

# NILPOTENT FAMILIES OF ENDOMORPHISMS OF $(\mathcal{P}(V)^+, \cup)$

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**Abstract.** An oriented graph  $G$  is *nice* if there exists a positive integer  $k$  such that for every two vertices  $u, v$  (allowing  $u = v$ ), and for every orientation of edges of the path of length  $k$ , there exists a walk of length  $k$  in  $G$  beginning at  $u$  and ending at  $v$  whose orientation of edges coincides with the given one. (Such a graph is also called  $k$ -nice.) We generalize this notion using the notion of a nilpotent semigroup of endomorphisms of  $(\mathcal{P}(V)^+, \cup)$ , and consider two basic problems:

- (1) find bounds for the nilpotency class of such semigroups in terms of their generators (in graph-theoretical language: provided that a graph  $G$  on  $n$  vertices is nice, find the smallest  $k$  such that  $G$  is  $k$ -nice);
- (2) find a way to demonstrate non-nilpotency of such semigroups (find as simple as possible characterization of non-nice graphs).

**Keywords.** Nice graphs, Nilpotent semigroup of endomorphisms.

## 1 Introduction

The notion of a nice graph first was implicitly used in the papers [1, 2, 5] as a useful tool for studying oriented chromatic number of graphs. Later, in [3], nice graphs were studied for their own sake, and some further generalizations were introduced.

An oriented graph  $G$  is called  $k$ -nice if for every two vertices  $u, v$  (allowing  $u = v$ ), and for every orientation of edges of the path of length  $k$ , there exists a walk of length  $k$  in  $G$  beginning at  $u$  and ending at  $v$  whose orientation of edges coincides with the given one. An oriented graph  $G$  is *nice* if it is  $k$ -nice for some positive integer  $k$  (Note that every  $k$ -nice graph is  $(k + 1)$ -nice). Similarly, a non-oriented (multi)graph  $G$  whose edges are coloured by  $c$  colours is called  $k$ -nice if for every two vertices  $u, v$  (allowing  $u = v$ ), and for every edge colouring of the path of length  $k$ , there exists a walk of length  $k$  in  $G$  beginning at  $u$  and ending at  $v$  whose colouring coincides with the given one. Also, such a (multi)graph is *nice* if it is  $k$ -nice for some positive integer  $k$ .

In [3], characterizations of non-nice graphs in terms of so called ‘black holes’ (see the next section) were found. It was proved that in every non-nice graph there exist black holes of relatively simple structure (especially simple for non-coloured oriented graphs).

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In this paper we show that “niceness” of graphs is a special case of a more general and natural notion of nilpotency of semigroups of endomorphisms of the upper semilattice  $(\mathcal{P}(V)^+, \cup)$  of all non-empty subsets of a finite set  $V$  of size  $n$ . In fact, this notion corresponds to nice edge-coloured oriented (multi)graphs.

In Section 3 we establish some bounds for the nilpotency class of such semigroups in terms of their generators, and the size  $n$  of the ground set. We show that the nilpotency class can be exponentially large if the number of generators is  $n$ , and give a polynomial in  $n$  upper bound on the nilpotency class when the generators have a special form related to the original graph-theoretical formulation: given a nice (oriented or simple multicoloured) graph on  $n$  vertices, what is the smallest  $k$  for which it is  $k$ -nice? As a corollary we get polynomial polynomial in  $n$  upper bounds on  $k$  for non-coloured oriented nice graphs and 2-edge-coloured non-oriented (multi)graphs.

In Section 4 we study the structure of shortest words in generators of a non-nilpotent semigroup, which demonstrate its non-nilpotency; that is, which have a black hole. We prove that there are no restrictions on these words; except that such a word cannot be a composite power.

Finally we pose some problems and conjectures.

