

Homomorphisms of sparse signed graphs

Clément Charpentier Reza Naserasr * Éric Sopena †

January 10, 2019

Abstract

The notion of homomorphisms of signed graphs, introduced quite recently, provide better interplay with notion of minor and thus of high importance in graph coloring. A newer, but equivalent, definition of homomorphisms of signed graph, proposed jointly by authors of this paper and Tom Zaslavsky, leads to a basic no homomorphism lemma. According to this definition, a signed graph (G, σ) maps to a signed graph (H, π) if there is a mapping ϕ of vertices and edges of G to vertices and edges of H (respectively) which preserves adjacencies, incidences, and signs of closed walks. For $ij = 00, 01, 10, 11$, let $g_{ij}(G, \sigma)$ be the length of a shortest closed walk of (G, σ) which is, positive and of even length for $ij = 00$, positive and of odd length for $ij = 01$, negative and of even length for $ij = 10$, negative and of odd length for $ij = 11$. For each ij if there is no closed walk of corresponding type, we define $g_{ij}(G, \sigma) = \infty$. If G is bipartite, then $g_{01}(G, \sigma) = g_{11}(G, \sigma) = \infty$. In this case, $g_{10}(G, \sigma)$ is certainly realized by a cycle of G , and it will be referred to as *unbalanced-girth* of (G, σ) .

It then follows that if (G, σ) admits a homomorphism to (H, π) , then we have $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ for $ij \in \{00, 01, 10, 11\}$.

Studying the restriction of homomorphisms of signed graphs on sparse families, in this paper we first prove that: for a given signed graph (H, π) , there is a positive value of ϵ such that if G is connected graph of maximum average degree less than $2 + \epsilon$, and if σ is a signature of G such that $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ for all $ij \in \{00, 01, 10, 11\}$, then (G, σ) admits a homomorphism to (H, π) .

For (H, π) being the signed graph on K_4 with exactly one negative edge, we show that $\epsilon = \frac{2}{3}$ works and that this is the best possible value of ϵ . For (H, π) being the negative cycles of length $2g$, denoted UC_{2g} , we show that $\epsilon = \frac{1}{2g-1}$ works.

As a bipartite analogue of Jaeger-Zhang conjecture we conjecture that every signed bipartite planar graph (G, σ) satisfying $g_{ij}(G, \sigma) \geq 4g - 2$ admits a homomorphism to UC_{2g} . We show that $4g - 2$ cannot be strengthened, and, supporting the conjecture, we prove it for (G, σ) satisfying the weaker condition of $g_{ij}(G, \sigma) \geq 8g - 2$.

In the course our work we provide a duality theorem to decide if a 2-edge-colored graph admits a homomorphism to certain class of 2-edge-colored signed graphs.

Keywords: Signed graphs; homomorphism, sparse graphs, planarity.

*CNRS, IRIF, UMR 8243, Université Paris Diderot - Paris 7 (France). E-mail: reza@irif.fr

†Univ. Bordeaux, Bordeaux INP, CNRS, LaBRI, UMR5800, F-33400 Talence.

1 Introduction

One of the earliest observation in the theory of (proper) vertex coloring of graphs is that if any subgraph of a graph G has a vertex of degree at most k , then G is $(k + 1)$ -colorable. Many efforts are devoted to improve, generalize, and strengthen this early observation. Brook's theorem is one such result which classifies the case when $k + 1$ colors are actually needed. The four color theorem could also be seen as an improvement as, by the Euler formula, there is a vertex of degree at most five in every planar graph, and thus the trivial bound of 6 is decreased to 4. So is Grötzsch's theorem, as every triangle-free planar graph has a vertex of degree at most 4, and is thus 5-colorable, but the theorem decreases this bound to 3.

Further studies and recent developments are based on the notions such as minor, *maximum average degree* of graph, denoted $\text{mad}(G)$, various coloring notions such as circular and fractional coloring and more generally the notion of graph homomorphism.

Considering the general notion of homomorphisms of graphs, the following is easily observed (we denote by C_n the cycle of order n).

Lemma 1. *There exists a homomorphism of $C_{2\ell+1}$ to C_{2k+1} if and only if $\ell \geq k$.*

Therefore, if there is a homomorphism of G to H , then the odd-girth of G is at least the odd-girth of H . This provides an easy condition upon which one can guarantee the non-existence of a homomorphism from G to H . While this condition is not always sufficient, under certain conditions it might be. Some of the well-known coloring results or conjectures in graph theory can be viewed in this context. A notable example is the following conjecture, whose first case, and the only case proven so far, is Grötzsch's theorem.

Conjecture 2 (Jaeger-Zhang[9]). *Every planar graph of odd-girth $4k + 1$, $k \geq 1$, admits a homomorphism to C_{2k+1} .*

For required definitions we refer to Section ??, but we would like to note that the same conjecture with a girth condition rather than an odd-girth condition is the dual statement of a conjecture of Jaeger on theory of flows on graphs, when restricted to the class of planar graphs. That the girth condition can be relaxed to a condition on the odd-girth is proposed by C.Q. Zhang. The folding lemma of W. Klostermeyer and C.Q. Zhang [9] is supporting evidence of this suggestion and implies a similar condition on the average degree. Partial results are obtained in [?, ?].

Results of this type can normally be rephrased in the following general framework: "Given a non bipartite graph H , any sufficiently enough sparse graph G either admits a homomorphism to H , or has a small odd cycle."

For each specific problem, working on a given family of graphs (such as planar graphs, or a minor-closed classes of graphs) and with a fixed H or a fixed family of H 's, one would like to know precisely "how sparse is enough". For a specific conjecture in this regard when H is just an odd cycle and some related question we refer to in [5] (Section 2).

A notion of graph theory which provides room for extensions and strengthening of classical graph theory results is the notion of *signed graph*. This notion, among others, provide a stronger and more natural connection between theories of minor and coloring. Using this notion, colorings and homomorphisms of (signed) bipartite graphs are no longer

a matter of triviality. Indeed, as we will describe later, the notion of homomorphism restricted to signed bipartite graphs captures the classical theory of homomorphisms of graphs.

In this work, we initiate the study of coloring and homomorphism problems for sparse signed graphs, with particular emphasize on the bipartite case. Among other things, we propose a bipartite analogue of Jaeger-Zhang conjecture, and we provide partial results supporting our conjecture.

2 Notation and terminology

Graphs are finite and simple, i.e., without loops and without parallel edges. For classical graph theory we use the standard notation, mainly following [1].

The main objects of this study are signed graphs. As the theories on signed graphs are under development from a wide range of interests, there are some differences in terminology in the existing literature. In this paper we adopt an improved notation introduced in [4] which fits well with respect to theory of homomorphisms and match most other views (compared to the recent introduction of the theory in [13]). We start with basic notation of graphs for clarity.

A graph is a pair (V, E) where V is a set, which normally and certainly in this paper, is finite, and E is a collection of 2-subsets of V (E could be considered as a multiset when we speak of a multigraph). Elements of V are referred to as *vertices* and elements of E are *edges*. Thus, using this definition, we do not allow loops. We follow the standard terminology of graphs. Some precisions are as follows. A vertex of degree k maybe referred to as a *k-vertex*. A *walk* of length k in a graph G is a sequence of (not necessarily distinct) vertices $v_0 \dots v_k$, such that $v_i v_{i+1}$ is an edge in G for every i , $0 \leq i < k$. If $v_0 = v_k$, we say the walk is a *closed walk*. A *path* is a walk with the additional property that no vertex appears twice. A close walk where the first vertex is the only repeated vertex is a *cycle*.

A *thread* is a path whose internal vertices are all of degree 2 in G . If x and y are the end vertices of the walk, the path or the thread, we call it an *xy-walk*, *xy-path* or *xy-thread*, respectively. Length of this path is the length of the thread. Observe that every edge in G is a walk, a path, and a thread of length 1.

A *2-edge-colored graph* is a graph whose edges are assigned one of the two possible colors (this coloring is not necessarily proper). Assuming E_1 and E_2 are the color classes, we will denote the corresponding 2-edge-colored graph by a triplet, namely (V, E_1, E_2) or (G, E_1, E_2) , in order to distinguish it from a signed graph.

A *signed graph* is a graph G together with an assignment of one of the two possible signs (we mean $+$ or $-$) to each edge of G . This notion is thus different from that of a 2-edge-colored graph, since $\{+, -\}$ is a 2-element multiplicative group, which allows us to speak of *positive* or *negative* objects (e.g. subgraphs), using the operation of this group.

One may then use one of the two natural notations to denote a signed graph, namely either (G, σ) , with σ being the function that assigns signs to edges, or (G, Σ) , with Σ being the set of negative edges. We rather use the latter in this paper.

One notion of importance, which distinguishes signed graphs from 2-edge-colored graphs, is the notion of *resigning* (also called *switching* by many authors). Given an edge-cut $[X, Y]$ of a signed graph (G, σ) , resigning at $[X, Y]$ consists of multiplying all edges of $[X, Y]$ by the negative sign, in other words, it is switching the sign of each edge

with one end in X and one end in Y . If $[X, Y]$ is of the form $[\{x\}, V(G) \setminus \{x\}]$ for some vertex x of G , we will say that we resign at x rather than we resign at $[\{x\}, V(G) \setminus \{x\}]$. It is then easy to observe that resigning at any cut $[X, Y]$ is equivalent to resigning at all vertices of X or resigning at all vertices of Y .

