

Rainbow Connection in Oriented Graphs

Paul Dorbec^{a,b}, Ingo Schiermeyer^c,
Elżbieta Sidorowicz^d and Éric Sopena^{a,b}

^a Univ. Bordeaux, LaBRI, UMR5800, F-33400 Talence

^b CNRS, LaBRI, UMR5800, F-33400 Talence

^c TU Bergakademie Freiberg, Institut für Diskrete Mathematik and Algebra, Freiberg, Germany

^d Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Zielona Góra, Poland

December 3, 2013

Abstract

An edge-coloured graph G is said to be *rainbow-connected* if any two vertices are connected by a path whose edges have different colours. The rainbow connection number of a graph is the minimum number of colours needed to make the graph rainbow-connected. This graph parameter was introduced by G. Chartrand, G.L. Johns, K.A. McKeon and P. Zhang in 2008. Since, the topic drew much attention, and various similar parameters were introduced, all dealing with undirected graphs.

Here, we initiate the study of rainbow connection in oriented graphs. An early statement is that the rainbow connection number of an oriented graph is lower bounded by its diameter and upper bounded by its order. We first characterize oriented graphs having rainbow connection number equal to their order. We then consider tournaments and prove that (i) the rainbow connection number of a tournament can take any value from 2 to its order minus one, and (ii) the rainbow connection number of every tournament with diameter d is at most $d + 2$.

1 Introduction

We consider finite and simple graphs only, and refer to [1] for terminology and notations not defined here.

In an edge-coloured graph G , a path is said to be *rainbow* if it does not use two edges with the same colour. Then the graph G is said to be *rainbow-connected* if any two vertices are connected by a rainbow path. This concept of rainbow connection in graphs was recently introduced by Chartrand et al. in [3]. An application of rainbow connection for the secure transfer of classified information between agencies in communication networks was presented in [4]. Along with it, rainbow paths are generally used in the concept of onion routing, using layered encryption [12]. For onion routing, one enciphers a message once by hop on the path, always with different keys (corresponding to the colors of the edges). This layered encryption is used e.g. by the anonymous networks TOR and I2P.

In the following, we are interested in the corresponding optimization parameter. The *rainbow connection number* of a connected graph G , denoted $rc(G)$, is the smallest number of colours that are needed in order to make G rainbow connected. The computational complexity of rainbow connectivity was studied in [2], where it is proved that the computation of $rc(G)$ is NP-hard. In fact it is already NP-complete to decide if $rc(G) = k$ for any fixed $k \geq 2$ or to decide whether a given edge-coloured (with any number of colours) graph is rainbow connected.

Additionally, Chartrand et al. computed the precise rainbow connection number of several graph classes including complete multipartite graphs [3]. The rainbow connection number was studied for further graph classes in [5, 6, 7, 9, 10, 14] and for graphs with fixed minimum degree in [5, 8, 13, 15]. Also, different other parameters similar to rainbow connection were introduced such as strong rainbow connection, rainbow k -connectivity, k -rainbow index and rainbow vertex connection. See [9] for a survey about these different parameters.

In this paper, we extend the problem of rainbow connection to oriented graphs. Whereas it was easily observed that a graph of order n has rainbow connection number at most $n - 1$ (giving different colours to all the edges in a spanning tree), the rainbow connection number of an oriented graph can be equal to its order. In this paper, we characterize oriented graphs with rainbow connection number equal to their order.

We proceed as follows. We start with useful definitions in Section 2. Then, in Section 3, we prove that the only minimally strong oriented graphs that have rainbow connection number exactly their order are cycles. In Section 4, we propose a characterization of all oriented graphs with rainbow connection number equal to their order. Finally, we prove in Section 5 that the rainbow connection number of a tournament can take almost any value in terms of its order, but is upper bounded by the tournament diameter plus 2.

2 Definitions, notation and basic results

2.1 Definitions and notation

For a given digraph G , we denote by $V(G)$ and $A(G)$ respectively its sets of vertices and of arcs. By an *oriented graph* we mean an antisymmetric digraph, that is where $yx \notin A(G)$ whenever $xy \in A(G)$. Given an arc xy in G , we say y is an *out-neighbour* of x while x is an *in-neighbour* of y . Moreover, we call x the *tail* of xy and y the *head* of xy .

By $N_G^+(x)$ (resp. $N_G^-(x)$) we denote the set of out-neighbours (resp. in-neighbours) of x in G . The *out-degree* (resp. the *in-degree*) of x is the order of its out-neighbourhood $d_G^+(x) = |N_G^+(x)|$ (resp. in-neighbourhood $d_G^-(x) = |N_G^-(x)|$), and the *degree* of x is simply $d_G(x) = d_G^+(x) + d_G^-(x)$.

For X a subset of $V(G)$, we denote by $G[X]$ the subgraph of G *induced by* X , given by $V(G[X]) = X$ and $A(G[X]) = A(G) \cap (X \times X)$. A *spanning subgraph* H of G is a subgraph of G with $V(H) = V(G)$.

A *path* of length $k \geq 1$ in an oriented graph G is a sequence $x_0 \dots x_k$ of vertices such that $x_i x_{i+1} \in A(G)$ for every i , $0 \leq i \leq k-1$. Such a path P , going from x_0 to x_k , is referred to as an $(x_0 - x_k)$ -*path*. Any vertex in $V(P) \setminus \{x_0, x_k\}$ is an *internal vertex* of P . If X and Y are two subsets of $V(G)$, an $(X - Y)$ -*path* is an $(x - y)$ -path linking a vertex $x \in X$ to a vertex $y \in Y$. A path P is *elementary* if no vertex appears twice in P . An elementary path *induced by* a path Q is any elementary path P obtained from Q by repeatedly deleting cycles, that is replacing a sequence of the form $u_1 \dots u_k x v_1 \dots v_\ell x w_1 \dots w_m$ by $u_1 \dots u_k x w_1 \dots w_m$ as many times as necessary. Given two paths $P_1 = x_1 \dots x_i$ and $P_2 = x_i \dots x_{i+j}$, we denote by $P_1 \cup P_2$ the path $x_1 \dots x_i \dots x_{i+j}$.

An *ear* in an oriented graph G is an $(x - y)$ -path Q such that $d_G(x) > 2$, $d_G(y) > 2$ and $d_G(z) = 2$ for every internal vertex z of Q .

The *distance* from a vertex x to a vertex y in an oriented graph G , denoted by $dist_G(x, y)$, is the length of a shortest $(x - y)$ -path in G (if there is no such path, we say $dist_G(x, y) = \infty$). The *diameter* of G , denoted by $diam(G)$, is the maximum distance between any pair of vertices in G . Two vertices at distance $diam(G)$ are *antipodal vertices*. The *eccentricity* of a vertex x in G , denoted by $ecc_G(x)$, is the maximum distance from x to any other vertex y of G .

Let G be an oriented graph. For an arc xy in $A(G)$, we denote by $G - xy$ the oriented graph defined by $G - xy = (V(G), A(G) \setminus \{xy\})$. For a vertex u in $V(G)$, we denote by $G - u$ the oriented graph $G - u = (V(G) \setminus \{u\}, (A(G) \setminus (\{u\} \times V(G))) \setminus (V(G) \times \{u\}))$. For G' an oriented graph, we denote by $G \cup G'$ the oriented graph given by $G \cup G' = (V(G) \cup V(G'), A(G) \cup A(G'))$.

An oriented graph G is *strongly connected* (*strong* for short) if there exists an $(x - y)$ -

path in G for every two vertices x and y . The graph G is *minimally strongly connected* (MSC for short) if G is strong and, for every arc xy in G , the graph $G - xy$ is not strong.