## 2 Nilpotent families

Let  $V$  be a finite set,  $|V| = n$ . By  $\mathcal{P}(V)^+$  we denote the set of all non-empty subsets of  $V$ . We consider endomorphisms of the semi-lattice  $(\mathcal{P}(V)^+, \cup)$ , that is, mappings  $\varphi : \mathcal{P}(V)^+ \rightarrow \mathcal{P}(V)^+$  satisfying the identity

$$\varphi(X \cup Y) = \varphi(X) \cup \varphi(Y).$$

Every such mapping  $\varphi$  is uniquely determined by the values it takes on one-element subsets:

$$\varphi(X) = \cup_{x \in X} \varphi(\{x\}).$$

It is often convenient to view such mappings as oriented graphs on the vertex set  $V$  in which  $(x, y)$  is an arc if and only if  $y \in \varphi(\{x\})$ . Occasionally we shall abuse the notation by writing  $\varphi(x)$  instead of  $\varphi(\{x\})$ .

The set  $M(V)$  of all such endomorphisms is a semigroup. It has a right zero element  $\Omega$ ,  $\Omega(X) = V$  for all  $X \in \mathcal{P}(V)^+$ . Let  $M_\Omega(V) \subset M(V)$  be the sub-semigroup of those elements  $\varphi$  for which  $\varphi(V) = V$ .  $\Omega$  is the unique two-sided zero of  $M_\Omega(V)$ .

Let  $\Lambda = \{\varphi_1, \dots, \varphi_r\} \subseteq M_\Omega(V)$  be any collection of endomorphisms. By  $\Lambda^k$  we denote the set of all products  $x_1 \dots x_k$  of  $k$  elements from  $\Lambda$ . The collection  $\Lambda$  is called *nilpotent of class  $k$* , or  $k$ -nilpotent, if  $\Lambda^k = \{\Omega\}$  for some  $k > 0$ . This terminology is in accordance with traditional usage of the word “nilpotent”. Obviously, if  $\Lambda$  is  $k$ -nilpotent then it is  $m$ -nilpotent for every  $m > k$ .

We say that  $\emptyset \neq X \subseteq V$  is a *black hole* for  $\varphi$  if  $\varphi(X) = X$ ; if  $X \neq V$  then the black hole is called *non-trivial*. Note that if for some  $X$  we have  $\varphi(X) \subseteq X$  then some non-empty subset of  $\varphi(X)$  is a black hole. In particular, if  $\varphi(V) \neq V$  then  $\varphi$  has a non-trivial black hole.

It is easy to see that  $\Lambda$  is nilpotent if and only if no composition of mappings from  $\Lambda$  has a black hole. If  $\Lambda$  is nilpotent then, starting from any non-empty subset  $X \subseteq V$  and applying mappings from  $\Lambda$  in any sequence, we eventually shall reach the whole set  $V$ , and in not more than  $2^n - 2$  steps (because each proper subset of  $V$  can appear at most once; otherwise it would be a black hole for some composition of mappings from  $\Lambda$ ). Thus, every nilpotent family is  $2^n - 2$ -nilpotent.

For some nilpotent collections  $2^n - 2$  steps may be necessary. Here is an example. Let  $X_1, \dots, X_{2^n - 1}$  be any linear extension of  $\mathcal{P}(V)^+$  ordered by inclusion. For  $i = 1, \dots, 2^n - 2$  the mapping  $\varphi_i$  is defined by  $\varphi_i(x) = X_{i+1}$  if  $x \in X_i$ ,  $\varphi_i(x) = V$  if  $x \notin X_i$ . Then the system  $\Lambda = \{\varphi_1, \dots, \varphi_{2^n - 2}\}$  is nilpotent, and it takes exactly  $2^n - 2$  steps to transform  $X_1$  into  $X_{2^n - 1} = V$  if we take the mappings in their order. Later, in Example 3, we shall see that by being more careful we can obtain the same number of steps for a nilpotent family of only  $n$  different mappings.

### 3 Bounds for nilpotency class

In this section we shall try to obtain better upper bounds for the nilpotency class when the collection  $\Lambda$  is small and/or satisfies some additional properties.