Let u and v be two (not necessarily distinct) vertices of a signed graph (G, σ) , and W be a uv -walk in G (recall that edges can be used more than once in a walk). The sign of such a walk is simply the product of the sign of its edges, each edge appearing more than once being counted with the corresponding multiplicity.

Observe that resigning at an internal vertex of walk does not change the sign of the walk. Thus sign of closed walks are invariant under the operation of resigning. However, if W is not a closed walk, then resigning at exactly one end will change the sign of W .

Together with parity of the length, this leads to four distinguishable types of closed walks: positive and of even length (type 00), positive and of odd length (type 01), negative and of even length (type 10), negative and of odd length (type 11).

This type notation is convenient in the following sense: if W_1 is a closed walk of type ab starting at u and W_2 is a closed walk of type cd also starting at u , then the uv -walk $W_1 \cup W_2$, obtained by concatenating W_1 and W_2 , is of the type $ab + cd$ (where the addition is taken in the additive group $\mathbb{Z}_2 \times \mathbb{Z}_2$).

A positive cycle is said to be *balanced* whereas a negative cycle is referred to as *unbalanced*. An unbalanced cycle of length l is denoted by UC_l .

Structures whose sign is invariant under resigning, specially closed walks, are the key to our study of homomorphisms. Two signatures σ_1 and σ_2 on a same graph G are said to be *equivalent* if one is obtained from the other by a resigning. It is easily observed that this induces an equivalence relation on the class of all signatures on G . For each family of equivalent signatures, the set of balanced cycles is fixed. A key lemma of Zaslavsky is that the converse is also true, i.e., the set of balanced cycles uniquely determines the equivalent classes of signatures.

Lemma 3 (Zaslavsky [17]). *Given two signatures σ_1 and σ_2 of a graph G , σ_1 is a resigning of σ_2 if and only if the sets of balanced (or, equivalently, unbalanced) cycles of (G, σ_1) and (G, σ_2) are the same.*

We may now define the key notions of this work: homomorphisms of graphs, of 2-edge-colored graphs and of signed graphs. We emphasize on the difference between homomorphisms of 2-edge-colored graphs and of signed graphs, but we will indeed take advantage of the strong relation between these two notions when proving our results.

Definition 4. *Given two graphs G and H , a homomorphism of G to H is a mapping ϕ which maps vertices of G to vertices of H , edges of G to edges of H , and such that it preserves adjacency and incidence, i.e., $\phi(x)\phi(y) \in E(H)$ whenever $xy \in E(G)$, and $\phi(x) \in \phi(e)$ (in H) whenever $x \in e$ (in G).*

If G and H are both 2-edge-colored graphs, then a homomorphism of G to H is a homomorphism of the underlying (uncolored) graphs which preserves the colors as well.

Given signed graphs (G, σ) and (H, π) , a homomorphism of (G, σ) to (H, π) is a homomorphism of G to H which preserves the balance of all closed walks.

In all of three of the definitions above if the underlying graphs are simple (which will be the case in this study), then we can define a homomorphism to be a mapping of vertices

which preserves adjacency. For 2-edge-colored graphs we require, furthermore, that the mapping preserves the color of edges, and for signed graphs we require that the mapping preserves the signs of closed walks. Next we show a strong connection between these two conditions.

Theorem 5. *A signed graph (G, σ) admits a homomorphism to a signed graph (H, π) if and only if there is a resigning (G, σ') of (G, σ) and a homomorphism of G to H which preserves the sign of the edges with respect to σ' and π .*

One direction of this theorem is easy, a sign preserving mapping of (G, σ') to (H, π) certainly preserves signs of closed-walks and these signs are the same in (G, σ) and (G, σ') . On the other hand if there is a homomorphism ϕ of (G, σ) to (H, π) , defining $\sigma'(e) = \pi(\phi(e))$ we get a signature σ' which induces a same sign as σ on all closed walks. Thus, by Lemma 3, σ' is a resigning of σ .

We note that one can then take the condition of this theorem as the definition of homomorphisms of signed graphs. This is indeed the original definition given in [13], and in the rest of this work we will use this definition. Thus a mapping of signed (simple) graph (G, σ) to (H, π) will be denoted by a function $f = (f_1, f_2) : \{+, -\} \times V(G) \rightarrow V(H)$, where for each vertex x of G , f_1 specifies if a resigning is done at x or not and f_2 specifies to which vertex of $V(H)$ the vertex x is mapped.

A rather surprising result of [13] is that the restriction of homomorphism on signed bipartite graphs captures the classic notion of homomorphism of graphs as a special case. This is shown through the following construction: given a signed graph G , a signed bipartite graph $S(G)$ is built as follows. For each edge uv of G , first add a parallel edge, and then subdivide both edges in order to form a 4-cycle (if G has n vertices and m edges, then $S(G)$ has thus $n + 2m$ vertices and $4m$ edges). Finally, for each such a 4-cycle, we assign one negative and three positive signs to its edges. With this construction in mind, the following is proved in [13].

Theorem 6. *Given graphs G and H , there is a homomorphism of G to H if and only if there is a homomorphism of $S(G)$ to $S(H)$.*

It is thus of special interest to study the homomorphism relation on the subclass of signed bipartite graphs.

A common notion in the theory of homomorphisms is the notion of *core* which is defined analogously for each of the structural subjects. A core is a graph (analogously, a signed or a 2-edge-colored graph) which do not admit a homomorphism to any of its proper subgraphs. The *core of a graph G* is then the smallest subgraph of G (with respect to subgraph inclusion) to which G admits a homomorphism. It is not difficult to show, in each case, that the core of a graph is unique up to isomorphism.

An *automorphism* of a signed graph (G, σ) is a homomorphism of (G, σ) to itself which is one-to-one. A signed graph is said to be *vertex transitive* if for each pairs x and y of vertices there is an automorphism which maps x to y . Most important example of vertex transitive signed graph for this work are unbalanced cycles.

3 Girth and maximum average degree conditions

An advantage of the new definition of homomorphisms of signed graph is an immediate no-homomorphism lemma which is based on the following definition.

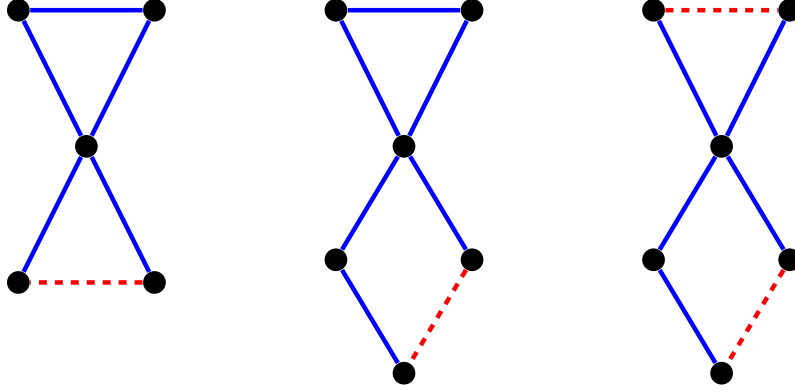


Figure 1: Signed graphs where g_{ij} is given by a closed walk which is not a cycle.

Definition 7. Given a signed graph (G, σ) and $ij \in \mathbb{Z}_2^2$, the ij -walk-girth of (G, σ) , denoted $g_{ij}(G, \sigma)$, is the length of a shortest closed walk of type ij in (G, σ) . When there is no such a walk, we write $g_{ij}(G, \sigma) = \infty$.

Observe that $g_{00}(G, \sigma) = 2$ unless G has no edges in which case $g_{00}(G, \sigma) = \infty$. We note that $g_{ij}(G, \sigma)$ for $ij \neq 00$ might not be realized by a cycle of G as shown by examples of Figure 1, however it can be show that for each connected graph, of the three values, at least two will be realized by a cycle. Furthermore, as shown in [4], in a connected signed graph (G, σ) , of the three values of $g_{01}(G, \sigma)$, $g_{10}(G, \sigma)$ and $g_{11}(G, \sigma)$ we cannot have exactly one value being ∞ . This leads to three special subclasses where exactly two of the values are ∞ . Of these three, the case that $g_{01}(G, \sigma) = g_{11}(G, \sigma) = \infty$ is the class of signed bipartite graph which is of special important for this work.

We now state a basic no-homomorphism lemma for signed graph:

Lemma 8. If a signed graph (G, σ) admits a homomorphism to a signed graph (H, π) , then we have:

$$g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$$

for each $ij \in \mathbb{Z}_2^2$.

It is easy to observe that the conditions of this lemma are far from being sufficient. However, as we will show in this work, for graphs of small maximum average degree, these conditions are also sufficient. The bound on the maximum average degree will be provided as a function of (H, π) . This is stated more precisely in the next theorem, but we first state a folklore lemma on the structure of graphs with small maximum average degree.

Lemma 9. If G is a connected **2-connected?** graph with minimum degree at least 2 and maximum average degree less than $2 + \frac{2}{2+3(d-1)}$, then either G is a cycle or G contains a thread of length d .

Theorem 10. For every connected signed graph (H, π) , there exists an ϵ , such that for each graph G with $\text{mad}(G) < 2 + \epsilon$ and any signature of G such that $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ we have $(G, \sigma) \rightarrow (H, \pi)$.

Proof. To prove Theorem 10 one may first observe that vertices of degree 0 or 1 are of no importance. Thus, by Lemma 9, either G is a cycle or it contains a long thread.