A *cycle* of length $k \geq 3$ in an oriented graph G is a sequence $x_0 \dots x_{k-1} x_0$ of vertices such that $x_i x_{i+1} \in A(G)$ for every i , $0 \leq i \leq k-2$, and $x_{k-1} x_0 \in A(G)$. For every i , $0 \leq i \leq k-1$, x_{i+1} (resp. x_{i-1}) is the *successor* (resp. *predecessor*) of x_i in C (subscripts are taken modulo k).

2.2 Rainbow connection of oriented graphs

Let G be an oriented graph. A *k-arc-colouring* of G , $k \geq 1$, is a mapping $\varphi : A(G) \rightarrow \{1, \dots, k\}$. Note that adjacent arcs may receive the same colour. An *arc-coloured* oriented graph is then a pair (G, φ) where G is an oriented graph and φ an arc-colouring of G . A path P in (G, φ) is *rainbow* if no two arcs of P are coloured with the same colour. An arc-coloured oriented graph (G, φ) is *rainbow connected* (or, equivalently, φ is a *rainbow arc-colouring* of G) if any two vertices in G are connected by a rainbow path. Note that in order to admit a rainbow arc-colouring, an oriented graph must be strong.

The *rainbow connection number* of an oriented graph G , denoted by $\vec{rc}(G)$, is defined as the smallest number k such that G admits a rainbow k -arc-colouring.

Note that in a rainbow connected graph, there must be a path with at least $\text{diam}(G)$ colours between antipodal vertices. We thus have the following proposition.

Proposition 1 *If G is a strong oriented graph with diameter d , then $\vec{rc}(G) \geq d$.*

Moreover, we have:

Proposition 2 *If G is a strong oriented graph on n vertices, then $\vec{rc}(G) \leq n$.*

Proof. Let $V(G) = \{x_1, \dots, x_n\}$. We define an n -arc-colouring φ of G by setting $\varphi(x_i x_j) = j$ for every arc $x_i x_j$ in $A(G)$. Obviously, every elementary path in G is rainbow and, therefore, φ is a rainbow arc-colouring of G . \square

Observe that this upper bound is tight since $\vec{rc}(C_n) = n$ for the cycle C_n on n vertices. Since every path in a subgraph of G is also a path in G , we have the following:

Proposition 3 *If H is a strong spanning subgraph of an oriented graph G , then $\vec{rc}(G) \leq \vec{rc}(H)$.*

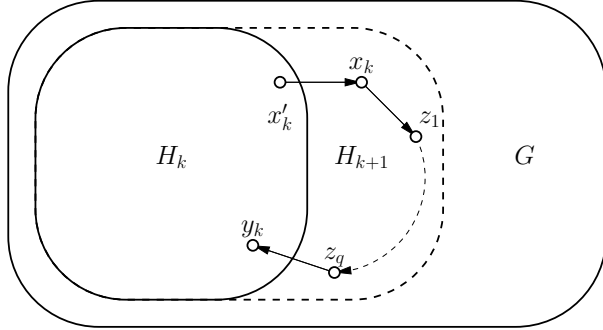


Figure 1: Illustration of proof of Lemma 4

3 Rainbow connection number of minimally strongly connected oriented graphs

In this section, we prove that the only MSC oriented graph on n vertices with rainbow connection number n is the cycle (Theorem 8). We first start with a few useful lemmas.

Lemma 4 *Let G be an MSC oriented graph. If G is not a cycle, then there is a strong subgraph H of G and a vertex $x \in V(G) \setminus V(H)$ such that $N_G^-(x) = \{y\}$ and $y \in V(H)$.*

Proof. Let G be an MSC oriented graph not reduced to a cycle. We construct a sequence of subgraphs $(H_i)_{i \geq 0}$ such that

- (i) for every $i \geq 0$, H_i is a strong subgraph of G ,
- (ii) for every $i \geq 1$, H_i contains H_{i-1} as a proper subgraph,
- (iii) for every $i \geq 0$, $V(G) \setminus V(H_i)$ is not empty.

until we find a subgraph H_k such that there exists a vertex $x_k \in V(G) \setminus V(H_k)$ with $N_G^-(x_k) = \{x'_k\}$ where $x'_k \in V(H_k)$, thus proving the lemma.

We first choose any elementary cycle C in G and set $H_0 = C$. Since G is not a cycle, we have $V(G) \setminus V(H_0) \neq \emptyset$. Properties (i) and (iii) are thus satisfied by H_0 .

Suppose now that we have a graph H_k , $k \geq 0$ satisfying properties (i) and (iii). If there exists $x_k \notin V(H_k)$ such that $N_G^-(x_k) = \{y\}$ and $y \in V(H_k)$ we are done. Otherwise, let x_k be any vertex having an in-neighbour x'_k in H_k . Since G is strong, there exists a path in G from x_k to H_k . Let $P_k = x_k z_1 \dots z_q y_k$, $q \geq 0$ be any such $(x_k - H_k)$ -shortest path (see Figure 1). We then set $H_{k+1} = H_k \cup x'_k x_k \cup P_k$. By construction, H_{k+1} is a

strong oriented graph containing H_k , and thus satisfies (i) and (ii). Moreover, since G is MSC, there is no arc zx_k with $z \in H_{k+1}$, $z \neq x'_k$ (otherwise, since there is a $(z - x_k)$ -path in G using only arcs from $H_k \cup x'_k x_k \cup P_k$, $G - zx_k$ would still be strong, contradicting the fact that G is MSC). Thus $d_{\overline{H_{k+1}}}(x_k) = 1$, and since $N_{\overline{G}}(x_k)$ is not a single vertex, $V(G) \setminus V(H_{k+1}) \neq \emptyset$, and (iii) is satisfied. We can then iterate the construction.

Since the graph G is finite, the process eventually ends up with a subgraph H_k for which there exists a vertex x_k with $N_{\overline{G}}(x_k) = \{x'_k\}$ and $x'_k \in V(H_k)$. This proves the lemma. \square

Since the subgraph H in the above lemma is strong, the vertex y necessary lies on a cycle of H . Therefore we have as a corollary:

Corollary 5 *Let G be an MSC oriented graph. If G is not a cycle, then there is a cycle C in G and a vertex $x \in V(G) \setminus V(C)$ such that $N_{\overline{G}}(x) = \{y\}$ with $y \in V(C)$.*

Lemma 6 *Let G be a strong oriented graph on n vertices, $x'x$ and $y'y$ be two arcs in G with $x \neq y$, $d_{\overline{G}}(x) = 1$ and $d_{\overline{G}}(y) = 1$. If G contains an $(x' - y)$ -path P_1 not containing $x'x$ and a $(y' - x)$ -path P_2 not containing $y'y$, then $\vec{rc}(G) \leq n - 1$.*

Proof. First note that by the degree condition, the paths P_1 and P_2 necessarily contain the arcs $y'y$ and $x'x$ respectively.

Let $c : V(G) \rightarrow \{1, 2, \dots, n - 1\}$ be a mapping such that $c(x) = c(y) = 1$, $c(v) \neq 1$ for $v \in V(G) \setminus \{x, y\}$, and $c(v) \neq c(w)$ for $v, w \in V(G) \setminus \{x, y\}$. Let now $\varphi : A(G) \rightarrow \{1, 2, \dots, n - 1\}$ be the arc-colouring given by $\varphi(uv) = c(v)$ for every arc uv . We claim that φ is a rainbow arc-colouring of G .

