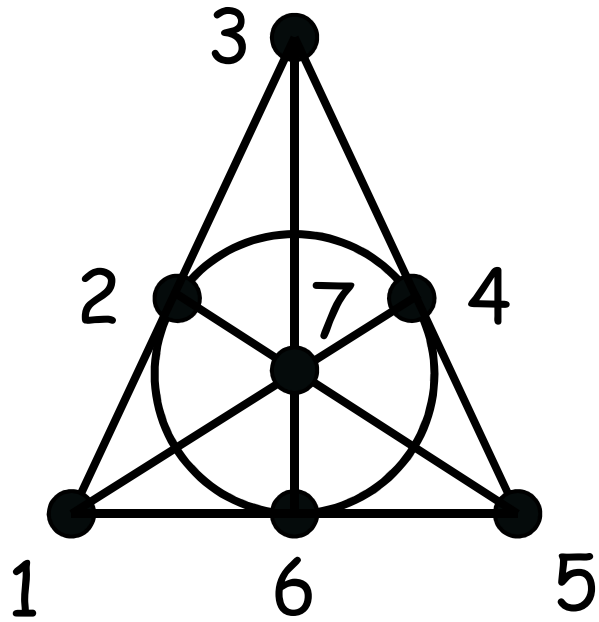
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*We know that such a conjecture does not hold for  $n \geq 7$ ...*