If G is a cycle, then (G, σ) is a cycle of type, say, ij . As $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$, this cycle can be mapped to walk of type ij in (H, π) . Therefore, by Lemma 9, we may assume that G has a thread of length d where d is a function of ϵ which increases when ϵ tends to zero.

For the remaining part of the proof we assume $g_{ij}(H, \pi) < \infty$ for all $ij \in \mathbb{Z}_2^2$. The other cases are similar and simpler, in each case one may consider (H, π) to be just a cycle and then apply a simpler version of the following arguments. Recall that $diam(H)$ is the largest possible distance between two vertices of H . As H is a connected graph, this is a finite number and a function of H . We take a d which is at least as $2diam(H) + \max\{g_{ij}(H, \pi)\}$ and, using Lemma 9, we choose an ϵ so that $mad(G) < 2 + \epsilon$ implies existence of a thread P of length d in G . Let x and y be the two ends of this thread. Consider the signed graph (G', σ') obtained from (G, σ) by removing internal vertices of P . We may map (G', σ') to (H, π) by induction, let ϕ be such a mapping. We would like to extend ϕ to a mapping of (G, σ) to (H, π) . We pay attention that x and y are already mapped and we are not allowed to resign at these two vertices but we are allowed to resign at internal vertices of P after which we may choose where to map them. Observe that resigning at internal vertices of P does not change the sign of P (that is the product of signs of all edges of P), furthermore, parity of P is given, thus, employing the terminology of types of closed walks, we may say P is of type ij for some $ij \in \mathbb{Z}_2^2$. We may now consider a shortest path Q in (H, σ) connecting $\phi(x)$ and $\phi(y)$. If Q is of the same type as P , then we map P on Q , that is possible because parity and sign of P permits this. Otherwise we choose a shortest closed walk W of type $i'j'$ starting at a vertex v of Q such that the walk Q' , starting from $\phi(x)$, going to v on the path Q , then traversing W , then moving to $\phi(y)$ on the path Q is of the same type as P . As we have assumed (H, π) has each type of the closed walks, and as H is connected, this is possible. Furthermore, by taking a shortest path and a shortest closed walk, we have insured that Q' has length at most d . We may now extend the mapping ϕ to a mapping of P to Q' . \square

A challenging question then is to determine the best value of ϵ for a given (H, π) . This value may be improved by further restriction on the graph (G, σ) . For example what if we consider only planar graphs? Note that, planarity already imposes a condition of maximum average degree being strictly less than 6. Further conditions on lengths of facial cycles of a planar graph may improve this bound on the average degree. A sort of dual question then is the following question of high interest:

Problem 11. *Given c_{ij} and l_{ij} , $ij \in \{01, 10, 11\}$, satisfying $c_{ij} \geq l_{ij}$ what is a smallest signed graph (H, π) satisfying $g_{ij}(H, \pi) \geq l_{ij}$ such that every planar signed graph (G, σ) satisfying $g_{ij}(G, \sigma) \geq c_{ij}$ admits a homomorphism to (H, π) ?*

For example for $c_{10} = c_{11} = l_{10} = l_{11} = \infty$, $c_{01} = l_{01} = 3$, it is a restatement of the four color theorem that $(K_4, +)$ works. Similarly, if we take $c_{10} = c_{11} = l_{10} = l_{11} = \infty$, $c_{01} = 5$ and $l_{01} = 3$, it is a restatement of the Grötzsch theorem that $(K_3, +)$ works. Furthermore, the Jaegr-Zhang conjecture can be restated to the claim that if $c_{10} = c_{11} = l_{10} = l_{11} = \infty$, $c_{01} = 4k + 1$ and $l_{01} = 2k + 1$, then $(C_{2k+1}, +)$ is the answer.

For $c_{10} = l_{10} = 4$, $c_{11} = l_{11} = c_{01} = l_{01} = 3$, while a lower bound of 10 on the order of H is given in [13], an upper bound of 40 is proved in [15], where the authors propose a

specific signed graph of order 10 as a conjecture.

For $c_{10} = 4$, $c_{01} = c_{11} = 3$ and $l_{10} = 2$, $l_{01} = 3$, $l_{11} = 1$, it is a conjecture of Naserasr and Raspaud that the signed graph on two vertices, consisting of a positive and a negative edges between two vertices and a negative loop on each vertex works.

The main results of this work can then be seen as providing optimal values in Theorem 10 for specific cases. More precisely, we prove in Section 6 the following result.

Theorem 12. *If G is a graph with $\text{mad}(G) \leq \frac{8}{3}$ then, for any signature $\Sigma \subseteq E(G)$, $(G, \sigma) \rightarrow (K_4, \{e\})$. Moreover the bound of $\frac{8}{3}$ is best possible.*

And then in Section 7 studying the case signed bipartite graphs we prove the following theorem.

Furthermore, as the bipartite analogue of Jaeger-Zhang conjecture, we propose the conjecture stated next and provide some supporting result for it.

Theorem 13. *If G is a bipartite graph with $\text{mad}(G) < 2 + \frac{1}{2g-1}$ then, for any signature σ such that $g_{01}(G, \sigma) \geq 2g$, $(G, \sigma) \rightarrow (C_{2g}, \{e\})$.*

Conjecture 14. *Any signed bipartite planar graph (G, σ) of unbalanced-girth at least $4g - 2$ admits a homomorphism to UC_{2g} .*

The rest of the paper is organized as follows. In the next section, we provide a duality theorem for mapping 2-edge-colored graphs into $(C_{2g}, \{e\}, E - e)$. In Section 5 we prove some properties of minimum counterexamples for our statements. In Section 6 we prove Theorem 12 and in Section 7 we prove Theorem 13. In Section 8, applying our results on planar graphs, we provide some support for Conjecture 14. Finally, in Section 9, we give some examples proving lower claims on some of our results and conjectures being the best possible.

4 A duality theorem for mapping to a 2-edge-colored cycle

As a 2-edge-colored homomorphism problem, given a fixed 2-edge-colored cycle C , the corresponding C -coloring problem, by Feder-Vardi dichotomy conjecture [6], is expected to be either solvable in polynomial time or to be NP-complete, see [2, 3].

For the case of $(C_{2g}, \{e\}, E - \{e\})$ we show that a duality theorem holds, which in particular implies a polynomial time algorithm. A polynomial time algorithm is also given in [2], but our interest is in theoretical applications of the simple and nice duality theorem which we provide here.

Let $(P_{n+1}, \{e_1, e_n\}, E - \{e_1, e_n\})$ be a path of length n whose first and last edges are colored red (or 1), while all other edges are colored blue (or 2). It is easily observed that there exists a homomorphism of this 2-edge-colored graph to the 2-edge-colored cycle $(C_{2g}, \{e\}, E - \{e\})$ if and only if either n is even, or n is odd and $n \geq 2g + 1$. Therefore, if a 2-edge-colored graph (G, E_1, E_2) admits a homomorphism from $(P_{2\ell-1}, \{e_1, e_{2\ell-1}\}, E - \{e_1, e_{2\ell-1}\})$, with $\ell \leq g$, then it does not admit any homomorphism to $(C_{2g}, \{e\}, E - \{e\})$. Our claim is that this necessary condition is also sufficient for an input whose underlying graph is bipartite.

Theorem 15. *Given a bipartite graph G , a 2-edge-colored graph (G, E_1, E_2) , admits a homomorphism to $(C_{2g}, \{e\}, E - \{e\})$ if and only if there is no homomorphism of $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$ to (G, E_1, E_2) .*

Proof. Observe that for every $i, i \leq g$, $(P_{2i-1}, \{e_1, e_{2i-1}\}, E - \{e_1, e_{2i-1}\})$ and $(C_{2i}, \{e\}, E - \{e\})$ are both homomorphic images of $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$. Moreover, any bipartite image of $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$ must contain either $(P_{2i-1}, \{e_1, e_{2i-1}\}, E - \{e_1, e_{2i-1}\})$ or $(C_{2i}, \{e\}, E - \{e\})$ as a subgraph for some $i \leq g$.

Thus (G, E_1, E_2) does not admit any homomorphism from $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$ if and only if it contains neither $(P_{2i-1}, \{e_1, e_{2i-1}\}, E - \{e_1, e_{2i-1}\})$ nor $(C_{2i}, \{e\}, E - \{e\})$ as a subgraph, for every $i \leq g$. In other words, (G, E_1, E_2) does not admit any homomorphism from $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$ if and only if no two vertices, each incident with a red edge, are connected by a blue path of odd length $2\ell + 1$ where $\ell < g$. In particular, this implies that $(C_{2g}, \{e\}, E - \{e\})$ does not admit any homomorphism from $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$, which proves the “only if” part of our theorem.

For the converse, denote by $x_0y_0x_1y_1 \dots x_{g-1}y_{g-1}$ the cycle C_{2g} and assume x_0y_0 is the red edge of $(C_{2g}, \{e\}, E - \{e\})$. Consider a 2-edge-colored graph (G, E_1, E_2) , where G is a bipartite graph with bipartition (X, Y) and such that $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$ does admit a homomorphism to (G, E_1, E_2) . We need to find a mapping of (G, E_1, E_2) to $(C_{2g}, \{e\}, E - \{e\})$. Let X_0 (respectively Y_0) be vertices in X (respectively Y) each of which is incident with a red edge. Let $X_i, i \leq g - 2$ be the set of vertices in X at distance i from $X_0 \cup Y_0$. Thus a vertex v of X is in X_i if a closest vertex to it in $X_0 \cup Y_0$, say $f(v)$, is at distance i from it. Observe that, because of the bipartition of G , for odd values of i , $f(v)$ is in Y_0 and for even values of i , $f(v)$ is in X_0 . Let X_{g-1} be all the remaining vertices in X , thus vertices in X_g are at distance $g - 1$ or more from all vertices in $X_0 \cup Y_0$. We define $Y_i, i \leq g - 1$ similarly.