To see that, observe that the two arcs $x'x$ and $y'y$ are the only arcs coloured 1. Therefore, every elementary path containing at most one of these two arcs is rainbow. Assume that there exist two vertices u and v that are not linked by a rainbow path and let P be any elementary $(u - v)$ -path. Without loss of generality, assume that the arc $x'x$ appears before the arc $y'y$ in P . Let Q be the $(u - v)$ -path obtained from P by replacing the subpath from x' to y by the path P_1 . Any elementary $(u - v)$ -path induced by Q is clearly a rainbow path, a contradiction. The arc-colouring φ is thus a rainbow colouring of G . \square

By taking $x' = y'$ in Lemma 6 we get the following corollary:

Corollary 7 *Let G be a strong oriented graph on n vertices. If there is a vertex z in G having two distinct out-neighbours x and y with $d_{\overline{G}}(x) = 1$ and $d_{\overline{G}}(y) = 1$, then $\vec{rc}(G) \leq n - 1$.*

We now are ready to prove the main result of this section.

Theorem 8 *Let G be an MSC oriented graph on n vertices. If G is not a cycle then $\vec{rc}(G) \leq n - 1$.*

Proof. Let G be an MSC oriented graph not reduced to a cycle. By Corollary 5 there is a cycle C in G and a vertex $x \in V(G) \setminus C$ such that $N_G^-(x) = \{y\}$ and $y \in V(C)$. Let z be the successor of y in C .

We now construct a sequence of pairs $\{(G_i, Q_i)\}_{i \geq 1}$ such that

- (i) G_i is a strong subgraph of G for every $i \geq 1$,
- (ii) Q_i is an ear in G_i for every $i \geq 1$,
- (iii) G_{i-1} is a proper subgraph of G_i for every $i \geq 2$ (more precisely, $G_i = G_{i-1} \cup Q_i$),
- (iv) denoting by $y_i z_i$ the first arc of Q_i , there exist a $(y - z_i)$ -path in G_i not containing the arc yx and a $(y_i - x)$ -path in G_i not containing the arc $y_i z_i$ for every $i \geq 1$,

until we reach a pair with $d_G^-(z_i) = 1$. Then the arcs yx and $y_i z_i$ satisfy the assumptions of Lemma 6, which allows us to conclude.

We first construct the pair (G_1, Q_1) . Let P be a shortest $(x - C)$ -path and $t \in V(C)$ its last vertex. We then set $G_1 = C \cup yx \cup P$ and Q_1 as the $(y - t)$ -path contained in C (thus, $y_1 = y$ and $z_1 = z$). The subgraph G_1 is clearly strong and Q_1 is an ear in G_1 , giving (i) and (ii). Moreover, the two arcs $yz = yz_1$ and $yx = y_1x$ give the two required paths of (iv).

Assume now that a pair (G_i, Q_i) satisfying (i) to (iv) is constructed. If $d_G^-(z_i) = 1$ then the two arcs yx and $y_i z_i$ satisfy the assumptions of Lemma 6 and we are done. Otherwise, we construct the next pair (G_{i+1}, Q_{i+1}) as follows. Since $d_G^-(z_i) \geq 2$, there exists an in-neighbour v_i of z_i distinct from y_i . Since G is MSC and the subgraph G_i is strong, we have $v_i \notin V(G_i)$. Let $P_{i+1} = y_{i+1} z_{i+1} \dots v_i$ be a shortest $(G_i - v_i)$ -path in G (it may happen that $y_{i+1} = z_i$), and set $Q_{i+1} = P_{i+1} \cup v_i z_i$. Since G is MSC and the subgraph G_i is strong, we have $y_{i+1} \in V(Q_i)$ (otherwise, the arc $y_i z_i$ could be deleted from G since there would be a $(y_i - z_i)$ -path in G going through G_{i-1} and Q_{i+1}). We then set $G_{i+1} = G_i \cup Q_{i+1}$. The subgraph G_{i+1} is clearly strong and items (i) to (iii) are satisfied. Since G_i is strong, there is a $(y_{i+1} - x)$ -path in G_i , and thus in G_{i+1} , not containing the arc $y_{i+1} z_{i+1}$ ($z_{i+1} \notin V(G_i)$). Since there is a $(y - y_i)$ -path in G_i not containing yx as well as a $(y_i - y_{i+1})$ -path along Q_i , it follows that there is a $(y - z_{i+1})$ -path not containing the arc yx . Hence, item (iv) is also satisfied, and the pair (G_{i+1}, Q_{i+1}) is well constructed.

Since the graph G is finite, we eventually construct a pair (G_k, Q_k) such that $d_G^-(z_k) = 1$, and then the two arcs yx and $y_k z_k$ satisfy the assumptions of Lemma 6, implying that $\vec{rc}(G) \leq n - 1$. \square

Using Proposition 3, we then get the following

Corollary 9 *Let G be an oriented graph on n vertices. If G has a strong spanning subgraph that is not Hamiltonian, then $\vec{rc}(G) \leq n - 1$.*

Proof. Let G' be a strong spanning subgraph of G that is not Hamiltonian and G'' an MSC spanning subgraph of G' (G'' is thus an MSC spanning subgraph of G that is not Hamiltonian). By Theorem 8, we have $\vec{rc}(G'') \leq n - 1$. By Proposition 3 we then get $\vec{rc}(G) \leq \vec{rc}(G'') \leq n - 1$. \square

Since every spanning subgraph of a non-Hamiltonian graph is non-Hamiltonian, we get:

Corollary 10 *Let G be a strong oriented graph on n vertices. If G is not Hamiltonian, then $\vec{rc}(G) \leq n - 1$.*

4 Oriented graphs with maximum rainbow connection number

In Theorem 8, we proved that MSC graphs with rainbow connection number exactly their order are cycles. As a consequence, every graph (not necessarily MSC) having rainbow connection number n must be Hamiltonian. This section is dedicated to better characterizing these graphs, the main result being Theorem 16. We first introduce some definitions and notation.

Let $C = x_0 \dots x_{k-1} x_0$ be a cycle in an oriented graph G . Since cycles are denoted similarly all along the following, it is taken for granted that when we use notations x_i , subscript are taken modulo k . By $C[x_i, x_j]$ we denote the set of vertices of the $(x_i - x_j)$ -path contained in C , that is $C[x_i, x_j] = \{x_i, x_{i+1}, \dots, x_j\}$. We also denote $C(x_i, x_j) = C[x_i, x_j] \setminus \{x_i\}$, $C(x_i, x_j) = C[x_i, x_j] \setminus \{x_j\}$, $C(x_i, x_j) = C[x_i, x_j] \setminus \{x_i, x_j\}$ (hence $C(x_i, x_j) = V(C) \setminus C(x_j, x_i)$). An arc $x_i x_j$, $0 \leq i, j \leq k - 1$ is a *chord* of C whenever $x_j \neq x_{i+1}$. Note that we necessarily have $x_j \neq x_{i-1}$ since G is antisymmetric.

Definition 11 (Path-property) Let $C = x_0 \dots x_{k-1} x_0$ be a cycle in an oriented graph G . A pair of (not necessarily distinct) arcs $\{x_i x_{i+1}, x_j x_{j+1}\}$ has the *Path-property* if there exist an $(x_i - x_{j+1})$ -path in G not containing $x_i x_{i+1}$ and an $(x_j - x_{i+1})$ -path in G not containing $x_j x_{j+1}$.

Definition 12 (Special Hamiltonian cycle) Let G be a Hamiltonian oriented graph and $C = x_0 \dots x_{n-1} x_0$ be a Hamiltonian cycle in G . We say the cycle C is *special* if it contains a pair of (not necessarily distinct) arcs $\{x_i x_{i+1}, x_j x_{j+1}\}$ having the Path-property.