We say that a mapping  $\psi$  is *opposite to*  $\varphi$  if  $y \in \varphi(\{x\})$  exactly when  $x \in \psi(\{y\})$ ; and that  $\varphi$  is *symmetric* if it is opposite to itself. These notions naturally arise in the graph-theoretical setting: taking the opposite of a mapping corresponds to inverting all arcs of an oriented graph; and symmetric mappings correspond to simple (non-oriented) graphs.

A mapping  $\varphi$  is called *increasing* if  $X \subseteq \varphi(X)$  for every  $X \subseteq V$ . Note that if  $\varphi$  is increasing and nilpotent then it is strictly increasing; i.e.  $X$  is a proper subset of  $\varphi(X)$ . Also, if  $\varphi^+$ ,  $\varphi^-$  are opposite then  $\varphi^+ \varphi^-$  is increasing.

**Proposition 1** *Let  $\Lambda = \{\varphi\}$ . If  $\Lambda$  is nilpotent then it is  $n(n-1)$ -nilpotent. If, moreover,  $\varphi$  is symmetric then  $\Lambda$  is  $(2n-2)$ -nilpotent.*

**Proof.** Take a longest sequence  $X_0, X_1, \dots, X_m$  such that  $X_{i+1} = \varphi(X_i)$  and  $X_m \neq V$ , and suppose, by way of contradiction, that  $m \geq n(n-1)$ . We can assume that  $X_0 = \{x\}$  is a one-element set. In the oriented graph corresponding to  $\varphi$  the vertex  $x$  lies in some oriented cycle (otherwise the set  $\cup_{i=1}^{\infty} \varphi^i(x)$  would be a non-trivial black hole). So, for some  $k \leq n$  we have  $x \in X_k$ , or  $X_0 \subseteq X_k$ . The mapping  $\varphi$  preserves the relation of one set being a subset of another; therefore, applying it several times, we obtain that  $X_j \subseteq X_{j+k}$  for every  $j$ . In particular,

$$X_0 \subseteq X_k \subseteq X_{2k} \subseteq \dots \subseteq X_{(n-1)k}.$$

All inclusions in this chain are strict (otherwise we would have a black hole); and so  $|X_{(n-1)k}| \geq n$ . Thus,  $X_{(n-1)k} = V$ , contrary to our assumption about  $m$ . The first part is proved.

If  $\varphi$  is symmetric then  $\varphi^2$  is an increasing mapping; so the above argument holds with the value  $k = 2$ , which proves the second claim.  $\square$

After submitting this present paper, the authors have learned that both bounds of this proposition have been previously obtained by other people in a different language. The general case was first studied by Wielandt in the paper [6]. There, the exact upper bound of  $n(n-2)$  on the nilpotency class was obtained. The (exact) bound  $2n-2$  for the symmetric case was obtained in the paper [4].

**Proposition 2** *Let  $\Lambda = \{a, b, c_1, \dots, c_k\}$  be a nilpotent collection such that the mappings  $ab$ ,  $ba$ , and all  $c_i$  are increasing. Then  $\Lambda$  is  $n^3$ -nilpotent.*

**Proof.** Take an arbitrary sequence  $(f_1, f_2, \dots, f_m)$  of mappings from  $\Lambda$ , an arbitrary non-empty  $X_0 \subseteq V$ , and let  $X_i = f_i(X_{i-1})$  for  $i = 1, 2, \dots, m$ . To each set  $X_i$  we assign a level, an integer value  $l_i$ , defined as follows:

$$\begin{aligned} l_0 &= 0; \\ \text{if } f_i &= a \text{ then } l_i = l_{i-1} + 1; \\ \text{if } f_i &= b \text{ then } l_i = l_{i-1} - 1; \\ \text{if } f_i &= c_j \text{ then } l_i = l_{i-1}. \end{aligned}$$

Now we shall prove two claims, from which the proposition will immediately follow. Let  $0 \leq p < q \leq m$ .

*Claim 1.* If  $l_q = l_p$  and  $X_p \neq V$  then  $X_p$  is a proper subset of  $X_q$ .