We may now define a homomorphisms of (G, E_1, E_2) to $(C_{2g}, \{e\}, E - \{e\})$ as follows: all vertices in X_i are mapped to the vertex x_i and all vertices in Y_i are mapped to the vertex y_i . It remains to show that red edges of (G, E_1, E_2) (edges in E_1) are mapped to red edge of $(C_{2g}, \{e\}, E - \{e\})$ (the edge $e = x_1y_1$) and that blues edges are mapped to the blue edges. A red edge in (G, E_1, E_2) must have, by definition and because of the bipartition, one end in X_0 and the other end in Y_0 . Thus it maps to x_0y_0 which is the red edge of $(C_{2g}, \{e\}, E - \{e\})$. Let $e = uv$ be a blue edge, and assume $u \in X_i, i < g - 1$. Let $f(u)$ be a closest vertex in $X_0 \cup Y_0$ to u . Recall that whether $f(u)$ is in X_0 or in Y_0 only depends on the parity of i . Observe that v is a vertex in Y part of G because of bipartition of G . We claim that v cannot be in Y_i . Otherwise, $f(v)$, a closest vertex to v in $X_0 \cup Y_0$, must be in the part which $f(u)$ is not. Then the walk composed of a shortest connection from $f(u)$ to u , edge uv and a shortest connection from v to $f(v)$ fits the parity condition to be the image of $(P_{2g-1}, \{e_1, e_{2g-1}\}, E - \{e_1, e_{2g-1}\})$, but this path cannot map to (G, E_1, E_2) . Thus, considering the triangular inequality, the vertex v has to be either in Y_{i-1} or Y_{i+1} . Since both $y_{i-1}x_i$ and x_iy_{i+1} are blue edges of $(C_{2g}, \{e\}, E - \{e\})$, uv is mapped to a blues edge and this this type of edges are fine as well. It only remains to consider a blue uv where $u \in X_{g-1}$. In this case, again by the triangular inequality, either $v \in Y_{g-2}$ or $v \in Y_{g-1}$, as $y_{g-2}x_{g-1}$ and $x_{g-1}y_{g-1}$ are both blues edges of $(C_{2g}, \{e\}, E - \{e\})$ we are done with this case as well. \square

In relation to signed graph homomorphisms, we observed that any unbalanced cycle

can be resigned in such a way that it contains a single negative edge. Therefore, the question of deciding whether a signed bipartite graph admits a homomorphism to an unbalanced cycle UC_{2g} amounts to finding an equivalent signature which does not induce a $(P_{n+1}, \{e_1, e_n\}, E - \{e_1, e_n\})$ structure.

However, unless $P = NP$, this would not be an easy task as it is shown in [3, 7] that this homomorphism question, in contrast to its 2-edge-colored counterpart, is an NP-complete problem.

As an application of Theorem 15, we have the following result.

Theorem 16. *A signed bipartite graph (G, Σ) maps to $(C_4, \{e\})$ if and only if it maps to $(K_4, \{e\})$.*

Proof. Since $(C_4, \{e\})$ is a subgraph of $(K_4, \{e\})$, if (G, Σ) maps to $(C_4, \{e\})$ then it also maps to $(K_4, \{e\})$. Conversely, suppose that (G, σ) maps to $(K_4, \{e\})$ under the signature Σ . It follows that (G, Σ) cannot contain any positive edge xy such that both x and y are incident with a negative edge. Assuming G is bipartite, this property corresponds to the hypothesis of Theorem 15 for $g = 2$, and thus (G, σ) , with this particular signature, maps to $(C_4, \{e\})$. \square

5 Minimal elements

A standard technique to prove results of the type we consider in this work is to consider a minimum counterexample, prove some properties that such an graph must satisfy, and finally derive a contradiction. The minimality of such counterexample can be viewed in two ways. The first one is to say that no subgraph of our minimal counterexample is a counterexample, which, in particular, says that a minimal counterexample is a core. The second one is to say that no smaller member of the class of graphs we are working with is a counterexample. This, in particular, implies that no (proper) homomorphic image of our minimal counterexample belongs to the considered class of graphs.

In view of the former case, i.e., a minimum counterexample necessarily being a core, we will show here that certain subdivisions of K_4 cannot be a core. In view of the latter case, Klostermyer and Zhang developed in [9] a so-called “folding lemma” for homomorphism problems on the class of planar graphs, which implies that every planar graph of odd girth $2k + 1$ has a planar homomorphic image for which in every planar embedding every face is a $(2k + 1)$ -cycle. An analogue of this lemma for the class of signed bipartite planar graphs is developed in [12] which would also be of importance for the part of our work which deals with this subclass of signed graphs. We thus restate this lemma and its corollary below.

Lemma 17 (Naserasr, Rollová and Sopena [12]). *Let (G, σ) be a signed bipartite graph of unbalanced-girth $2g$, together with a planar embedding. If (G, σ) has a facial cycle C which is not an unbalanced cycle of length $2g$, then there exist two vertices x and y of C , at distance two from each other, such that the planar signed graph obtained from (G, σ) by identifying x and y is a homomorphic image of (G, σ) with unbalanced-girth $2g$.*

Corollary 18. *Every signed planar bipartite graph of unbalanced-girth $2g$ admits a planar homomorphic image of unbalanced-girth $2g$ where in every planar embedding every face is an unbalanced cycle of length exactly $2g$.*

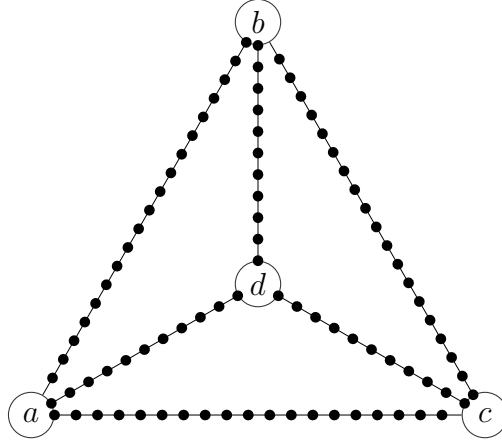


Figure 2: Subdivision of K_4

Thus, for instance, a minimal counterexample to Conjecture 14 should be a signed bipartite planar graph such that every facial cycle of any of its planar embeddings is an unbalanced cycle of length exactly $4g - 2$.

A *subdivision* of K_4 is a graph obtained by replacing some or all edges of K_4 by threads connecting their end points. A *signed subdivision* of K_4 is any signed graph based on a subdivision of K_4 . A planar drawing of a subdivision of K_4 has four facial cycles. As the number of unbalanced faces must be even (since each negative edge changes the balance of two incident faces), there are essentially three different types of signed subdivisions of K_4 , namely those having two, four, or no unbalanced faces. We are interested in the case where the number of unbalanced faces is two, and we want to determine when such a signed subdivision of K_4 is a core. The next two lemmas give an answer to this question for two particular cases, and will be used in Section 7 to prove Theorem 13.

Let \overline{K}_4 be a subdivision of K_4 whose (main) vertices are a, b, c and d (see Figure 2). For every $x, y \in \{a, b, c, d\}$, $x \neq y$, let P_{xy} be the path which represents the edge xy , and let L_{xy} be the length of P_{xy} . Moreover, let $C_{xyz} = P_{xy} + P_{yz} + P_{zx}$.

In [14] all subdivisions of K_4 which are cores are classified. The following case of that result is used in this work.

Lemma 19. *Let \overline{K}_4 be a bipartite subdivision of K_4 satisfying $|C_{abc}| = 2g$ and $L_{ad} + L_{bd} + L_{cd} \geq 4g$, and σ be a signature of \overline{K}_4 such that C_{abc} is an unbalanced cycle, and (\overline{K}_4, σ) is of unbalanced-girth $2g$. Then, the core of (\overline{K}_4, σ) is UC_{2g} .*

Let K_3^{++} be the multigraph obtained from K_3 , with vertices a, b and c , by adding a parallel edge to each of the edges ab and ac . Similarly to the proof of the previous lemma, we can prove the following result which claims that certain signed graphs, build upon specific subdivisions of K_3^{++} (see Figure 3) are not cores. The following has also been proved in [14].