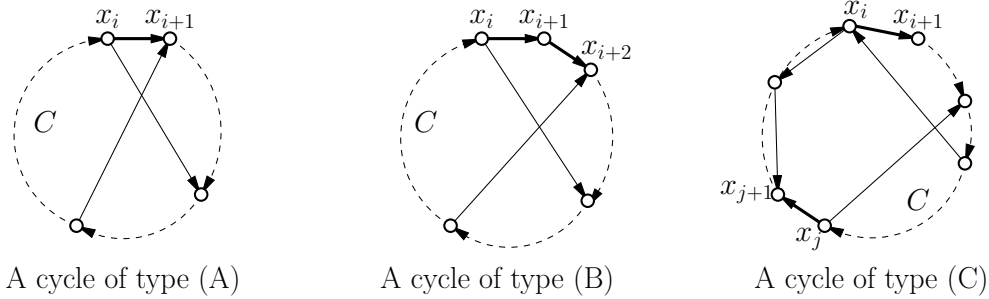


Figure 2: The three types of special Hamiltonian cycles

A special Hamiltonian cycle C thus satisfies one of the following properties (see Figure 2):

- (A) there are two consecutive vertices x_i and x_{i+1} in C joined by a path not containing the arc $x_i x_{i+1}$, (case $i = j$),
- (B) there are three consecutive vertices x_i , x_{i+1} and x_{i+2} in C such that there exists an $(x_i - x_{i+2})$ -path in G containing neither $x_i x_{i+1}$ nor $x_{i+1} x_{i+2}$, (case $j = i + 1$),
- (C) there are two disjoint arcs $x_i x_{i+1}$ and $x_j x_{j+1}$ in C such that there exist an $(x_i - x_{j+1})$ -path in G not containing $x_i x_{i+1}$ and an $(x_j - x_{i+1})$ -path in G not containing $x_j x_{j+1}$.

We hereafter refer to such special Hamiltonian cycles as Hamiltonian cycles of type (A), (B) or (C), respectively.

Definition 13 (Head-Tail-property) Let $C = x_0 \dots x_{k-1} x_0$ be a cycle in an oriented graph G . A vertex $x_i \in V(C)$ is said to satisfy the *Head-Tail-property with respect to C* if, when going along the cycle C from x_i to x_{i-1} , we meet the head of each chord before its tail (see Figure 3 for an example). In particular, if G is itself a cycle, then every vertex of G has the Head-Tail-property with respect to G .

Definition 14 (Strong pair) Let $C = x_0 \dots x_{k-1} x_0$ be a cycle in an oriented graph G . A pair of distinct vertices $\{x_i, x_j\}$ in C is a *strong pair of C in G* if both the induced subgraphs $G[C[x_i, x_{j-1}]]$ and $G[C[x_j, x_{i-1}]]$ are strong.

As a direct consequence of the definition, we get the following observation:

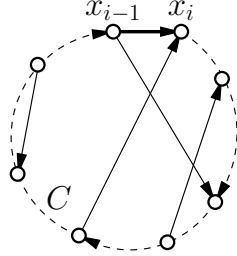


Figure 3: The vertex x_i has the Head-Tail-property with respect to the cycle C

Observation 15 *Let $C = x_0 \dots x_{k-1}x_0$ be a cycle in an oriented graph G . If $\{x_i, x_j\}$ is a strong pair of C in G , then the pair $\{x_{i-1}x_i, x_{j-1}x_j\}$ has the Path-property.*

The remaining of this section is dedicated to the proof of its main theorem, stated below.

Theorem 16 *Let G be a strong oriented graph on n vertices. The following statements are equivalent:*

- (1) $\vec{rc}(G) = n$,
- (2) G is Hamiltonian but has no special Hamiltonian cycle,
- (3) G has a Hamiltonian cycle $C = x_0x_1 \dots x_{n-1}$ and two distinct vertices x_i and x_j having the Head-Tail-property with respect to C but not forming a strong pair of C in G .

The proof of this theorem relies on the following series of lemmas. We first prove that (1) \Rightarrow (2), by showing that the rainbow connection number of a graph is no more than $n - 1$ whenever it has a cycle of type (A) (Lemma 17), of type (B) or of type (C) (Lemma 18). Then, we show with Lemma 19 that (2) \Rightarrow (3), and finally that (3) \Rightarrow (1) with Lemma 20.

Lemma 17 *Let G be a Hamiltonian oriented graph on n vertices. If there exists a special Hamiltonian cycle C of type (A) in G , then $\vec{rc}(G) \leq n - 1$.*

Proof. Let C be a Hamiltonian cycle of type (A) in G , with say x_{n-1} and x_0 the two consecutive vertices of C that are joined by a path not containing the arc $x_{n-1}x_0$. Choose such an $(x_{n-1} - x_0)$ -path P with the smallest number of chords. Let H be the subgraph

of G formed by $C[x_0, x_{n-1}] \cup P$. Clearly, H is a strong spanning subgraph of G , and by Proposition 3, $\vec{rc}(G) \leq \vec{rc}(H)$.

Let us prove that H is MSC. By our assumption on P , for any chord $x_i x_j$ in H , $H - x_i x_j$ contains no path from x_{n-1} to x_0 and thus is not strong. Let $x_i x_j$ be a chord in P . If $i \leq j$, the arc $x_{n-1} x_0$ does not lay in $C[x_i, x_j]$ and this chord can be replaced in P by the path $C[x_i, x_j]$ to obtain an $(x_{n-1} - x_0)$ -path with fewer chords, contradicting our assumption on P . Thus, for each chord $x_i x_j$ in P , $j < i$. As a consequence, we get that $C[x_0, x_{n-1}]$ is the only elementary path from x_0 to x_{n-1} . Therefore, no subgraph $H - x_i x_{i+1}$ is strong, and H is MSC. Since H is not a cycle, Theorem 8 gives $\vec{rc}(H) \leq n - 1$ and the result follows. \square

Lemma 18 *Let G be a Hamiltonian oriented graph on n vertices. If C is a special Hamiltonian cycle of type (B) or (C) in G , then $\vec{rc}(G) \leq n - 1$.*

Proof. Let C be a Hamiltonian cycle of type (B) or (C) in G , with say $x_{n-1} x_0$ and $x_j x_{j+1}$ ($j \neq n - 1$) the two arcs in C such that there exist an $(x_{n-1} - x_{j+1})$ -path P_1 not containing $x_{n-1} x_0$ and an $(x_j - x_0)$ -path P_2 not containing $x_j x_{j+1}$ (possibly, $j = 0$). If P_1 contains the vertex x_0 (yet not the arc $x_{n-1} x_0$) or P_2 contains the vertex x_{j+1} (yet not the arc $x_j x_{j+1}$), then the cycle C is also of type (A) and the result follows from Lemma 17.

Otherwise, consider the arc colouring $\varphi : A(G) \rightarrow \{1, \dots, n-1\}$ defined by $\varphi(x_k x_\ell) = \ell$ for all $1 \leq \ell \leq n - 1$ and $\varphi(x_k x_0) = j + 1$ for all $0 \leq k \leq n - 1$. We claim that φ is a rainbow arc-colouring of G . Consider any two vertices x_k and x_ℓ in G , we now prove that they are rainbow connected. Note that an elementary path is rainbow if and only if it does not contain both vertices x_0 and x_{j+1} as internal or last vertices. Thus, if $k \leq \ell$, then the subpath of the cycle $C[x_k, x_\ell]$ may contain vertex x_0 only as initial vertex and is rainbow, we are fine. Suppose now that $k > \ell$. If $k \geq j + 1 > \ell$, then $C[x_k, x_\ell]$ may contain x_{j+1} only as initial vertex and is also rainbow. Now, if $k > \ell \geq j + 1$, then the path obtained by starting on $C[x_k, x_{n-1}]$, then following P_1 , and finally continuing along C on $C[x_{j+1}, x_\ell]$ does not contain x_0 and is rainbow. Finally, if $j + 1 > k > \ell$ then the path obtained by starting on $C[x_k, x_j]$, then following P_2 , and finally continuing along C on $C[x_0, x_\ell]$ does not contain x_{j+1} and is rainbow. \square

We now prove that (2) \Rightarrow (3) with the following lemma.