We shall prove this claim by induction on  $q-p$ . If  $q-p = 1$  then  $f_q = c_j$  is increasing, as required. If  $l_{p+1} \neq l_{q-1}$  then there is an  $r$  such that  $p < r < q$  and  $l_r = l_p$ , and we apply the induction hypothesis to  $(p, r)$  and  $(r, q)$ . Finally, let  $l_{p+1} = l_{q-1} \neq l_p$ . We can assume that  $f_{p+1} = a$ ,  $f_q = b$  (the other case is similar). By induction, we have  $X_{p+1} \subseteq X_{q-1}$ . So,

$$X_p \subset b(a(X_p)) = b(X_{p+1}) \subseteq b(X_{q-1}) = X_q,$$

and the claim is proved. (Here the first inclusion is proper since  $X_p$  is not a black hole.)

*Claim 2.* If  $|l_q - l_p| \geq n(n-1)$  then  $X_q = V$ .

Let  $l_q = l_p + n(n-1)$ ; the other case is treated similarly. For  $i = 0, \dots, n(n-1)$  let  $p_i$  be the smallest index such that  $p \leq p_i \leq q$  and  $l_{p_i} = l_p + i$ . In particular,  $p_0 = p$ . For every  $i = 1, \dots, n(n-1)$  we have  $l_{p_{i-1}} = l_p + i - 1 = l_{p_{i-1}}$ . So, by Claim 1, we have  $X_{p_{i-1}} \subseteq X_{p_i-1}$ ; and  $a(X_{p_{i-1}}) \subseteq a(X_{p_i-1}) = X_{p_i}$ . All these inclusions together imply that  $a^{n(n-1)}(X_p) \subseteq X_{p_{n(n-1)}}$ , and by Proposition 1,  $X_{p_{n(n-1)}} = V$ . The claim is proved.

Now, if some value of the level is assigned to  $n$  or more sets then Claim 1 implies that the last of these sets is equal to  $V$ . On the other hand, if we have more than  $n(n-1)$  different values of the level, Claim 2 implies that we have reached  $V$ . Therefore we shall reach  $V$  after at most  $n(n-1)(n-1)$  steps.  $\square$

**Proposition 3** *Let  $\Lambda = \{\varphi_1, \varphi_2\}$ . If  $\Lambda$  is nilpotent and either both  $\varphi_i$  are symmetric, or they are opposite to each other, then  $\Lambda$  is  $(c \cdot n^3)$ -nilpotent.*

**Proof.** When  $\varphi_1$  and  $\varphi_2$  are opposite, Proposition 2 applies immediately, and  $\Lambda$  is  $n^3$ -nilpotent.

When both mappings are symmetric, define four new mappings:  $a = \varphi_1\varphi_2$ ,  $b = \varphi_2\varphi_1$ ,  $c_1 = \varphi_1\varphi_1$ , and  $c_2 = \varphi_2\varphi_2$ . It is easy to check that these mappings satisfy the conditions of Proposition 2. Now, every sequence of  $2N$  mappings  $\varphi_i$  can be considered as a sequence of  $N$  mappings  $a, b, c_i$ . Therefore  $\Lambda$  is  $2n^3$ -nilpotent.  $\square$

The case of  $\varphi_1$  and  $\varphi_2$  being opposite corresponds precisely to the original notion of nice graphs [3]. Thus, Proposition 3 in this case asserts that "every nice graph on  $n$  vertices is  $n^3$ -nice".

We now give some examples. We do not have examples which would give lower bounds comparable with upper bounds from Propositions 2 and 3. The next two examples provide such bounds for Proposition 1. In all examples, we let  $V = \{0, 1, \dots, n-1\}$ .

**Example 1.** Let the mapping  $a$  correspond to the path  $0-1-\dots-(n-1)$  with a loop at the vertex  $n-1$ ; this mapping is symmetric. The single-element collection  $\{a\}$  is nilpotent, and if we start with the set  $X = \{0\}$ , we reach  $V$  only after  $2n-2$  steps.