Lemma 20. *Let \overline{K}_3^{++} be a bipartite subdivision of K_3^{++} such that the outer cycle is of length $2g$ and the total length of the three threads incident to b is at least $4g$, and σ be a signature of \overline{K}_3^{++} such that $(\overline{K}_3^{++}, \sigma)$ is of unbalanced-girth $2g$. Then, the core of $(\overline{K}_3^{++}, \sigma)$ is UC_{2g} .*

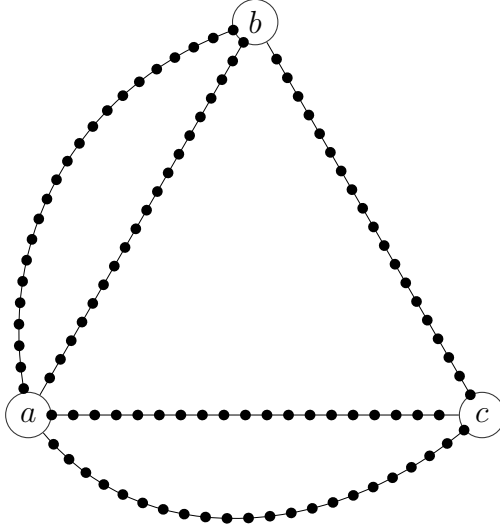


Figure 3: Subdivision of K_3^{++}

6 Mapping sparse signed graphs into $(K_4, \{e\})$

In this section, we prove Theorem 12, using standard discharging technique. A reducible configuration for this theorem is a signed graph (F, σ_1) with the following property: given any signed graph (G, σ) containing (F, σ_1) as a subgraph, any possible homomorphism of the signed graph induced by $G - F$ to $(K_4, \{e\})$ can be extended to a homomorphism of (G, σ) to $(K_4, \{e\})$.

To prove Theorem 12, we first exhibit a set of reducible configurations (Lemmas 24 to 27). Then, to complete the proof, we show that if a graph G has maximum average degree less than $\frac{8}{3}$, then G must contain at least one of these reducible configurations. This clearly implies that Theorem 12 does not admit any counterexample.

We start with some notation and terminology. In the rest of this section, (G, σ) is considered to be a minimal counterexample to Theorem 12: that is a counterexample with minimum possible number of vertices and among all such counterexamples having a smallest number of edges. That is to say, $\text{mad}(G) \leq \frac{8}{3}$, $(G, \sigma) \not\rightarrow (K_4, \{e\})$, and every signed graph with less vertices than G having maximum average degree less than $\frac{8}{3}$ maps to $(K_4, \{e\})$. Note that, since the maximum average degree is taken over all subgraphs, any subgraph of G has also maximum average degree less than $\frac{8}{3}$. Thus, any proper signed subgraph of (G, σ) maps to $(K_4, \{e\})$, which implies that G must be connected.

Recall that a k -vertex is a vertex of degree k .

Definition 21 (Weak vertex). A *weak vertex* is a 3-vertex adjacent to two 2-vertices.

Definition 22 (3-subgraph, 3-subtree). A *3-subgraph* of a graph G is a connected subgraph S of G all of whose vertices have degree 3 in G and which is maximal for this property. Every vertex of $G - S$ adjacent to a vertex of S is a *neighbour* of S . (Observe that no 3-vertex can be a neighbour of S .) If S is a tree, then S is a *3-subtree* of G .

We will label vertices of the signed graph $(K_4, \{e\})$, with $V(K_4) = \{1, 2, 3, 4\}$ where $e = 14$ is the negative edge (see Figure 4). This signed graph has the following easy to

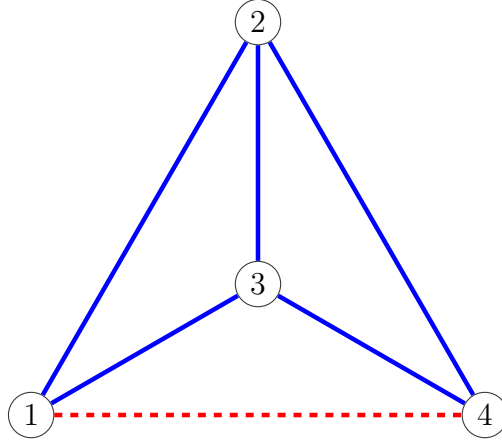


Figure 4: The signed graph $(K_4, \{e\})$

observe properties.

Observation 23. *For every two vertices (not necessarily distinct) x and y of $(K_4, \{e\})$,*

1. *there exists a vertex z such that the walk xzy is positive,*
2. *if $x \in \{a, c\}$ and $y \in \{b, d\}$, then there exists a vertex z such that the path xzy is negative,*
3. *there exist vertices z and z' such that the walk $xzz'y$ is positive,*
4. *there exist vertices z and z' such that the walk $xzz'y$ is negative.*

We now prove a series of lemmas giving a set of reducible configurations.

Lemma 24 (Reducible configuration 1). *The graph G has no 1-vertex, i.e., $\delta(G) \geq 2$.*

Proof. Assume to the contrary that u is a 1-vertex of G , and let v be the neighbour of u . By the minimality of G , there is a homomorphism ϕ of $(G - u, \sigma - \{uv\})$ to $(K_4, \{e\})$. We can then extend this mapping to a mapping of (G, σ) to $(K_4, \{e\})$ as follows. If $uv \in \sigma$, we first resign at u , so that the edge uv is positive. We can then map u to any vertex of $(K_4, \{e\})$ connected to $\phi(v)$ by a positive edge. We thus obtain a homomorphism of (G, σ) to $(K_4, \{e\})$, a contradiction. \square

Lemma 25 (Reducible configuration 2). *The graph G has no pair of adjacent 2-vertices.*

Proof. Assume to the contrary that u and u' are two adjacent 2-vertices in G , and let $G' = G - \{u, u'\}$. By the minimality of G , there is a homomorphism ϕ of (G', σ') to $(K_4, \{e\})$, where σ' is the signature induced by σ on G' . We prove that this mapping can be extended to a mapping of G to $(K_4, \{e\})$.

Let v and v' denote the other neighbour of u and u' , respectively. By Observation 23(3,4), there exist a positive 3-walk and an negative 3-walk from $\phi(v)$ to $\phi(v')$ in $(K_4, \{e\})$. If necessary, we can resign at u , or u' , or both, in such a way that the signs of the edges of the path $vu'u'v'$ and the corresponding, positive or negative, walk in $(K_4, \{e\})$ coincide. By mapping u and u' to the internal vertices of the corresponding walk, we get a homomorphism of (G, σ) to $(K_4, \{e\})$, a contradiction. \square

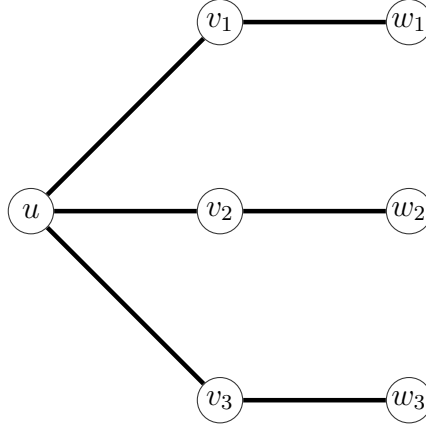


Figure 5: Reducible configuration 3

Lemma 26 (Reducible configuration 3). *The graph G has no 3-vertex adjacent to three 2-vertices.*

Proof. Assume to the contrary that u is a 3-vertex, adjacent to three 2-vertices v_1 , v_2 and v_3 (see Figure 5), and let $G' = G - \{u, v_1, v_2, v_3\}$. By the minimality of G , there is a homomorphism ϕ of (G', σ') to $(K_4, \{e\})$, where σ' is the signature induced by σ on G' . Again, we prove that this mapping can be extended to a homomorphism of G to $(K_4, \{e\})$, leading to a contradiction.

Let w_1 , w_2 and w_3 denote the other neighbour of v_1 , v_2 and v_3 , respectively. We consider two cases.

1. *The three paths uv_1w_1 , uv_2w_2 and uv_3w_3 are all positive or all negative.*

In this case, by resigning at v_1 , v_2 , v_3 and u , if necessary, we may get an equivalent signature where all the six edges incident to v_1 , v_2 and v_3 are positive. We then map u to any vertex of $(K_4, \{e\})$, and extend the mapping ϕ to v_1 , v_2 and v_3 thanks to Observation 23(1).

2. *The three paths uv_1w_1 , uv_2w_2 and uv_3w_3 do not have the same balance.*

In that case, we may resign at u in such a way that only one path, say uv_1w_1 , is negative. If $\phi(w_1) \in \{a, c\}$ (respectively, $\phi(w_1) \in \{b, d\}$), then we map u to b (respectively, to a). The mapping can then be extended to v_1 , thanks to Observation 23(2), and to v_2 and v_3 , thanks to Observation 23(1).

This concludes the proof. □

Lemma 27 (Reducible configuration 4). *The graph G has no 3-subtree T all of whose neighbours (in $G - T$) are 2-vertices in G .*

Consider a 3-subtree T of G which is only adjacent in $G - T$ to 2-vertices. Let T_2 be the subgraph of G containing T and all its adjacent 2-vertices, and $G' = G - T_2$. By the minimality of G , there is a homomorphism ϕ of (G', σ') to $(K_4, \{e\})$, where σ' is the signature induced by σ on G' . Again, we want to extend this mapping to a homomorphism of G to $(K_4, \{e\})$, leading to a contradiction. In what follows, we show that ϕ can be partially extended, with some degree of freedom which is made more precise by the following definitions.

Let Q be a connected (strict) subtree of T such that $T - Q$ is connected, and let $x = \text{end}(Q)$ be the unique vertex of Q having a neighbour in $T - Q$, called the *end vertex* of Q . (For instance, any leaf vertex of T can be chosen as Q .) Let Q_2 be the subgraph of G containing the vertices of Q and the 2-vertices of G adjacent to vertices of Q . Recall that the mapping $\phi = (\phi_1, \phi_2)$ decides, for each vertex v of G' , if there must be a resigning at v (determined by the component ϕ_1), and to which of the four vertices of $(K_4, \{e\})$ the vertex v is mapped to (determined by the component ϕ_2).