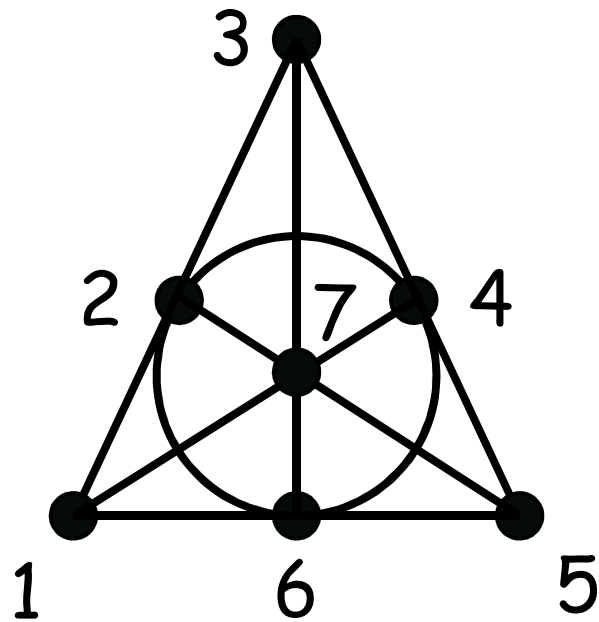


# The signed graph $F = (K_{7,7}, \Phi)$

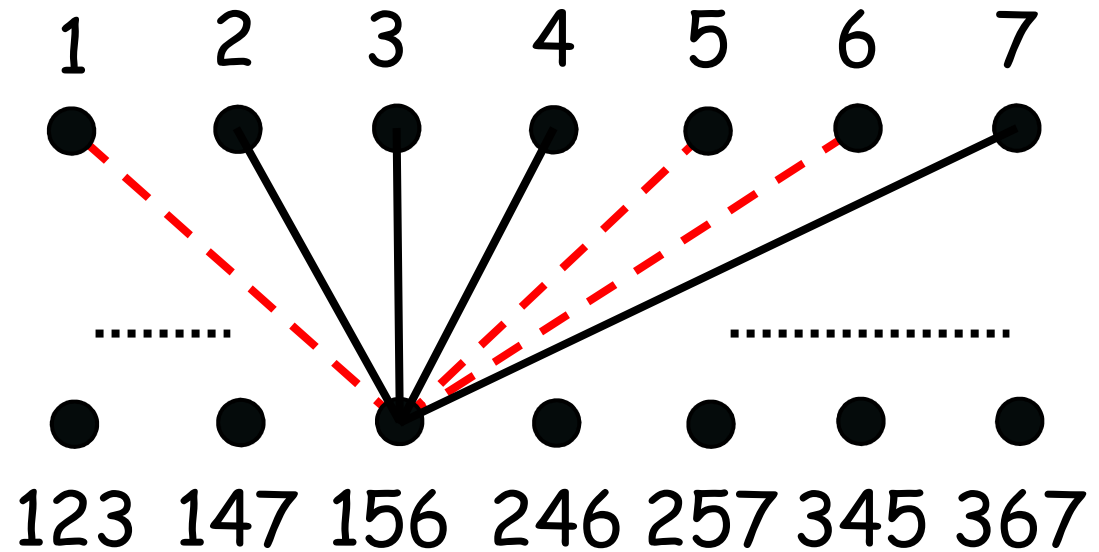


The Fano plane

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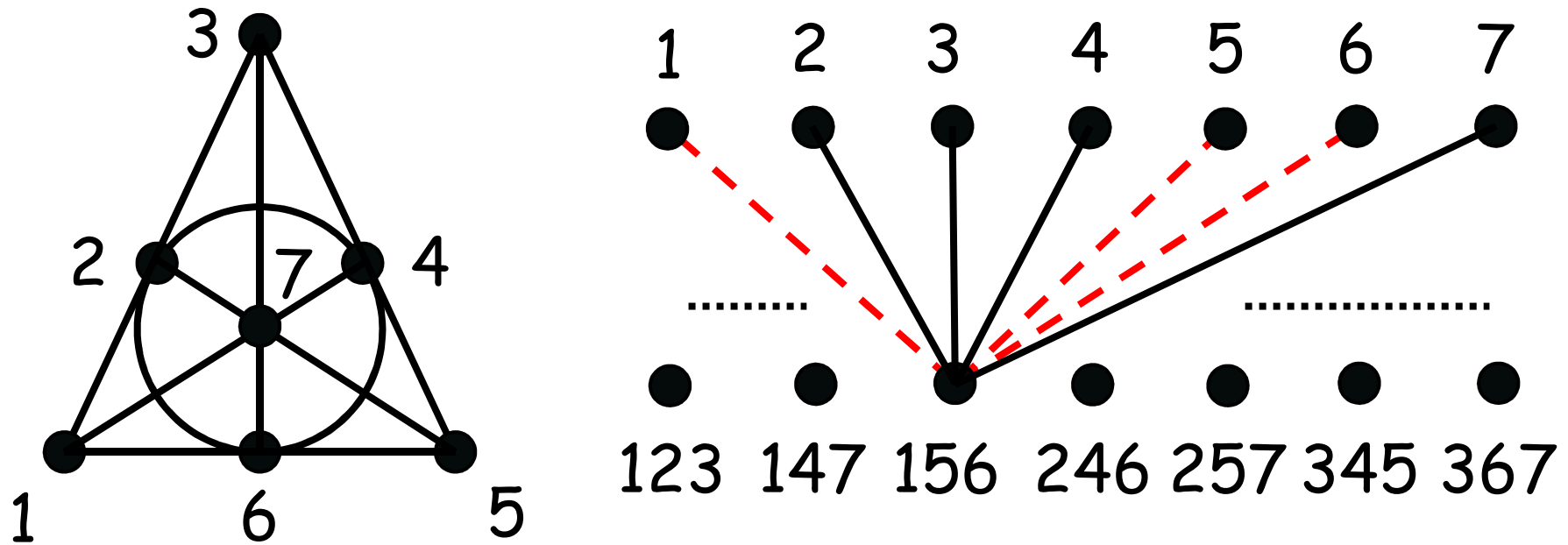


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The signed graph  $F = (K_{7,7}, \Phi)$   
 $(i, \{a,b,c\}) \in \Phi$  iff  $i \in \{a,b,c\}$

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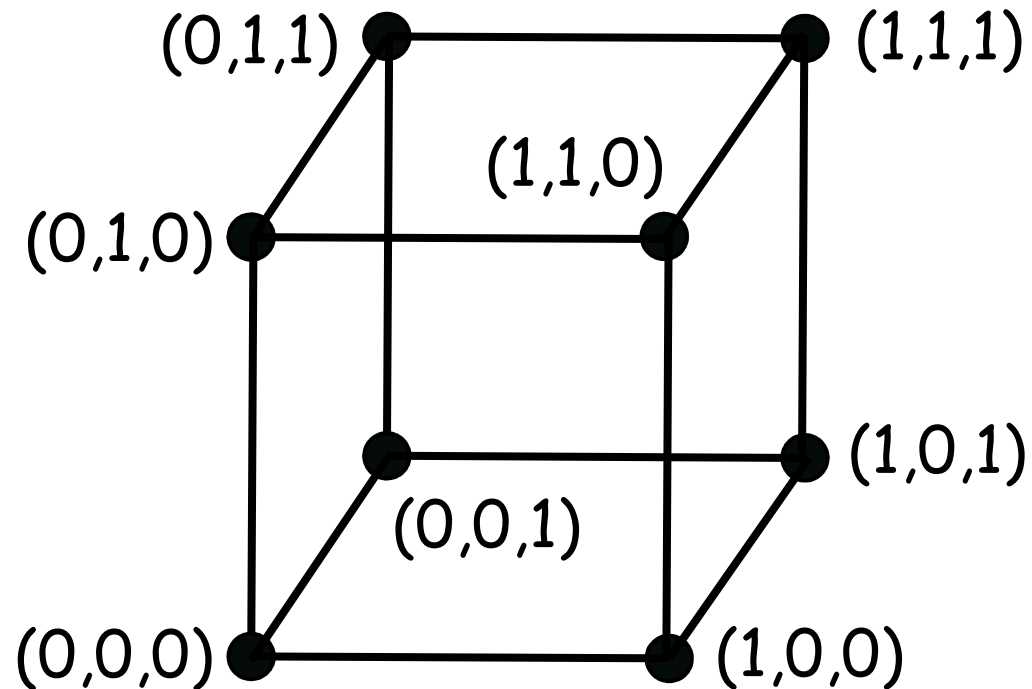
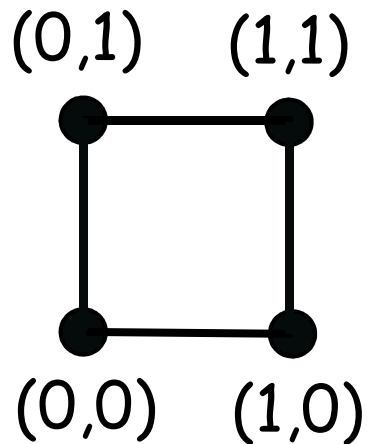