Lemma 19 *Let G be a Hamiltonian oriented graph on n vertices having no special Hamiltonian cycle. There exist a Hamiltonian cycle $C = \{x_0, x_1, \dots, x_{n-1}\}$ in G and two vertices x_i and x_j such that x_i and x_j have the Head-Tail-property with respect to C , and $\{x_i, x_j\}$ is not a strong pair of C in G .*

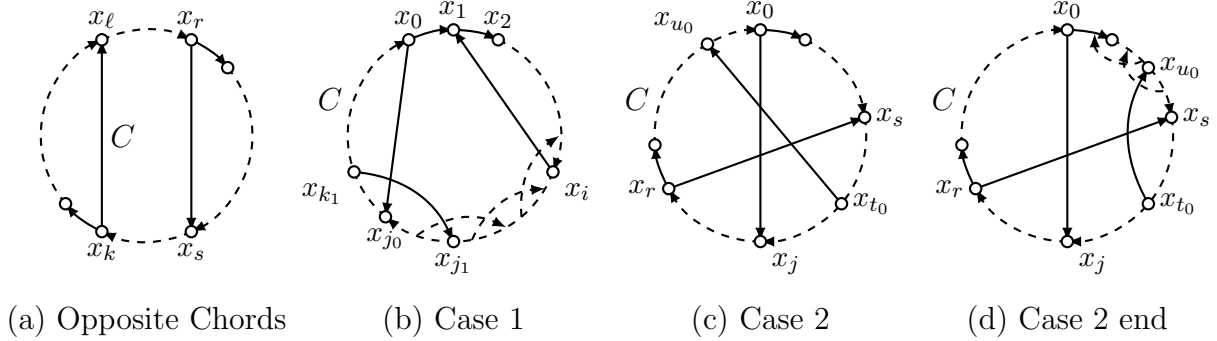


Figure 4: Configurations and cases for Lemma 19

Proof. Let G be a Hamiltonian graph with no special cycle. Let $C = x_0 x_1 \dots x_{n-1} x_0$ be a Hamiltonian cycle of G . First remark that if C has a strong pair x_i, x_j , then from Observation 15, $\{x_{i-1} x_i, x_{j-1} x_j\}$ has the Path-property and therefore, C is a special cycle. Thus, there is no strong pair of C in G .

Suppose now that the cycle C has two opposite chords, that is two chords $x_k x_\ell$ and $x_r x_s$, such that $x_s, x_r \in C[x_\ell, x_k]$ and $x_k, x_\ell \in C[x_s, x_r]$ (see Figure 4(a)). Then, the pair $\{x_k x_{k+1}, x_r x_{r+1}\}$ has the Path-property. Indeed, $x_r x_s \cup C[x_s, x_{k+1}]$ is an $(x_r - x_{k+1})$ -path not using the arc $x_r x_{r+1}$ and $x_k x_\ell \cup C[x_\ell, x_{r+1}]$ is an $(x_k - x_{r+1})$ -path not using the arc $x_k x_{k+1}$. Thus, the cycle C is special of type (C), contradicting our initial assumption.

Assume now that the cycle C has no two opposite chords. We consider two cases, depending on the number of vertices in G with the Head-Tail-property with respect to C .

Case 1. *The cycle C has exactly one vertex satisfying the Head-Tail-property.*

Say x_1 is the only vertex with the Head-Tail-property with respect to C in G . We now look for an $(x_0 - x_1)$ -path that does not use the arc $x_0 x_1$ to conclude that the cycle is special of type A, a contradiction.

The vertex x_2 does not satisfy the Head-Tail-property, thus going along the cycle from x_2 , we meet a chord by its tail x_i before meeting its head. Yet, from x_1 , we met its head before its tail. Thus, the chord's head has to be the vertex x_1 and the chord is $x_i x_1$ (see Figure 4(b)). Similarly, x_0 does not satisfy the Head-Tail-property, and we have a chord $x_0 x_{j_0}$ in the graph. If $j_0 \leq i$, the $(x_0 - x_1)$ -path $x_0 x_{j_0} \cup C[x_{j_0}, x_i] \cup x_i x_1$ does not go through $x_0 x_1$, as required. Assume then that $i < j_0$. Now the vertex x_{j_0} does not have the Head-Tail-property either. Thus there exists a chord $x_{k_1} x_{j_1}$ such that $k_1 \geq j_0$ and $1 \leq j_1 < j_0$, as x_1 does have the Head-Tail-property. If $j_1 \leq i$, then we find an

$(x_0 - x_1)$ -path $x_0x_{j_0} \cup C[x_{j_0}, x_{k_1}] \cup x_{k_1}x_{j_1} \cup C[x_{j_1}, x_i] \cup x_ix_1$ that does not use the arc x_0x_1 and the cycle is then special of type A. Otherwise, $i < j_1 < j_0$ and we can iterate with the statement that x_{j_1} does not satisfy the Head-Tail-property. Since there is a finite number of vertices in $C(x_i, x_{j_1})$, the process eventually ends up with an $(x_0 - x_1)$ -path of the form $x_0x_{j_0} \cup \bigcup_{1 \leq \alpha \leq t} (C[x_{j_{\alpha-1}}, x_{k_\alpha}] \cup x_{k_\alpha}x_{j_\alpha}) \cup C[x_{j_t}, x_i] \cup x_ix_1$.

Case 2. *The cycle C has no vertices satisfying the Head-Tail-property.*

Consider a chord x_ix_j such that $|C(x_i, x_j)|$ is minimum. Without loss of generality, say x_i is x_0 . Since x_j does not have the Head-Tail-property, there exists a chord x_kx_ℓ with $x_\ell \in C(x_k, x_j)$. If $x_k \in C(x_0, x_j)$, then so does x_ℓ , and $|C(x_k, x_\ell)| \leq |C(x_i, x_j)|$, contradicting our assumption. Thus we have $j \leq k \leq n - 1$. If $x_\ell \in C(x_k, x_0)$, then x_0x_j and x_kx_ℓ are opposite chords, contradicting our initial assumption. So $0 < \ell < j$, and any chord x_kx_ℓ preventing x_j from having the Head-Tail-property is a chord with $0 < \ell < j \leq k \leq n - 1$. Let x_rx_s be such a chord with s being as small as possible.

The vertex x_s does not have the Head-Tail-property, so there is another chord $x_{t_0}x_{u_0}$ with $x_{t_0} \in C[x_s, x_{u_0}]$. If x_{u_0} is in $C(x_s, x_r)$, then so does x_{t_0} , thus $x_{t_0}x_{u_0}$ and x_rx_s are opposite chords, a contradiction. So x_{u_0} is in $C(x_r, x_s)$. Suppose that x_{u_0} is in $C(x_r, x_0)$. If $x_{t_0} \in C[x_j, x_{u_0}]$, then the arcs x_0x_j and $x_{t_0}x_{u_0}$ are opposite chords, a contradiction. So $x_{t_0} \in C(x_s, x_j)$ (see Figure 4(c)), but then x_0x_1 and x_rx_{r+1} have the Path-property. Indeed, $x_0x_j \cup C[x_j, x_{r+1}]$ is an $(x_0 - x_{r+1})$ -path not containing x_0x_1 , and $x_rx_s \cup C[x_s, x_{t_0}] \cup x_{t_0}x_{u_0} \cup C[x_{u_0}x_1]$ is an $(x_r - x_1)$ -path not containing x_rx_{r+1} . Thus C is a special cycle (of type C), a contradiction with our first assumption.