Based on the following notation we show that one can extend the mapping ϕ to Q_2 , with a degree of freedom at x . We define the six following sets of images of the mapping ϕ .

$$\begin{aligned} A_1 &= \{(+, 1), (+, 4), (-, 1), (-, 4)\}, & A_2 &= \{(+, 2), (+, 3), (-, 2), (-, 3)\}, \\ A_3 &= \{(+, 1), (+, 4), (-, 2), (-, 3)\}, & A_4 &= \{(+, 2), (+, 3), (-, 1), (-, 4)\}, \\ A_5 &= \{(+, 1), (+, 4), (+, 2), (+, 3)\}, & A_6 &= \{(-, 1), (-, 4), (-, 2), (-, 3)\}. \end{aligned}$$

We now prove the following claim.

Claim. Given a connected subtree Q of T with end vertex x , and a homomorphism ϕ of $G' = G - T_2$ to $(K_4, \{e\})$, we claim that there exists a set A_i , $1 \leq i \leq 6$, such that for any of the four choices $(\alpha, \beta) \in A_i$, the mapping ϕ can be extended to Q_2 in such a way that $\phi(x) = (\alpha, \beta)$.

Proof of the claim. We will prove this claim by the methods of dynamic programming. First we show that the claim holds if Q is just a leaf vertex of T . Then we show that add a vertex u to Q as a new end vertex when u is of degree 2 in T . Finally we show that two end vertices can be merged to form a larger three with a new end vertex.

Suppose first that Q is any leaf u of T , in particular we have $\text{end}(Q) = u$. Since all vertices of T are of degree 3 in G and all neighbours of T outside T are of degree 2, u is a weak vertex of G . Let v_1 and v_2 denote the two neighbours of u not in T . Let $H = \{u, v_1, v_2\}$ and let w_1, w_2 be the other neighbours of v_1 and v_2 in G , respectively. Depending on the signs of the paths uv_1w_1 and uv_2w_2 , we consider two cases.

1. *The paths uv_1w_1 and uv_2w_2 are both positive or both negative.*

In the former case, by resigning at v_1 , or at v_2 , or at both v_1 and v_2 , we may assume that the four edges of these two paths are positive. Then, any choice of $\phi_2(u)$ (without resigning at u) can be extended to v_1 and v_2 by Observation 23(1). Thus, $\phi(u)$ can be any member of A_5 . In the latter case, after resigning at u , we are in the former case, and thus $\phi(u)$ can be any member of A_6 .

2. *The paths uv_1w_1 and uv_2w_2 do not have the same sign.*

Without loss of generality, assume uv_1w_1 is negative. If w_1 is mapped to 1 or 4, then each choice of $\phi(u) = (+, 2)$ and $\phi(u) = (+, 3)$ can be extended to v_1 and v_2 . If w_1 is mapped to 2 or 3, then each choice of $\phi(u) = (+, 1)$ and $\phi(u) = (-, 1)$ can be extended to v_1 and v_2 . But then, after a resigning at u , we may exchange the role of w_1 with w_2 to get two more possibilities for u . Therefore, depending on the values of $\phi_2(w_1)$ and $\phi_2(w_2)$, one of the four sets A_1, A_2, A_3 or A_4 works.

Suppose now that the claim holds for every connected subtree of T of order at most $k \geq 1$. Let Q be a connected subtree of T of order $k + 1$ with end vertex x , and ϕ be a homomorphism of $G' = G - T_2$ to $(K_4, \{e\})$.

To proceed with our goal, given a subtree Q with end vertex x which satisfies our claim, we want to extend Q to have a new end vertex which also satisfies our claim. This is done in the following setting: Let u be a 2-vertex of T and x be a neighbour of u . Let Q be the subtree of T such that $\text{end}(Q) = x$ and $u \notin Q$. Observe that u is the end vertex of the subtree $Q^+ = Q + \{u\}$. We claim that if x satisfies our claim as the end vertex of Q , then u also satisfies our claim as the end vertex of Q^+ .

Proof. Since all vertices of T are 3-vertices of G , u has one neighbour in G which is not in T , we call this vertex v . As we have assumed all neighbours of T are 2-vertices, v is a vertex of degree 2 in G , thus it has another neighbour which we name it w . We note that w could be a vertex of T or Q , but even then we will assume, in the rest of the proof, that $\phi(w)$ is already given. Depending on the balance of the path $xuvw$ and which four choices are available for x we have four possible cases:

Case 1 - $xuvw$ is positive. We first resign at u or v or both, if needed, such that all the three edges of the path $xuvw$ are positive. Then first we assume that of the four choices for x two are without a resigning at x , i.e., choices of the form $(+, i)$. For any such choice of $(+, i)$ all three choices of $(+, j)$, $j \neq i$ for u are then extendable at v , thus if we have not resigned at u , then A_5 works and if we have then A_6 works. If our assumption does not hold, then A_6 is the set of four choices for x . In such a case for each choice of $(-, i)$ the choice of color i for u is also extendable to v . Thus again depending on whether we have already resigned at u or not, the set A_5 or A_6 would work.

Case 2 - $xuvw$ is negative. In this case we may resign at u or v or both, if needed, such that ux is a negative edge and it is the only negative edges of the path $xuvw$. Then we consider a set of four possible choices for x . If for and $i \in \{1, 2, 3, 4\}$ we have $(-, i)$ as a choices, then every choice of $(+, j)$, $j \neq i$ for u can be extended to v as well. Observe that if for some j we have $(-, j) \in A_i$, then there is a $j' \neq j$ such that $(-, j')$ is also in A_i . Thus in this case depending on whether we have originally resigned at u or not, A_5 or A_6 is a collection of four possible choices for u . Finally if there are no negative choice available for x , then the corresponding set of four possible choices for x are the four members of A_5 . In this case, with respect to current signature, for each choice of $(+, i)$ for x , assigning u to same vertex, i.e., i , can be extend to v . Thus, again depending on whether we have already resigned at u or not the set A_5 or A_6 provide a list of four possible choice for u .

Next we want to merge to subtrees with distinct end points to get a larger subtree with a new end point, and to show that if we had the four choices (from sets A_i and A_j) for each of the two previous end points, then we have such a set of choices for the new one as well. This is done with the following precise notation: Let y be a 3-vertex of T , with vertices x_1 and x_2 of T being two of its neighbours. Let Q_1 and Q_2 be the subtrees of T with $\text{end}(Q_1) = x_1$, $\text{end}(Q_2) = x_2$, $y \notin Q_1$ and $y \notin Q_2$. Observe that y is an end vertex of the subtree Q induced by Q_1 , Q_2 and y . Assume that for a coloring ϕ of G' , (recall $G' = G - T_2$), there is set of A_i ($i \in \{1, 2, \dots, 6\}$) of four possible choices for x_1 with which ϕ can be extended to Q'_1 and a set A_j ($j \in \{1, 2, \dots, 6\}$) of four possible choices for x_2 with which ϕ can be extended to Q'_2 . Then we claim that there is a set A_l , $1 \leq l \leq 6$, such that any choice of an element of A_l for $\phi(y)$ can be extend to the Q' .

To prove this, first suppose that the corresponding set for one of x_1 or x_2 , say x_1 , is A_i for some $i \leq 4$. That means we have two choices of colors for x_1 without resigning at x_1 and two choices together with resigning. Thus over all, we have two choices of distinct colors for each of x_1 and x_2 such that the path x_1yx_2 is positive. Thus for each choice of color for y we can make it distinct from a choice of color for x_1 and choice of color for x_2 and such that the sign of the edges x_1y and yx_2 remain the same. Thus if the sign of these edges are positive, then A_5 is a set of possible choices for y and otherwise A_6 is such a choice set for y .

Thus we may assume that the set of four choices for each of x_1 and x_2 is either A_5 or A_6 . In such a case, after each of such assignment to x_1 and x_2 each of the two edges, x_1y and yx_2 , have a fixed sign. If these fixed signs matches i.e., if the path x_1yx_2 is positive, then A_5 or A_6 is a set of possible choices for y as before. Otherwise, one of the two edges, say x_1y is negative and the other, yx_2 remains positive. Then for each choice of (α, i) for x_1 , the choice of $(+, i)$ works with some choice for x_2 , thus in this case A_5 is a set of choices for y .

This concludes the proof of the claim.

Finally we are ready to prove that a 3-subtree of G all whose neighbours are 2-vertices of G is a reducible configuration.

If T has only one vertex, then this is done in reducible configuration 3. Thus we may assume that T has at least one edge xy . Let assume that x is an end vertex of Q_x , $y \notin V(Q_x)$, and that y is an end vertex of Q_y , $x \notin V(Q_y)$. We note that Q_x and Q_y each can be constructed starting from leaf vertices of T by the operations of adding degree 2-vertices of T or merging at degree 3 vertices as we described above. Thus there are sets A_i and A_j such that for each choice of $\phi(x)$ from A_i and $\phi(y)$ from A_j we have extension which is ok for all edges except possibly for the edge xy . Observe that for each element (α, l) in A_i there is an element (α, l') , $l \neq l'$, in A_i . Thus if for some such a choice of $\phi(x)$ and $\phi(y)$ the edges xy is of positive sign, then we may simply change the color of one end, if necessary, to have all edges satisfied. Otherwise, xy is a negative edge and we must have $i, j \in 5, 6$. In such a case then it would be enough to choose $\phi(x)$ and $\phi(y)$ vertices 1 and 4 of the K_4 .