**Theorem (Naserasr, Rollová, ES, 2012+).**  
 There is no value  $n$  for which  $F \rightarrow (K_{n,n}, M)$ .

# Homomorphisms to signed projective cubes

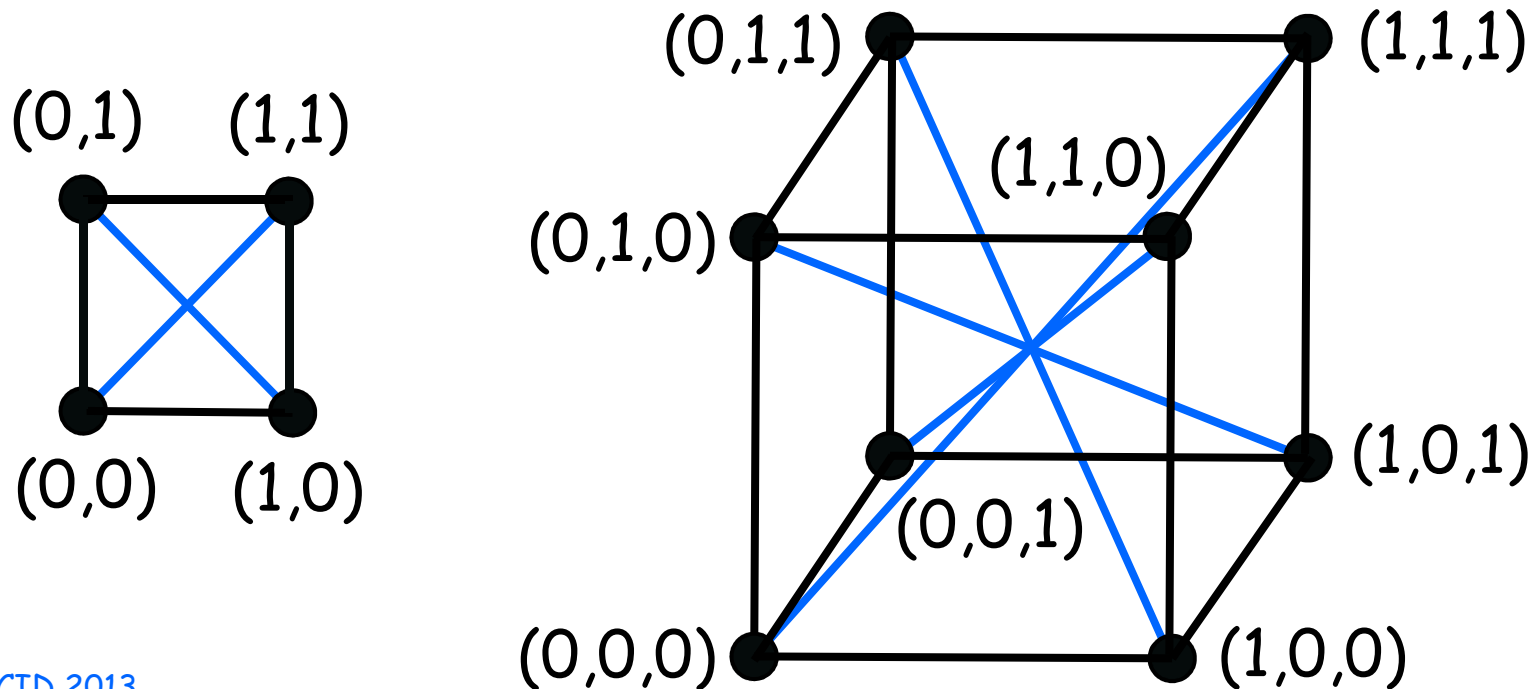
# Projective cubes

The **cube** of *dimension*  $d$  is the Cayley graph  $\mathcal{C}_d = ((\mathbb{Z}_2)^d, \{e_1, e_2, \dots, e_d\})$ , where  $e_i = (0, \dots, 0, 1, 0, \dots, 0)$



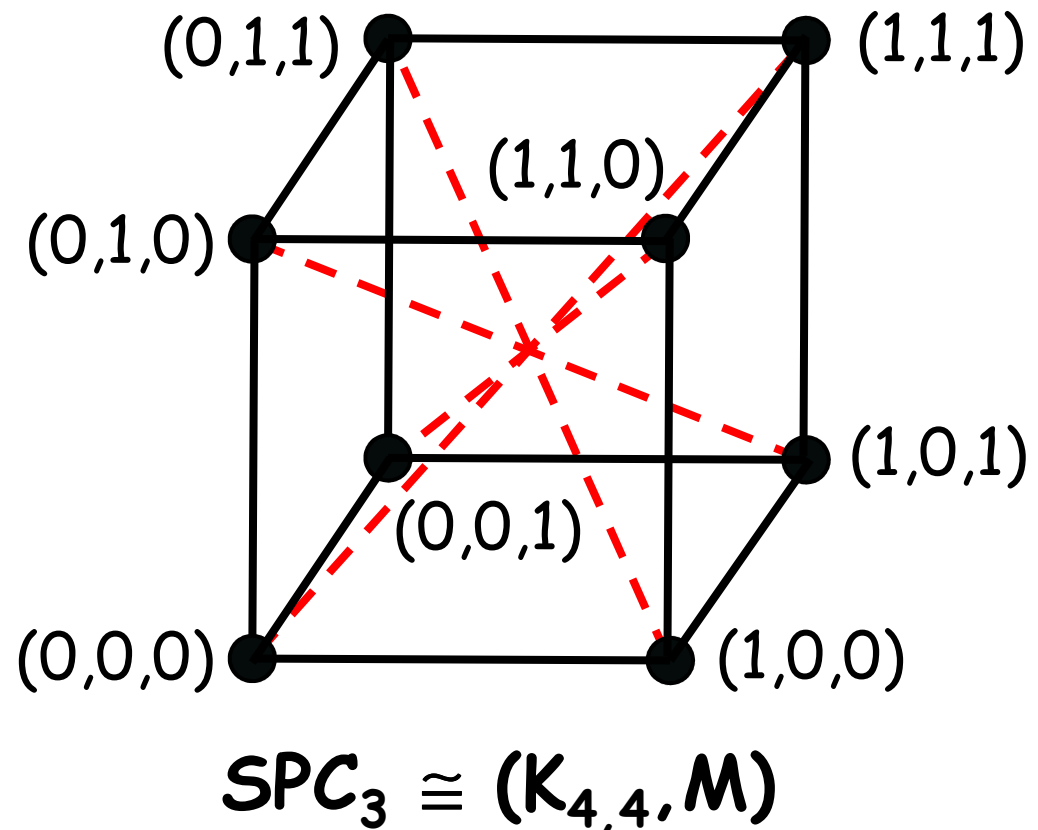
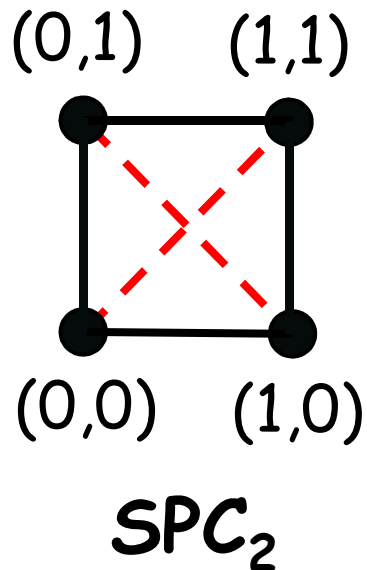
# Projective cubes

The **projective cube** of *dimension*  $d$  is the Cayley graph  $PC_d = ( (\mathbb{Z}_2)^d, \{ e_1, e_2, \dots, e_d, j \} )$ , where  $e_i = (0, \dots, 0, 1, 0, \dots, 0)$  and  $j = (1, 1, \dots, 1)$ .