Thus, x_{u_0} is in $C[x_1, x_s]$. From the minimality of s , if x_{u_0} is in $C[x_1, x_s]$, then $x_{t_0}x_{u_0}$ does not prevent x_j from having the Head-Tail-property, so $x_{t_0} \in C[x_s, x_j]$. Now the vertex x_{u_0} does not have the Head-Tail-property either, so there is another chord $x_{t_1}x_{u_1}$ with $x_{t_1} \in C[x_{u_0}, x_{u_1}]$. From the minimality of $|C(x_0, x_j)|$, we have that $x_{u_1} \notin C(x_{u_0}, x_j)$. This implies that $x_{u_1} \in C(x_j, x_0)$ or $x_{u_1} \in C(x_0, x_{u_0})$. In the first case we claim that C is a special cycle. If $x_{u_1} \in C(x_j, x_r)$, then x_rx_{r+1} and $x_{t_1}x_{t_1+1}$ have the Path-property, since $x_{t_1}x_{u_1} \cup C[x_{u_1}, x_{r+1}]$ is an $(x_{t_1} - x_{r+1})$ -path not containing $x_{t_1}x_{t_1+1}$, and $x_rx_s \cup C[x_s, x_{t_0}] \cup x_{t_0}x_{u_0} \cup C[x_{u_0}x_{t_1+1}]$ is an $(x_r - x_{t_1+1})$ path not containing x_rx_{r+1} . If $x_{u_1} \in C(x_r, x_0)$, then x_0x_1 and x_rx_{r+1} have the Path-property, since $x_0x_j \cup C[x_j, x_{r+1}]$ is an $(x_0 - x_{r+1})$ -path not containing x_0x_1 , and $x_rx_s \cup C[x_s, x_{t_0}] \cup x_{t_0}x_{u_0} \cup C[x_{u_0}x_{t_1}] \cup x_{t_1}x_{u_1} \cup C[x_{u_1}, x_1]$ is an $(x_r - x_1)$ -path not containing x_rx_{r+1} . Thus x_{u_1} is in $C(x_0, x_{u_0})$, i.e., $0 < u_1 < u_0$ and we can iterate with the statement that x_{u_1} does not satisfy the Head-Tail-property. Since there is a finite number of vertices in $C(x_1, x_{u_0})$, the process eventually ends up with one of the following cases:

1. The arcs x_0x_1 and x_rx_{r+1} have the Path-property, since $x_rx_s \cup C[x_s, x_{t_0}] \cup x_{t_0}x_{u_0} \cup$

$\bigcup_{1 \leq \alpha \leq p} (C[x_{u_{\alpha-1}}x_{t_\alpha}] \cup x_{t_\alpha}x_{u_\alpha}) \cup C[x_{u_p}, x_1]$ is an $(x_r - x_1)$ -path not containing $x_r x_{r+1}$, and $x_0 x_j \cup C[x_j, x_{r+1}]$ is an $(x_0 - x_{r+1})$ -path not containing $x_0 x_1$.

2. The arcs $x_{t_p}x_{t_{p+1}}$ and $x_r x_{r+1}$ have the Path-property, since $x_r x_s \cup C[x_s, x_{t_0}] \cup x_{t_0} x_{u_0} \cup \bigcup_{1 \leq \alpha \leq p-1} (C[x_{u_{\alpha-1}}x_{t_\alpha}] \cup x_{t_\alpha}x_{u_\alpha}) \cup C[x_{u_{p-1}}, x_{t_{p+1}}]$ is an $(x_r - x_{t_{p+1}})$ -path not containing $x_r x_{r+1}$, and $x_{t_p}x_{u_p} \cup C[x_{u_p}, x_{r+1}]$ is an $(x_{t_p} - x_{r+1})$ -path not containing $x_{t_p}x_{t_{p+1}}$.

Therefore, the cycle C has at least two vertices satisfying the Head-Tail-property, and this concludes the proof. \square

It remains to prove that (3) \Rightarrow (1), what we do with the following lemma.

Lemma 20 *Let G be an oriented graph on n vertices with a Hamiltonian cycle $C = \{x_0, x_1, \dots, x_{n-1}\}$. If G contains two distinct vertices x_i and x_j having the Head-Tail-property with respect to C such that $\{x_i, x_j\}$ is not a strong pair of C in G , then $\vec{rc}(G) = n$.*

Proof. Let G, C, x_i and x_j satisfy the lemma conditions. Assume by way of contradiction that $\vec{rc}(G) \leq n - 1$ and let φ be a rainbow $(n - 1)$ -arc-colouring of G . Since x_i has the Head-Tail-property with respect to C , there is exactly one elementary $(x_i - x_{i-1})$ -path in G , and this path goes along the cycle C . Thus all the arcs of $C - x_{i-1}x_i$ must be coloured with distinct colours so that $\vec{rc}(G) \geq n - 1$. Similarly, all the arcs of $C - x_{j-1}x_j$ must be coloured with distinct colours and, therefore, we must have $\varphi(x_{j-1}x_j) = \varphi(x_{i-1}x_i)$.

Moreover, since x_i satisfies the Head-Tail-property, there are no chords of C having their tail in $C[x_i, x_{j-1}]$ and their head in $C[x_j, x_{i-1}]$. Similarly, there are no chords of C having their tail in $C[x_j, x_{i-1}]$ and their head in $C[x_i, x_{j-1}]$. Thus, the only arcs joining a vertex in $C[x_j, x_{i-1}]$ and a vertex in $C[x_i, x_{j-1}]$ are $x_{i-1}x_i$ and $x_{j-1}x_j$. However, there exists a rainbow $(x_{j-1} - x_i)$ -path in G , that necessarily avoids one of the arcs $x_{i-1}x_i$ and $x_{j-1}x_j$. So that rainbow path may not contain any vertex in $C[x_j, x_{i-1}]$. Therefore, $G[C[x_i, x_{j-1}]]$ contains an $(x_{j-1} - x_i)$ -path as well as the path $C[x_i, x_{j-1}]$ which is Hamiltonian, so $G[C[x_i, x_{j-1}]]$ is strong. We get similarly that $G[C[x_j, x_{i-1}]]$ is strong, and thus that $\{x_i, x_j\}$ is a strong pair, contradicting our initial assumption. \square

This concludes the proof of Theorem 16. Note that even though this result gives a nice characterization of the oriented graphs having rainbow connection number their order, no obvious polynomial time algorithm for the associate decision problem can be deduced from it.

5 The rainbow connection number of tournaments

In this section, we consider the case of (strong) tournaments. We first show that the rainbow connection number of tournaments on $n \geq 4$ vertices is between 2 and $n - 1$,

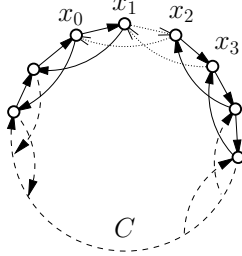


Figure 5: a strong spanning subgraph of T in Theorem 21

then that it can achieve any value from 3 to $n - 1$, and propose some construction for tournaments with rainbow connection number 2. We finally propose a better bound considering the diameter of T . Let us start by the general bounds for strong tournaments.

Theorem 21 *If T is a strong tournament with $n \geq 5$ vertices, then $2 \leq \vec{rc}(T) \leq n - 1$.*

Proof. Let T be a strong tournament on $n \geq 5$ vertices. The diameter of T is at least two, implying the lower bound. Recall that by Moon's Theorem [11], every strong tournament is Hamiltonian. Let $C = x_0 \dots x_{n-1}x_0$ be any Hamiltonian cycle in T . If C contains three consecutive vertices x_i, x_{i+1} and x_{i+2} such that $x_i x_{i+2} \in A(T)$ then C is a special Hamiltonian cycle of type (B) and the result follows from Theorem 16.