Lemma 28. *Let G be a graph. If $\text{mad}(G) < \frac{8}{3}$, then G contains one of the following configurations:*

C1: A 1-vertex.

C2: Two adjacent 2-vertices.

C3: A 3-vertex adjacent to three 2-vertices.

C4: A 3-subtree S whose neighbors are all 2-vertices of G .

Proof. We prove this lemma with a discharging argument. Suppose that this lemma is false, and let G be a counterexample, i.e., a graph with $\text{mad}(G) < \frac{8}{3}$ and with no subgraph isomorphic to configurations C1, C2, C3 or C4. We assign to each vertex v of G an *initial charge* which equals to $\mu(v) = d(v) - \frac{8}{3}$. As $\text{mad}(G) < \frac{8}{3}$, the sum of the initial charges of the graph is negative.

Then we redistribute these charges, according to several rules given below, verifying that the whole charge on the graph does not change. After this discharging, we compute

the *final charge* $\mu^*(v)$ at each vertex v and prove that the sum of all the final charges on the graph $\mu^*(G)$ is non-negative. This contradiction completes the proof.

Here are the discharging rules:

R1 - Each 3^+ -vertex gives $\frac{1}{3}$ to each adjacent 2-vertex.

R2 - Each 4^+ -vertex adjacent to a 3-vertex gives it $\frac{1}{3}$.

- Let v be a vertex with $d(v) = 2$. We have $\mu(v) = -\frac{2}{3}$, and since G has no C2 configuration, v is adjacent to two 3^+ -vertices. Thus v receives $2 \times \frac{1}{3}$ and, by R1, we have $\mu^*(v) = 0$.
- Let v be a vertex with $d(v) = 3$ which has no neighbour of degree 3, that is to say the 3-subgraph containing v consists of v only. We have $\mu(v) = \frac{1}{3}$ and since G has no (C3), v has at most two neighbours of degree 2 and so gives away at most $\frac{2}{3}$ (in the rule R1). Moreover, v receives at least $\frac{1}{3}$ by R2, thus $\mu^*(v) \geq 0$.
- Let v be a vertex with $d(v) \geq 4$, we have $\mu^*(v) \geq d(v) - \frac{8}{3} - \frac{d(v)}{3}$ by Rule R2, so $\mu^*(v) \geq \frac{2d(v)-8}{3} \geq 0$.

Recall that a 3-subgraph is a connected component of the subgraph induced by 3-vertices of G . We have already seen that the charge of each vertex not in this subgraph and the charge of each isolated vertex of this subgraph is positive. To complete the proof we show that the sum of charges on each component of this subgraph is also positive, this would mean that the total charge is positive, contradicting our assumptions.

Let S be such a component of the subgraph induced by 3-vertices, thus S has at least two vertices. Let v be a vertex S . Let us denote its degree in the subgraph S by $d_S(v)$. We consider all three possibilities:

- If $d_S(v) = 3$, then v is adjacent to three 3-vertices in G and $\mu^*(v) = \frac{1}{3}$.
- If $d_S(v) = 2$, then v is adjacent to two 3-vertices in G .
 - Either the third neighbour of v in G has degree 2, and $\mu^*(v) = 0$.
 - Or it has degree at least 4, and $\mu^*(v) = \frac{2}{3}$.
- If v is a leaf of S , i.e. $d_S(v) = 1$, then if v has two neighbours that are 2-vertices, then $\mu^*(v) = -\frac{1}{3}$, otherwise $\mu^*(v) \geq 0$.

For $i \in \{1, 2, 3\}$, we denote by $n_i(S)$ the number of vertices v with $d_S(v) = i$, and by $\mu_i^*(S)$ the sum of the charges on those vertices. Since G has no C4 configuration, either S is not a tree (case 1), or one leaf of S is not a weak vertex (case 2), or at least one internal vertex of S is adjacent to a 4^+ -vertex (case 3).

To complete the proof we prove that for every connected (multi)-graph S of maximum degree at most 3, $n_3(S) \geq n_1(S) - 2$ and that if S is not a tree, then $n_3(S) \geq n_1(S)$. This claim can be easily proved by induction on $|S|$. It is easily verified for small values of $|S|$. Assuming it is true $|S| \leq k$, we consider a graph on $k + 1$ vertices. If S has a vertex of degree 2, we contract one of the edges incident to it, the result has the same number of

vertices of degree 3 and 1 as the original graph, and it is still of maximum degree at most 3, so we are done. Otherwise, we only have vertices of degree 1 and 3. As S is connected and not a K_2 , each vertex of degree 1 is adjacent to a vertex of degree 3. If each vertex of degree 3 is adjacent to at most one vertex of degree 1, then we have $n_3(S) \geq n_1(S)$, otherwise a vertex x of degree three has two neighbours u and v each of degree 1. Delete both u and v , then now x is of degree 1, we have thus one less degree 3 and one less degree 1 vertices in the new graph. The claim then follows from inductive assumption.

Recall that $\mu^*(S) = \mu_3^*(S) + \mu_2^*(S) + \mu_1^*(S)$, and that $\mu_2^*(S) = 0$. We consider the possible cases:

Case 1 - In this case we have $\mu^*(S) \geq \frac{1}{3}n_3(S) + 0 - \frac{1}{3}n_1(S)$, and since S is not a tree, we have $\mu^*(S) \geq 0$.

Case 2 - In this case S is a tree, but a neighbour u of S is not of degree 2. As S is a connected component induced by degree 3 vertices, u is of degree at least 4. Thus first of all vertices in S send a maximum charge of $\frac{1}{3}(n_1(S) - 1)$ to their neighbours by R1 but then they receive a charge of $\frac{1}{3}$ from u . Therefore, over all we have $\mu^*(S) \geq \frac{1}{3}n_3(S) + 0 - \frac{1}{3}(n_1(S) - 1) + \frac{1}{3} = \frac{1}{3} \times (n_3(S) - (n_1(S) - 1) + 1)$, which is a nonnegative value as $n_3(S) \geq n_1(S) - 2$.

Case 3 - Let u be the internal vertex of S which is adjacent to a 4^+ -vertex of G . Then in the count of $n_3(S)$, $n_2(G)$ and $n_1(S)$, the vertex u is counted as a degree 2 vertex. However, following the discharging rules, not only u does not lose a charge but it also receives a minimum charge of $\frac{1}{3}$ from its 4^+ -neighbour. Thus to total charge on S is at least $\frac{1}{3}n_3(S) - \frac{1}{3}n_1(S) + \frac{2}{3} \geq 0$.

This concludes the proof of this lemma. We may now prove Theorem 12. □

Proof of Theorem 12. By contradiction, let (G, σ) be a counterexample to our claim with a minimum possible number of vertices. Since we have $\text{mad}(G) < \frac{8}{3}$, and by the Lemma 28, G must have one of the reducible configurations. However, this contradicts our earlier proofs that G cannot contain any of these reducible configurations. □

7 Mapping sparse signed bipartite graphs into unbalanced even cycles

Recall that the study of homomorphism of signed bipartite graphs captures the study of homomorphisms of graphs as a special case. Thus the restriction of the study on the class of bipartite graphs is of high interest. In the world of signed bipartite graphs, for (H, π) to admit a homomorphism from any signed bipartite graph (G, σ) with G having maximum average degree less than $2 + \epsilon$, where $\epsilon > 0$ depends on H , it is necessary and sufficient for H to have an unbalanced cycle. Here, proving Theorem 13, we give a lower bound on the best values of ϵ . Let us first recall the theorem.

Theorem 29. *For every bipartite graph G with $\text{mad}(G) < 2 + \frac{1}{2g-1}$ and for every signature σ of G such that unbalanced-girth of (G, σ) is at least $2g$ we have $(G, \sigma) \rightarrow (C_{2g}, \{e\})$.*

Proof. Note that as a signed graph but not a 2-edge-colored graph, the unbalanced-cycle of length l , UC_l , is vertex-transitive. Let (G, σ) be a minimum counterexample to our claim. Thus G is a bipartite graph of maximum average degree at most $2 + \frac{1}{2g-1}$, the unbalanced-girth of (G, σ) is at least $2g$, but (G, σ) does not map to UC_{2g} . Since (G, σ) is of unbalanced-girth at least $2g$, and by Lemma 9, G is not just a cycle. Our next conclusion is that G must be 2-connected: otherwise either G has a connected component G_1 which is a proper subgraph, or G has a 2-connected end block G_2 which is connected to the rest of the graph at a vertex x . By minimality of G , induced signed graph on G_1 or on G_2 maps to UC_{2g} . Similarly, if we remove all vertices of G_1 , or if we remove all vertices of G_2 except x , then the induced signed graph also maps to UC_{2g} . Note that to resign at a set X of vertices of any signed graph is the same as resigning at the complement of X . Since G_2 has only one common vertex with the rest of the graph and G_1 has non, in mapping of the two parts we may choose resignings that coincide on the possible common vertex. Finally since UC_{2g} is vertex-transitive, by composing one of the two homomorphisms with an automorphism of UC_{2g} we may assume that x is mapped to a same vertex by the two mappings. Hence they can be composed to a homomorphism of (G, σ) to UC_{2g} .