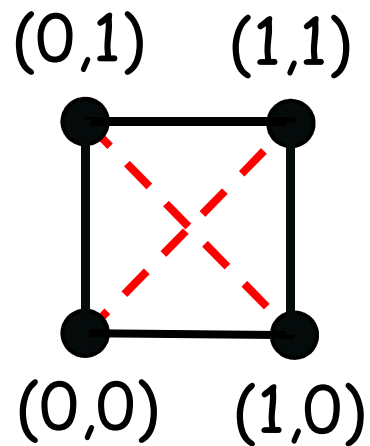
# Signed projective cubes

The **signed projective cube** of *dimension  $d$*  is defined by  $\mathbf{SPC}_d = (\mathbf{PC}_d, \mathbf{J})$ .



# Signed projective cubes

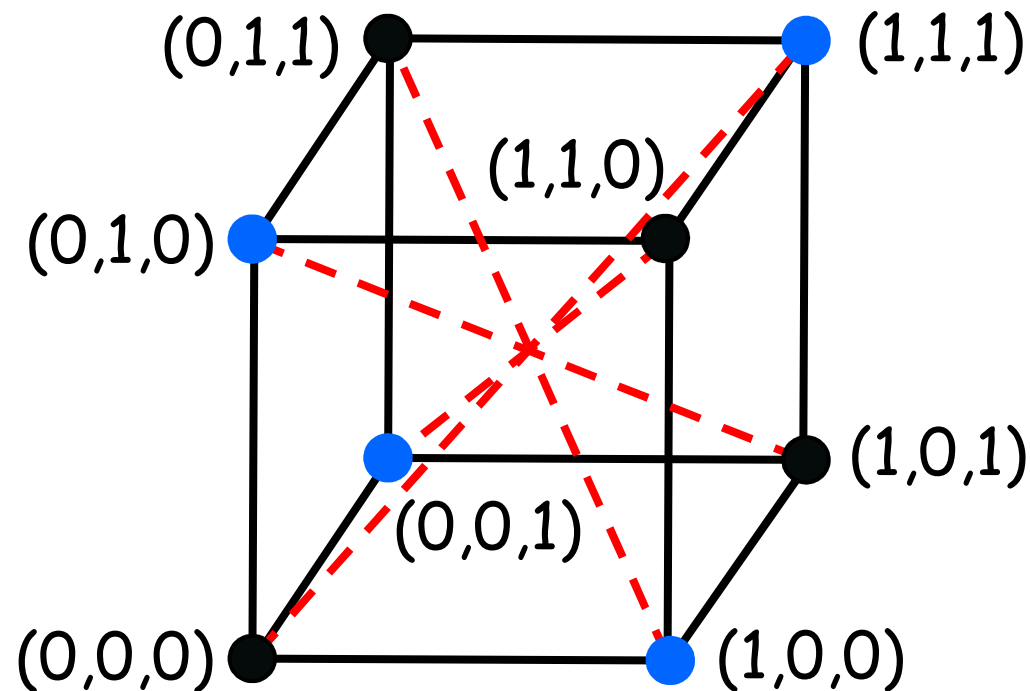
If  $d$  is even,  $SPC_d$  is an *odd signed graph*.





# Signed projective cubes

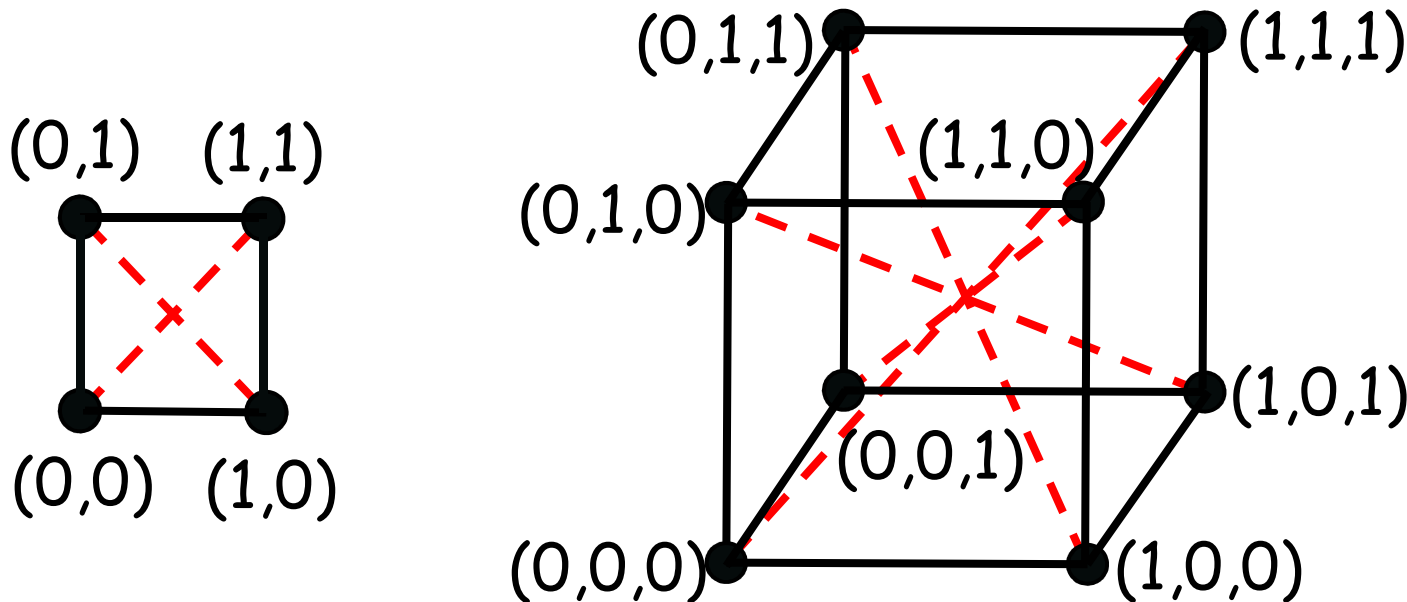
If  $d$  is odd,  $SPC_d$  is a *signed bipartite graph*.