Otherwise, T contains all the arcs $x_{i+2}x_i$ (that are distinct since $n \geq 5$). Consider the spanning subgraph T' of T given by $T' = C \cup \{x_{i+2}x_i, 0 \leq i \leq n - 1\} \setminus \{x_1x_2, x_3x_1, x_2x_0\}$ (see Figure 5). The subgraph T' is clearly strong but not Hamiltonian so the upper bound follows from Corollary 9. \square

We now construct tournaments on n vertices with rainbow connection number k for every n and k such that $3 \leq k \leq n - 1$.

Given two oriented graphs G and H , $u \in V(G)$, we call *graph obtained from G by expanding u into H* the graph G' obtained from G by replacing the vertex u by a copy of H , and replacing the arc xu (resp. ux) in G by all the arcs xv (resp. vx) for $v \in V(H)$. Note that the graph obtained from G by expanding every vertex into H is also known as the lexicographic product $G \circ H$.

Lemma 22 *Let G be a strong oriented graph, H be an oriented graph, $u \in V(G)$. Let G' be the graph obtained from G by expanding u into H . If there is a rainbow arc-colouring φ of G with $\vec{rc}(G)$ colours such that u belongs to some rainbow cycle C , then $\vec{rc}(G') \leq \vec{rc}(G)$.*

Proof. To prove this lemma, we just check that we can extend the colouring φ to G' . Let φ' be an extension of the arc-colouring φ to G' such that the colour of every arcs xv or vx with $x \in V(G) \setminus \{u\}$ and $v \in V(H)$ corresponds to the colour of the corresponding arcs xu or ux in G . The colours of H 's internal edges may be chosen freely among the $\vec{rc}(G)$ available colours. Observe that every rainbow path P in G (under φ) remains a rainbow path P' in G' (under φ'), and thus, every vertex of G remains rainbow connected to any other vertices. A rainbow path between any pair of vertices v and v' in H can be found using the rainbow cycle C : writing $C = ux_1 \dots x_\ell u$ in G , the path $vx_1 \dots x_\ell v'$ is rainbow in G' . Therefore, φ' is a rainbow arc-colouring of G' using at most $\vec{rc}(G)$ colours, and this concludes the proof of the lemma. \square

We are now ready to prove our theorem:

Theorem 23 *For every n and k such that $3 \leq k \leq n - 1$, there exists a tournament $T_{n,k}$ on n vertices such that $\vec{rc}(T_{n,k}) = k$.*

Proof. We first define $T_{k+1,k}$, which is used later on to build the tournament for other values of n .

For $k \geq 3$, let $T_{k+1,k}$ be the tournament on $k + 1$ vertices made of a directed path $v_1 v_2 \dots v_{k+1}$ with all other arcs heading backward. Formally, $T_{k+1,k}$ is the tournament on vertex set $\{v_1, \dots, v_{k+1}\}$ with arcs set $\{v_i v_{i+1}, 1 \leq i \leq k\} \cup \{v_j v_i, 1 \leq i \leq j - 2, 3 \leq j \leq k + 1\}$.

The only directed path from v_1 to v_{k+1} is $v_1 v_2 \dots v_{k+1}$, so $\text{diam}(T_{k+1,k}) = k$ and $\vec{rc}(T_{k+1,k}) \geq k$ by Proposition 1. Let now φ be the k -arc-colouring of $T_{k+1,k}$ defined by $\varphi(v_i v_{i+1}) = i$ for every i , $1 \leq i \leq k$, and $\varphi(v_j v_i) = 1$ whenever $v_i \neq v_{j+1}$. Let us check that the colouring φ is a rainbow arc-colouring of $T_{k+1,k}$: (i) if $i < j$, then $v_i v_{i+1} \dots v_j$ is a rainbow path, (ii) $v_j v_i$ is a rainbow path whenever $i \leq j - 2$, and (iii) either $v_i v_{i+1} v_{i-1}$ or $v_i v_{i-2} v_{i-1}$ is a rainbow path otherwise. Therefore, $\vec{rc}(T_{k+1,k}) = k$, as desired.

Let us now define $T_{n,k}$ for larger n . Consider any family T_2, \dots, T_{k+1} of non-empty tournaments such that $|V(T_2)| + \dots + |V(T_{k+1})| = n - 1$. Let $T_{n,k}$ be the tournament on n vertices obtained from $T_{k+1,k}$ by expanding each vertex v_i into T_i , $2 \leq i \leq k + 1$, and extend the colouring φ as in Lemma 22. Using the rainbow cycle $v_2 v_3 \dots v_{k+1}$ and applying Lemma 22, we get that $\vec{rc}(T_{n,k}) \leq k$. Moreover, the diameter of $T_{n,k}$ is still at least k , so $\vec{rc}(T_{n,k}) \geq k$. \square

The remaining question on that topic is for which values of n there exists a tournament on n vertices having rainbow connection number 2. By hand, it is easily checked that for $n = 4$ or 5 , there exist no such tournaments. However, such a tournament exist for any $n \equiv 8 \pmod{12}$. It can be constructed from the following arc-coloured circulant graphs:

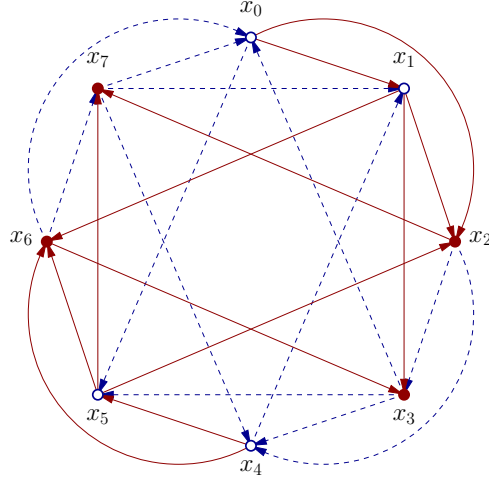


Figure 6: An oriented graph with a rainbow 2-colouring

let $n = 12k + 8$, $V = \{x_0, x_1, \dots, x_{n-1}\}$ and $E = \{x_i x_{i+1}, x_i x_{i+3j+2} \mid 0 \leq i \leq n-1, 0 \leq j \leq 4k+1\}$ (subscripts are taken modulo n). Set $c(i) = \lfloor \frac{i}{2} \rfloor \pmod{2}$ and colour the arcs $x_i x_{i+1}$ with colour $1 - c(i)$, and any other arc $x_i x_j$, $j \neq i+1$ with colour $c(j)$. The oriented graph obtained for $n = 8$ is drawn in Figure 6. The reader can easily check that this construction is well defined when $n \equiv 8 \pmod{12}$ and that it is rainbow connected (using only direct arcs or paths of type $x_i x_j x_{j+1}$ or $x_i x_j x_{j+2}$).

The previous results suggests a large range of possible values for the rainbow connection number of a tournament with a given order. However, it appears that the rainbow connection number of a tournament is closely related to its diameter, as the following results shows.

Theorem 24 *Let T be a tournament of diameter d . We have*

$$d \leq \vec{rc}(T) \leq d + 2.$$

Proof. Let T be a tournament of diameter d . The lower bound is a direct consequence of Proposition 1. For establishing the upper bound, we describe a rainbow arc-colouring of T , but we first give an appropriate decomposition of the graph.