**Example 2.** Now, let  $a$  correspond to the oriented graph formed from the oriented cycle  $0 \rightarrow 1 \rightarrow \dots \rightarrow n-1 \rightarrow 0$  and an extra arc  $(n-1) \rightarrow 1$ . Again,  $\{a\}$  is nilpotent, and if we start with the set  $X = \{0\}$ , we reach  $V$  only after  $n(n-2)$  steps.

It was proved in [6] that this is the only mapping which attains this bound.

**Example 3.** For each  $v \in V$  define the mapping  $a_v$  as follows:  $a_v(\{v\}) = V$ ,  $a_v(\{x\}) = \{v\}$  if  $x < v$ ,  $a_v(\{x\}) = \{x, v\}$  if  $x > v$ . All these mappings are symmetric. Let  $X_1, \dots, X_{2^n-1}$  be the lexicographic ordering of  $\mathcal{P}(V)^+$ ; each  $X_i$  is the set of positions at which the binary expansion of  $i$  has ones.

To make sure that the collection  $\{a_0, \dots, a_{n-1}\}$  is nilpotent, it is enough to check that in this ordering  $a_i(X) > X$  for every  $i$  and every  $X \subset V$ . This is straightforward.

On the other hand, for  $1 \leq i < 2^n - 1$ , let  $z(i)$  be the position of the first zero in the binary expansion of  $i$ ; or, equivalently,  $z(i) = \min\{j \mid j \in V \setminus X_i\}$ . Now it is straightforward to check that  $a_{z(i)}(X_i) = X_{i+1}$ .

Thus, the collection  $\{a_0, \dots, a_{n-1}\}$  is nilpotent of class  $2^n - 2$ .

## 4 Black holes

Here we consider families of endomorphisms which are not nilpotent. This means that some element of the semigroup generated by the family has a black hole. So, we say that the family  $\Lambda$  has a black hole  $X$  with a pattern  $w$  if  $w$  is a word over the alphabet  $\Lambda$ , and  $X \subset V$  is a black hole of the endomorphism corresponding to this word.

A non-nilpotent family can have many different patterns of black holes; in particular, every power of a pattern is again a pattern. So, the question arises: what can be said about the shortest pattern of a black hole of a non-nilpotent family?

**Proposition 4** *Let  $\Lambda$  be a finite alphabet, and  $w$  an arbitrary word over it. The following statements are equivalent:*

- (1)  $w$  cannot be represented in the form  $v^n$  for a composite number  $n$ .
- (2) There exists a set  $V$  and a collection of mappings denoted by elements of  $\Lambda$  such that  $w$  is a shortest pattern of any black hole of this collection.

**Proof.**

(1)  $\rightarrow$  (2). Let  $w = x_0x_1 \dots x_{N-1}$  (indices are considered modulo  $N$ , in order that  $w$  could be treated as a cyclic word). We take  $V = \{0, 1, \dots, N-1\}$ . Elements of  $V$  are also considered modulo  $N$ . For any subset  $X \subseteq V$ , we shall denote by  $X+i$  the set  $\{x+i \mid x \in X\}$  (addition modulo  $N$ ), the cyclic shift of  $X$ .

To define the mappings, we fix a number  $k$ , to be specified later. We shall define the mappings in such a way that every set of  $k$  consecutive elements of  $V$  (interval of length  $k$ ) shall be a black hole having some cyclic shift of  $w$  as a pattern.

For each  $v \in V$ , consider the subword  $x_{v-k+1} \dots x_{v-1}x_v$  of the cyclic word  $w$ . For any  $x \in \Lambda$ , let  $i$  and  $j$  be the first and the last, respectively, of the indices within this interval for which  $x_i = x_j = x$ . Define  $x(\{v\}) = \{j+1, \dots, i+k\}$ . If none of the letters of this interval is equal to  $x$ , we set  $x(\{v\}) = V$ .

We always have that  $v+1 \in x(\{v\})$ ; therefore for every subset  $A \subseteq V$  we have  $|x(A)| \geq |A|$ , and if  $|x(A)| = |A|$  then  $x(A) = A+1$ .