# Homomorphisms to $SPC_d$

If  $(G, \Sigma)$  admits a homomorphism to  $SPC_d$ , then:

- $(G, \Sigma)$  must be an *odd signed graph* if  $d$  is *even*,
- $(G, \Sigma)$  must be a *signed bipartite graph* if  $d$  is *odd*.



# Homomorphisms to $SPC_d$

If  $(G, \Sigma)$  admits a homomorphism to  $SPC_d$ , then:

- $(G, \Sigma)$  must be an *odd signed graph* if  $d$  is *even*,
- $(G, \Sigma)$  must be a *signed bipartite graph* if  $d$  is *odd*.

The  $SPC_d$ -homomorphism problem is equivalent to a packing problem:

**Theorem (Guenin, 2005).** A consistent signed graph admits a homomorphism to  $SPC_d$  iff it admits at least  $d$  edge-disjoint signatures.

# Homomorphisms to $SPC_d$

The **unbalanced girth** of  $(G, \Sigma)$  is the shortest length of an unbalanced cycle of  $(G, \Sigma)$ .

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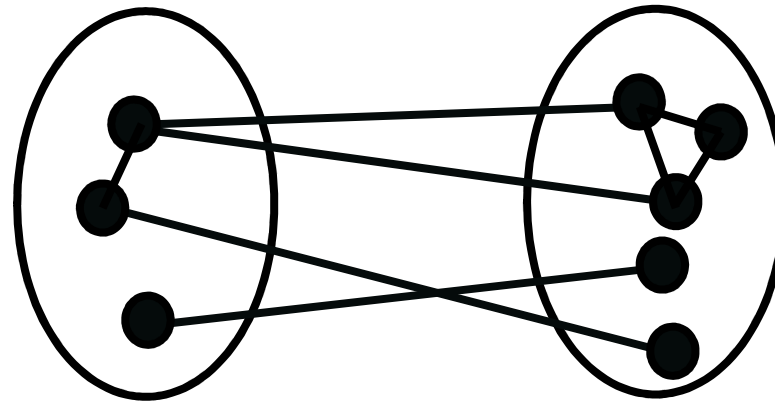
**Conjecture (Naserasr, 2007).** Every consistent planar signed graph with unbalanced girth  $k$  admits a homomorphism to  $SPC_{k-1}$ .

This conjecture is related to a conjecture of Seymour...

# Seymour's Conjecture

**Conjecture (Seymour, 1975).** Every  $k$ -regular planar multigraph with no odd edge-cut of less than  $k$  edges is  $k$ -edge-colourable.

An odd  
edge-cut:



$X$  with  $|X|$  odd

$V(G) \setminus X$

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This conjecture, if true, generalizes the Four-Colour Theorem: for  $k = 3$ , it claims that every bridgeless cubic planar multigraph is 3-edge-colourable, which is equivalent to the Four-Colour Theorem by the result of Tait (1880).



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This conjecture has been proved for

- $k = 4, 5$ : Guenin (2003),
- $k = 6$ : Dvořák, Kawarabayashi & Král' (2010),
- $k = 7$ : Edwards & Kawarabayashi (2011),
- $k = 8$ : Chudnovsky, Edwards & Seymour (2012).

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**Theorem (Naserasr, 2007 - Naserasr, Rollová, ES, 2013).** These two conjectures are equivalent.

# Seymour's Conjecture

Since Seymour's conjecture holds for  $k \leq 8$ , we get the following:

**Corollary.** Every planar signed bipartite graph of unbalanced girth 4 (resp. 6, 8) admits a homomorphism to  $SPC_3$  (resp.  $SPC_5$ ,  $SPC_7$ ).

# Seymour's Conjecture

In particular, since the unbalanced girth of every bipartite graph is at least 4:

**Corollary.** Every planar signed bipartite graph admits a homomorphism to  $SPC_3 \cong (K_{4,4}, M)$ .

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**Corollary.** Every planar signed bipartite graph admits a homomorphism to  $SPC_3 \cong (K_{4,4}, M)$ .

This is an extension of the Four-Colour Theorem, since  $\chi(G) \leq k$  if and only if  $S(G) \rightarrow (K_{k,k}, M)$ ...

The chromatic number  
of signed graphs

# Chromatic number

Recall that the chromatic number  $\chi(G)$  of a graph  $G$  can equivalently be defined as the smallest order of a graph  $H$  such that  $G \rightarrow H$ .



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Similarly, we can thus define the chromatic number  $\chi(G, \Sigma)$  of a signed graph  $(G, \Sigma)$  as the smallest order of a graph  $H$  such that  $(G, \Sigma) \rightarrow (H, T)$  for some signature  $T$  of  $H$ .

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Note that if  $(G, \Sigma) \rightarrow (G', \Sigma')$  then  $\chi(G, \Sigma) \leq \chi(G', \Sigma')$ .