Let a be a vertex of eccentricity d with maximum in-degree subject to that condition. For $1 \leq i \leq d$, denote by A_i the set of vertices at distance i from a . By the choice of a , note that none of the A_i are empty. Moreover, any vertex in A_1 has an out-neighbour

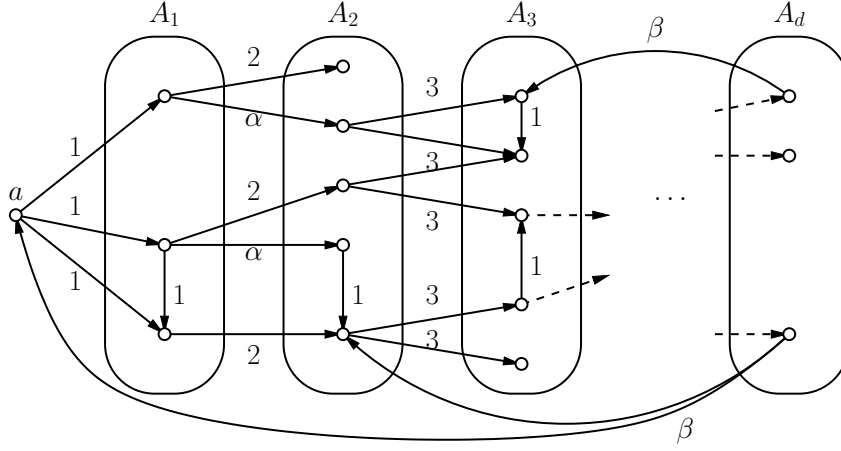


Figure 7: The decomposition and colouring of T

in A_2 . Otherwise, take a vertex u in A_1 with no out-neighbour in A_2 : it has no out-neighbours in A_i for any $i \geq 2$, so a shortest path from u to a vertex in A_d is of length at least d , and u is of eccentricity d . Also, every in-neighbour of a is an in-neighbour of u , as well as a itself. So u has larger in-degree than a . This contradicts our choice of a .

We now can define the colouring of the arcs with colours $\{1, 2, \dots, d, \alpha, \beta\}$ and prove that it does rainbow connect the tournament T (see Figure 7).

- Set colour i to all arcs uv where $u \in A_{i-1}$ and $v \in A_i$, for $1 \leq i \leq d$ and $i \neq 2$.
- Set colour 1 to all arcs uv where u and v belong to the same set A_i , $1 \leq i \leq d$.
- Set colour β (for backward) to all arcs from A_i to A_j where $j < i$.
- For arcs uv from A_1 to A_2 :
 - if v is the only out-neighbour of u in A_2 , set colour 2 to uv
 - otherwise, make sure there is one arc from u to A_2 coloured α , and colour all other arcs from u to A_2 with colour 2.

With this colouring, we show that the graph is rainbow connected. First observe that the shortest paths from a to any other vertex are rainbow, and do not use colour β . Let u be a vertex in A_i , $i \geq 2$. There is a rainbow path from u to any other vertex v starting by the arc ua (coloured β), followed by a shortest path from a to v (which is rainbow and does not use colour β).

We only have now to find rainbow paths from vertices in A_1 to other vertices. Let $u \in A_1$. For a path from u to vertex a or to a vertex $v \in A_1$, simply use any arc from u to A_2 (recall that there is one by our choice of a) and then the backward arc to a , plus possibly the arc av . We get a rainbow path with colours 2 or α and β and possibly 1. Now, let us describe a path to any other vertex $v \in A_i, i \geq 2$. Let P be a shortest path from a to v and $z = P \cap A_1$. Recall that P is rainbow and does not use colour β . If $u = z$, then P contains a rainbow path from u to v , and we are done. If uz is an arc, it is coloured 1 and the path starting with uz then going along P connects u to v and is rainbow. Assume now that T contains the arc zu . If u has at least two out-neighbours in A_2 , then consider the colour of the arc going out from z in P (it is either 2 or α), and take an arc of the second colour to A_2 , then go backward to a , and follow the path P . This path rainbow connects u and v . Finally, consider the case when u has only one out-neighbour in A_2 , say x . Recall our assumption that zu is an arc. So the in-neighbourhood of u contains all the in-neighbourhood of a but x (that is $\bigcup_{2 \leq i \leq d} A_i \setminus \{x\}$) as well as a and z . So u has larger in-degree than a . By our initial assumption, u is then of eccentricity at most $d - 1$. Thus, there is a shortest path through u from a to any vertices in A_d , and u is rainbow connected to any vertex in A_d . Also, u is rainbow connected to any vertex v in A_i for $2 \leq i \leq d - 2$ by going along a shortest path to some vertex in A_d and then backward to v (using colour β that was not in the path). Finally, to get to a vertex v in A_{d-1} , a shortest path is at least of length $d - 2$ and at most of length $d - 1$. If it is of length $d - 2$, then it has to go along a shortest path from a to v and thus to be rainbow. Otherwise, it is of length $d - 1$, and thus it uses only arcs from A_i to A_{i+1} , $1 \leq i \leq d - 2$ (coloured $i + 1$) except for exactly one arc that must be from A_j to A_j for some $1 \leq j \leq d - 2$ (coloured 1), and thus this path is also rainbow. This concludes the proof. \square

Note that there exist tournaments of diameter d with rainbow connection number $d + 1$. An example is the vertex transitive tournament on 5 vertices (there is only one up to isomorphism). It has diameter 2 but its rainbow connection number is 3. However, we don't know if there exist tournaments that have rainbow connection number equal to their diameter plus two.

References

- [1] J. A. Bondy and U.S.R. Murty, *Graph Theory*, Springer, 2008.
- [2] S. Chakraborty, E. Fischer, A. Matsliah, and R. Yuster, *Hardness and algorithms for rainbow connectivity*, J. Comb. Optim. 21 (2011), 330–347.

- [3] G. Chartrand, G.L. Johns, K.A. McKeon, and P. Zhang. *Rainbow connection in graphs*, Math. Bohem. 133 (2008), 85–98.
- [4] G. Chartrand, G.L. Johns, K.A. McKeon, and P. Zhang. *The rainbow connectivity of a graph*. Networks 54 (2) (2009), 75–81.
- [5] Y. Caro, A. Lev, Y. Roditty, Z. Tuza, and R. Yuster *On rainbow connection*, Electron. J. Combin. 15 (2008), #57.
- [6] J. Ekstein, P. Holub, T. Kaiser, M. Koch, S. Matos Camacho, Z. Ryjáček, I. Schiermeyer, *The rainbow connection number of 2-connected graphs*, Discrete Math. 313(19) (2013), 1884–1892.
- [7] A. Kemnitz and I. Schiermeyer, *Graphs with rainbow connection number two*, Discuss. Math. Graph Theory 31 (2) (2011), 313–320.
- [8] M. Krivelevich and R. Yuster, *The rainbow connection of a graph is (at most) reciprocal to its minimum degree*, J. Graph Theory 63 (3) (2009), 185–191.
- [9] X. Li, Y. Shi and Y. Sun, *Rainbow Connections of Graphs: A survey*, Graphs Combin. 29 (2013), 1–38.
- [10] X. Li and Y. Sun, *Rainbow Connections of Graphs*, Springer Briefs in Mathematics, Springer 2012.
- [11] J.W. Moon. *Topics on Tournaments*. Holt, Rinehart and Winston Eds., New-York 1968.
- [12] M.G. Reed, P.F. Syverson and D.M. Goldschlag, *Anonymous Connections and Onion Routing*, IEEE Journal on Selected Areas in Communications 16 (4) (1998), 482–494.
- [13] I. Schiermeyer, *Rainbow connection in graphs with minimum degree three*, IWOCA 2009, LNCS 5874 (2009), 432–437.
- [14] I. Schiermeyer, *On minimally rainbow k -connected graphs*, Discrete Appl. Math. 161(4-5) (2013), 702–705
- [15] I. Schiermeyer, *Rainbow connection and minimum degree*, Discrete Appl. Math. 161(12) (2013), 1784–1787.