Let  $I = \{v, v+1, \dots, v+l-1\}$  be an arbitrary (cyclic) interval of length  $l < N$ . Its image  $x(I)$  under any mapping  $x \in \Lambda$  is again an interval, because the image of each element is an interval and these intervals respectively contain consecutive elements  $v+1, \dots, v+l$ . It follows that  $x(I) = I+1$  if and only if  $v \notin x(I)$  and  $v+l+1 \notin x(I)$ . From the definition of  $x$ , we have:

- $v \notin x(I)$  if and only if  $x_v = x$ ;
- $v+l+1 \notin x(I)$  if and only if  $x_{v+l-k} = x$ .

Conversely, if these two conditions are satisfied then  $x(I) = I+1$ ; unless there is a subinterval of  $I$  of length  $k$  containing no letter  $x$  — then  $x(I) = V$ .

Let  $A$  be an arbitrary subset of  $V$ .  $A$  is the union of disjoint intervals; let their initial vertices be  $v_1, \dots, v_m$ , and their lengths  $l_1, \dots, l_m$ . From the above remarks we have that  $|x(A)| = |A|$  for at most one mapping  $x$ , and if  $x$  is such then  $x_{v_i} = x_{v_i+l_i-k} = x$  for all  $i = 1, \dots, m$ .

Thus every interval of length  $k$  is a black hole whose pattern is uniquely determined, and this pattern is a cyclic shift of  $w$ . On the other hand, if a black hole contains an interval of length  $l \neq k$ , or two disjoint intervals with initial vertices  $v_1, v_2$ , then the word  $w$  is periodic with period  $|l-k|$ , resp.  $|v_1-v_2|$ .

Thus, if the word  $w$  is not periodic, we can choose  $k=1$ , so the only black holes of the resulting collection of mappings will be one-vertex subsets, and their shortest patterns will be cyclic shifts of  $w$ . If  $w$  is periodic,  $w = w_0^p$  for a prime  $p > 1$  and a non-periodic word  $w_0$ , then we can take  $k = |w_0|$ , and the resulting collection of mappings will satisfy the required property.

(2)  $\rightarrow$  (1) Suppose the contrary. Let  $w = u^{pq}$  for some word  $u$  and  $p, q > 1$ ; suppose that  $w$  is a shortest pattern of a black hole, and let  $A$  be a black hole of minimum size with pattern  $w$ . Consider the sets  $A_0 = A$ ,  $A_1 = u(A)$ ,  $\dots$ ,  $A_{pq-1} = u^{pq-1}(A)$ . These sets are all distinct, and all non-empty. We claim that they are also pairwise disjoint. Indeed, if  $B = A_i \cap A_j \neq \emptyset$  then  $u(B) \subseteq A_{i+1} \cap A_{j+1}$ , etc. (indices taken modulo  $pq$ ), and  $w(B) \subseteq B$  — therefore, some non-empty subset of  $B$  is a black hole with pattern  $w$ , contrary to our choice of  $A$ .

But now we see that the sets

$$X_i = \bigcup_{j=0}^{q-1} A_{i+pj}$$

for  $i = 0, \dots, p - 1$ , form an orbit of length  $p$  of the mapping  $u$ , and so  $X_0$  is a black hole with pattern  $u^q$ , contrary to our assumption about  $w$ .  $\square$

Thus, there are no serious restrictions on shortest patterns of non-nilpotent families, apart from exception of composite powers.

We conclude this note with several open questions.

First, nothing is known about computational complexity of deciding whether a set of mappings is nilpotent or not. It is not even clear whether this problem belongs to NP and/or co-NP.

Second, Propositions 1 and 3 together with Example 3 lead to the following natural conjecture:

**Conjecture 5** *For every natural  $k$  there exists an exponent  $f(k)$  and a constant  $c(k)$  such that every nilpotent family of  $k$  endomorphisms of the semilattice  $(\mathcal{P}(n)^+, \cup)$  of all non-empty subsets of an  $n$ -element set has nilpotency class at most  $c(k)n^{f(k)}$ .*

This conjecture remains open even for  $k = 2$ .

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