# $K_4$ -minor free graphs

**Theorem (Naserasr, Rollová, ES, 2012+).**

For every  $K_4$ -minor free graph  $G$  and signature  $\Sigma$  on  $G$ ,  $(G, \Sigma) \rightarrow ST_5$ , so that  $\chi(G, \Sigma) \leq 5$ .

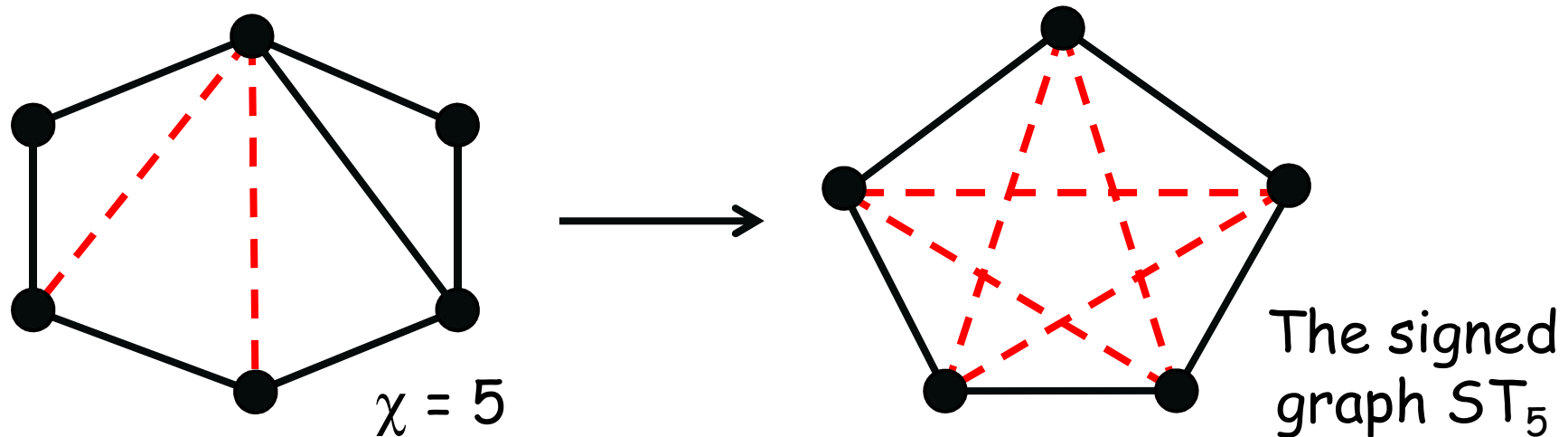
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Moreover, this bound is tight.



# Planar graphs

An **acyclic colouring** of a graph  $G$  is a proper colouring of  $G$  such that any two colours induce a *forest* (in other words, each *cycle* of  $G$  uses *at least 3 colours*).

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**Theorem (Ochem, Pinlou, Sen, 2012+).**

If  $G$  is acyclically  $k$ -colorable then, for every signature  $\Sigma$  on  $G$ ,  $\chi(G, \Sigma) \leq k \cdot 2^{k-2}$ .



# Planar graphs

An **acyclic colouring** of a graph  $G$  is a proper colouring of  $G$  such that any two colours induce a *forest* (in other words, each *cycle* of  $G$  uses *at least 3 colours*).

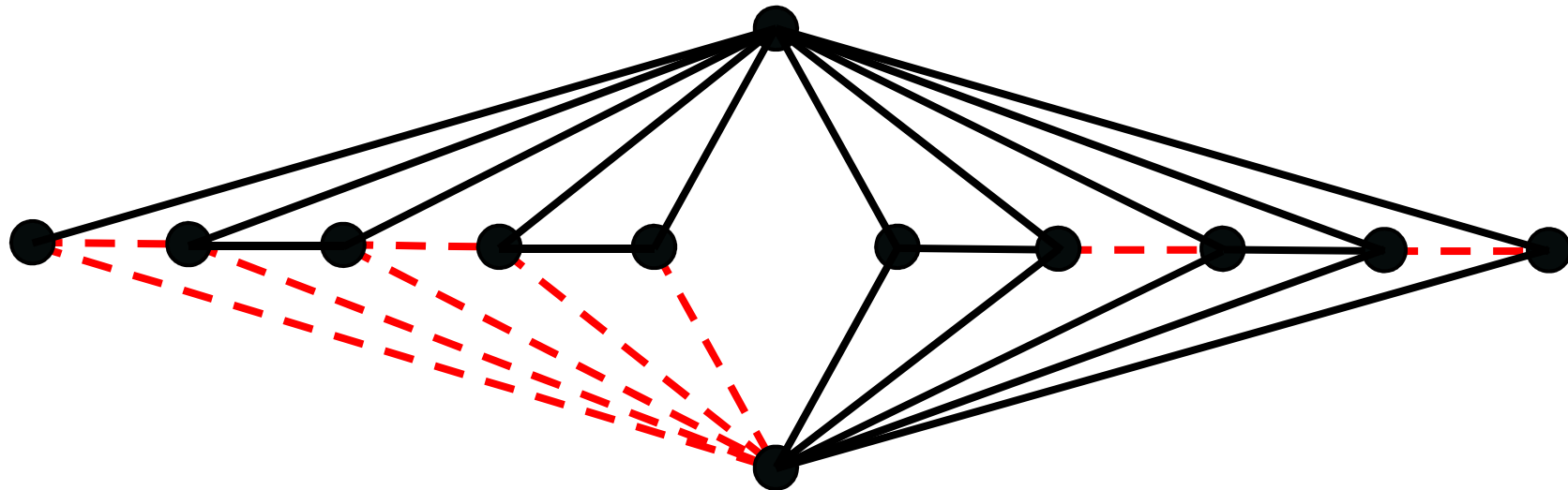
**Theorem (Ochem, Pinlou, Sen, 2012+).**

If  $G$  is acyclically  $k$ -colorable then, for every signature  $\Sigma$  on  $G$ ,  $\chi(G, \Sigma) \leq k \cdot 2^{k-2}$ .

**Corollary.** If  $G$  is planar, then  $\chi(G, \Sigma) \leq 40$ .

# Planar graphs

The following planar signed graph has 12 vertices and chromatic number 10:





# Complexity

**Problem:** K-CHROMATIC-SIGNED-GRAPH

**Input:** A signed graph  $(G, \Sigma)$ .

**Question:** Is  $\chi(G, \Sigma) \leq k$ ?

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**Proposition.** The problem k-CHROMATIC-SIGNED-GRAPH is polynomial for  $k = 1, 2$  and NP-complete for  $k \geq 3$ .

- $\chi(G, \Sigma) \leq 2$  iff  $G$  is bipartite and  $(G, \Sigma)$  contains no unbalanced cycle
- k-CHROMATIC-SIGNED-GRAPH contains k-CHROMATIC-GRAPH

# S-cliques

An **S-clique** is a signed graph  $(G, \Sigma)$  such that  
$$\chi(G, \Sigma) = |V(G)|.$$

# S-cliques

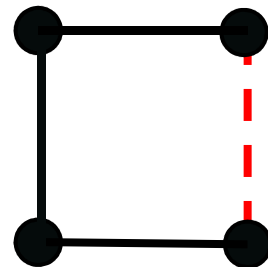
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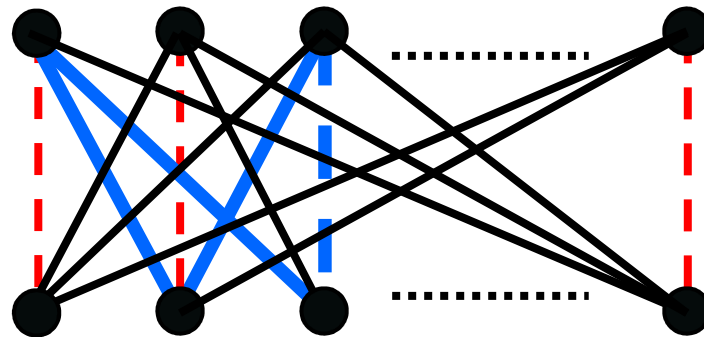
**Proposition.** A signed graph  $(G, \Sigma)$  is an S-clique if and only if, for any two vertices  $u$  and  $v$ , either  $uv$  is an edge or  $u$  and  $v$  belong to some unbalanced 4-cycle.



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Every  $(K_{n,n}, M)$ ,  $k \geq 3$ , is thus an S-clique...

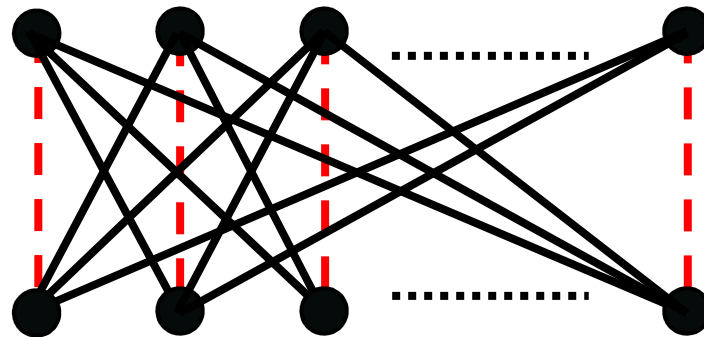


# S-cliques

This example shows that the difference between  $\chi(G)$  and  $\chi(G, \Sigma)$  can be arbitrarily large...

$$\chi(K_{n,n}) = 2$$

$$\chi(K_{n,n}, \mathcal{M}) = 2n$$





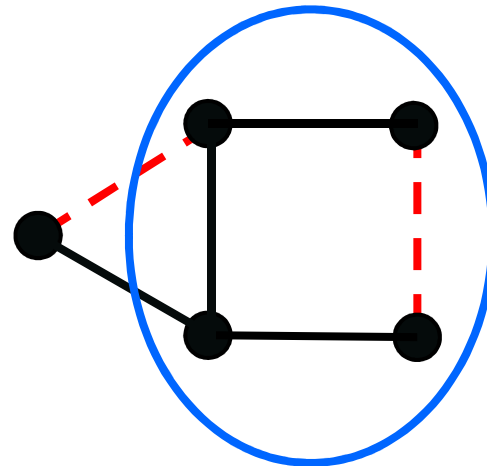
# S-clique numbers

The **absolute S-clique number**  $\chi_{as}(G, \Sigma)$  is the maximum order of an *S-clique subgraph* of  $(G, \Sigma)$ .

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$$\chi_{as}(G, \Sigma) = 4$$



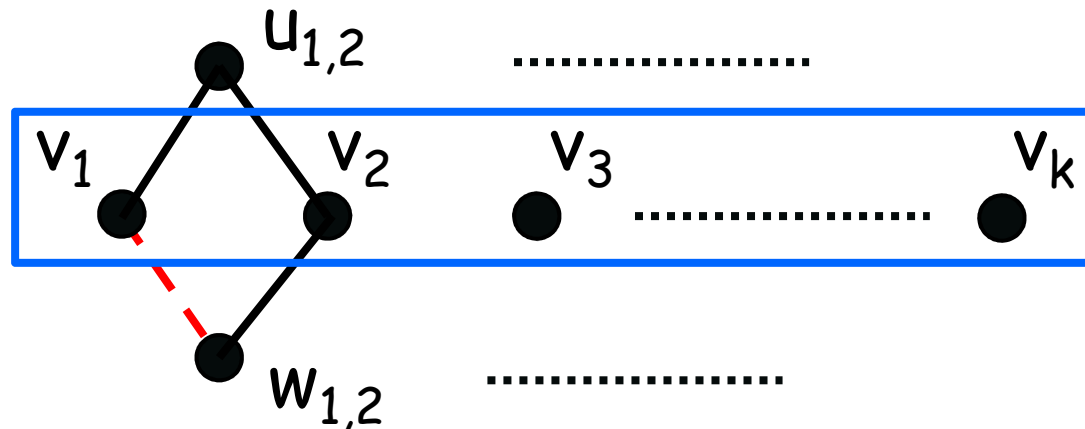
# S-clique numbers

The **relative S-clique number**  $\chi_{rs}(G, \Sigma)$  is the maximum size of a subset  $S$  of  $V(G)$  such that no two vertices in  $S$  can be *identified* in any homomorphic image of  $(G, \Sigma)$ .

# S-clique numbers

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$$\chi_{rs}(G_k, \Sigma) = k$$



*For every  $v_i, v_j$ , create a  $UC_4$  using new vertices  $u_{i,j}$  and  $w_{i,j}$ ...*

# S-clique numbers

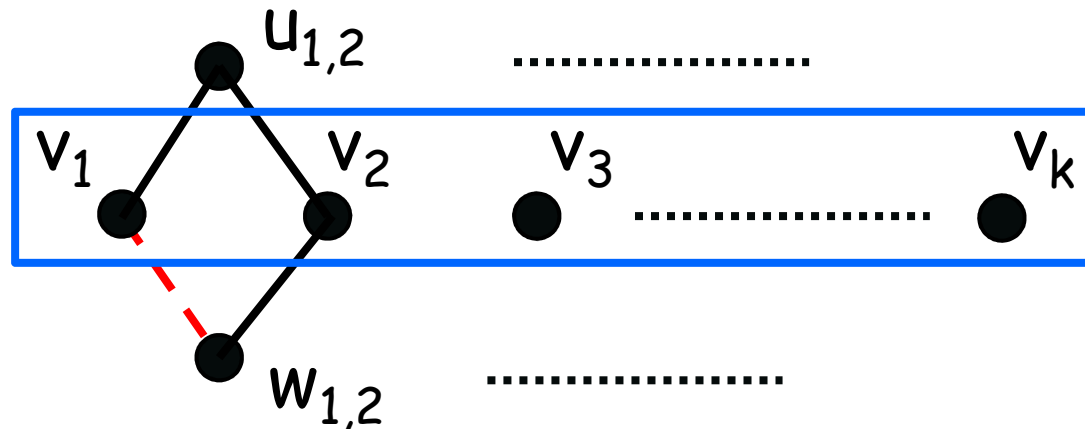
The difference

$$\chi_{rs}(G, \Sigma) - \chi_{as}(G, \Sigma)$$

can be arbitrarily large...

$$\chi_{as}(G_k, \Sigma) = 4$$

$$\chi_{rs}(G_k, \Sigma) = k$$



*For every  $v_i, v_j$ , create a  $UC_4$  using new vertices  $u_{i,j}$  and  $w_{i,j}$ ...*

# S-clique numbers

## Proposition.

For every signed graph  $(G, \Sigma)$ :

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# S-clique numbers

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Note that computing any of these S-clique numbers is NP-hard, since both these problems contain the problem of computing the clique number of a graph (simply set  $\Sigma = \emptyset$ )...

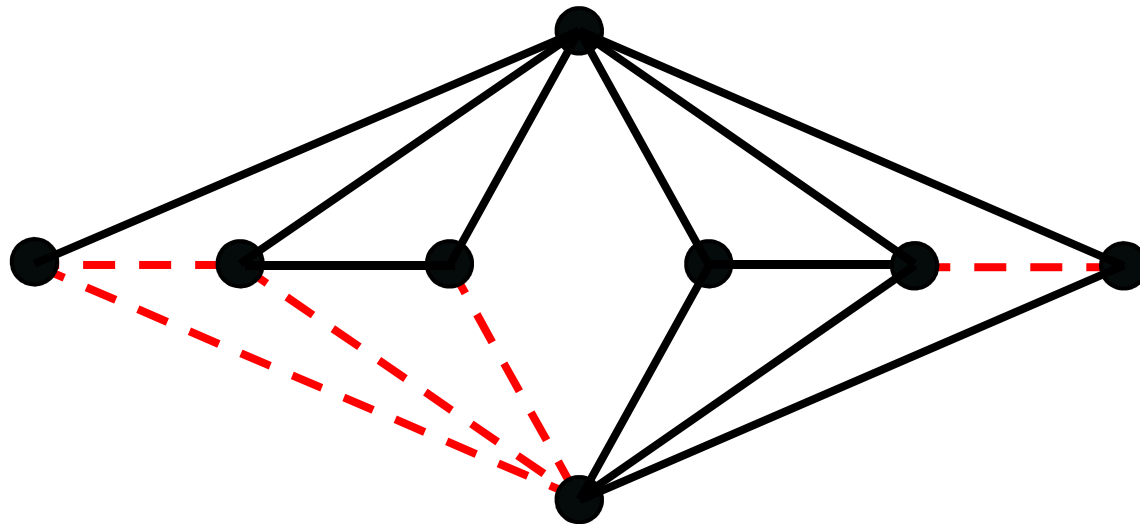
# Planar $S$ -cliques

**Theorem (Naserasr, Rollová, ES, 2012+).**  
The maximum order of a planar  $S$ -clique is 8.



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A planar  $S$ -clique of order 8

Open questions

# A few open questions...

1. Determine the maximum chromatic number of a planar signed graph (lies between 10 and 40).
2. Determine the optimal bound for  $\chi_{rs}(G, \Sigma)$  when  $G$  is planar (the optimal bound for  $\chi_{as}$  is 8).
3. Is it true that if  $G$  is bipartite and  $(G, \Sigma)$  has no  $(K_4, E(K_4))$  as a minor, then  $(G, \Sigma) \rightarrow (K_{3,3}, M)$ ?
4. Is it true that every consistent planar signed graph with unbalanced girth  $k$  admits a homomorphism to  $SPC_{k-1}$ ? (equivalent to Seymour's conjecture...)

Thank you for your  
attention...