That G must be 2-connected implies, in particular, that it has minimum degree at least 2. A vertex of degree 2 then is an internal vertex of a maximal thread. Our next claim is that no such thread can have a length $2g - 1$ or higher. Assume to the contrary that G has a thread T of length exactly $2g - 1$ (not necessarily a maximal thread) with x and y as its end points. Let G' be the subgraph obtained from deleting all internal vertices of this thread and let σ' be the signature induced by σ on G' . Observe that since G was 2-connected, G' is a connected graph. By minimality of (G, σ) , the signed bipartite graph (G', σ) maps to UC_{2g} . Let ϕ be such a mapping. Since G' is connected, ϕ , as a mapping of underlying bipartite graph G' to bipartite graph C_{2g} , must preserve bipartition. Thus $\phi(x)$ and $\phi(y)$ partition UC_{2g} into two paths of odd length: one containing the negative edge, the other composed of only positive edges. Then depending on the parity of the number of negative edges of T and after a suitable resigning of internal vertices of it, we may extend the mapping ϕ to the vertices of T , thus mapping (G, σ) to UC_{2g} which contradict the choice of (G, σ) .

Next we consider vertices of degree 3. Let v be a 3-vertex of G and let x , y and z be the three ends of the maximal threads whose other end is v . Let l_x , l_y and l_z be the corresponding lengths of these threads. We claim that $l_x + l_y + l_z \leq 4g - 1$. Suppose the contrary and let G' be the subgraph obtained from G by removing v and all the internal vertices of the three threads $x - v$, $y - v$ and $z - v$. Let σ' be the induced signature on G' . By the minimality of G , we have a mapping $\phi = (\phi_1, \phi_2)$ of (G', σ') to UC_{2g} . Considering the size of the set $\{\phi_2(x), \phi_2(y), \phi_2(z)\}$ we have three possible cases:

First we consider the case when this set is of size one, that is to say $\phi_2(x) = \phi_2(y) = \phi_2(z)$. In this case we consider the image of G after applying ϕ on the subgraph G' . The result then is a graph obtained as from an unbalanced cycle of length $2g$ and the vertex v which is joined to a vertex $(\phi_2(x))$ of the cycle by three internally vertex disjoint paths of length l_x , l_y and l_z whose signature is induced by signature of (G, σ) and $\phi_1(x) = \phi_1(y) = \phi_1(z)$. Of these three paths then two are of a same sign and, since our graph is bipartite, all three of them are of same parity. Thus we can map the larger of the two to the smaller one. Noting the this larger thread is not of length greater than $2g$ and that the sum

three lengths were at least $4g$, the cycle formed by the two remaining threads between v and $\phi_2(x)$ must be of length at least $2g$. We may now use vertex transitivity of UC_{2g} to complete our mapping as in the previous case.

In the case that $|\{\phi_2(x), \phi_2(y), \phi_2(z)\}| \geq 2$ we similarly use ϕ to build a homomorphic image of (G, σ) , then we show that the image itself can be mapped to UC_{2g} . In this image first of all we apply the resigning of ϕ_1 and then any pair of vertices that are identified in the mapping ϕ_2 are also identified, no further resigning or identification is done. Let (G^*, σ^*) be the image. Since G is connected and ϕ_2 preserves bipartition, the image, G^* is also bipartite. Furthermore, any cycle using v must use two of the threads incident to it, and thus is of length at least $2g$. The only cycle not using v is an unbalanced cycle of length exactly $2g$. Depending on if $\{\phi_2(x), \phi_2(y), \phi_2(z)\}$ is of size 3 or 2, the signed graph (G^*, σ^*) satisfies the conditions of the Lemma 19 or Lemma 20 (respectively). Thus in both case (G^*, σ^*) and therefore, (G, σ) maps to UC_{2g} .

Finally to complete our proof we show that the four conditions of (1) G is 2-connected, (2) G is not cycle (3) it has no thread of length $2g - 1$ or longer, (4) it has no 3-vertex whose total length of incident threads is larger than $4g - 1$, would imply an average degree of at least $2 + \frac{1}{2g-1}$ which contradicts our assumption on maximum average degree.

We use discharging technique for this step. Assign the degree of each vertex of G as a charge to this vertex. The average charge then is the average degree of G . For each vertex v of degree 2 consider the two ends x_v and y_v of the thread to which v belongs. Note that x_v and y_v do exist because G is not a cycle, and that x_v and y_v are distinct and that each have degree at least 3 because G is 2-connected (but not a cycle). We then do a discharging by the following rule:

discharging rule: Each vertex of degree 2 receives a charge of $\frac{1}{2(2g-1)}$ from each end of the thread it belongs to.

We claim that after applying this rule, all vertices have a charge of at least $2 + \frac{1}{2g-1}$ proving that G has average degree at least as much. We consider three possibilities for a given vertex based on its degree.

A vertex v of degree 2. The original charge of v is 2. Then, as mentioned above, there are two distinct vertices as the ends of the thread containing v . Each of them gives a charge of $\frac{1}{2(2g-1)}$ by the discharging rule. As vertex of degree 2 does not lose any charge, its final charge is increased to $2 + \frac{1}{2g-1}$.

A vertex x of degree 3. The original charge of x is 3. It may give a charge of $\frac{1}{2(2g-1)}$ to each vertex v which is on a thread with x as an end point. As the total length of the three threads with x as an end point is at most $4g - 1$, there are a maximum of $4g - 4$ number of such degree 2 vertices. Thus x loses a maximum charge of $(4g - 4) \times \frac{1}{2(2g-1)} = \frac{2g-2}{2g-1}$, leaving it with a charge of at least $3 - \frac{2g-2}{2g-1} = 2 + \frac{1}{2g-1}$.

A vertex y of degree k , $k \geq 4$. The original charge of y is k . The vertex y is an end point of k threads, each of length at most $2g - 1$. Thus in total y may give a charge of $\frac{1}{2(2g-1)}$ to a maximum of $k(2g - 2)$ vertices. Hence y loses at most a charge of $k(2g - 2) \times \frac{1}{2(2g-1)} = \frac{k(g-1)}{2g-1}$, leaving it with a charge of at least $k - \frac{k(g-1)}{2g-1} = \frac{kg}{2g-1}$ which is larger than $2 + \frac{1}{2g-1} = \frac{4g-1}{2g-1}$ for $k \geq 4$.

Thus at the end each vertex has a charge of $2 + \frac{1}{2g-1}$ or higher, implying the average charge, or, equivalently, the average degree is at least as much, contradicting our assumption that the maximum average degree of G is strictly less than $2 + \frac{1}{2g-1}$. \square

The result of this theorem is not tight. For the case of $g = 1$, allowing multi-edge, the unbalanced cycle of length 2 is the digon. It is then straight forward to check that any signed bipartite multi-graph maps to UC_2 .

For $g = 2$ while Theorem 29 implies that any signed bipartite graph (G, σ) of with no digon maps to UC_4 as long as G has maximum average degree less than $\frac{7}{3}$ an improved and tight bound of $\frac{8}{3}$ is proved below. Note that unlike the exceptional case of $g = 1$, in general case of the theorem we do not consider multi-graphs as any such of edges must be of same sign to satisfy the unbalanced-girth condition and the existence of such a homomorphism depends on only the underlying simple signed graph.

Theorem 30. *Let G be bipartite graph of maximum average degree less than $\frac{8}{3}$ and let σ be a signature of G . Then (G, σ) maps to UC_4 . Furthermore, the bound of $\frac{8}{3}$ cannot be improved.*

Proof. As G is a graph of maximum average degree less than $\frac{8}{3}$, and by Theorem 12, (G, σ) maps to $(K_4, \{e\})$. Then, as G is bipartite, by Theorem 16 it also maps to $(C_4, \{e\})$. An example of a bipartite graph of maximum average degree $\frac{8}{3}$ which does not map to $(C_4, \{e\})$ is given in Section 9. \square

For larger values of g , i.e., $g \geq 3$, we do not know the optimal bound on maximum average degree by which Theorem 13 would be valid and leave this as an open question.

One may also consider subclasses of signed bipartite graphs. One of special interest is the class of signed bipartite planar graphs. In Conjecture 14 we have introduced a condition on unbalanced-girth of signed bipartite planar graphs which may imply existence of a mapping to UC_{2g} . Using the folding lemma, Lemma 17 or it corollary, Corollary 18, and the Euler formula this can translate a condition on average degree. We will show on Section 9 that the proposed bound of this conjecture, if true, is tight.

8 Application on planar graphs

For the class planar signed graphs, a class of graphs of maximum average degree strictly smaller than 6, the first question is whether there exists a signed graph to which every planar signed graph admits a homomorphism? Using techniques developed in [RS94, AM98] such a signed graph of order 48 is built in [13] where a lower bound of 10 on the number of vertices of such a bound is presented as well. The upper bound of 48 is improve to 40 in [15] where they have, furthermore, eliminated all but one graph on 10 vertices as a candidate for a bound on 10 vertices (up to equivalence of signatures). This unique (up to equivalence of signatures) signed graph is a signed graph on K_{10} where the signature (i.e., the set of negative edges) induces a graph on 9 vertices which is isomorphic to the Cartesian product $K_3 \square K_3$. This signed graph on 10 vertices then seems to be a natural candidate and it is tempting to believe that 10 is the right number.

Here we address the problem for planar graphs of large girth, which, combined with Euler formula, is a translation of maximum average degree condition for planar graphs.

Recall that by the Euler formula for any connected planar graph G we have $n - e + f = 2$ where n is the number of vertices of G , e is the number of edges and f is the number of faces in a planar embedding of G . For the sake of our study we make assume G is 2-connected which implies each edge is exactly on 2 distinct faces (counting the outer face). If in planar embedding of G each face is of length at least g , then there are at least $\frac{f \times g}{2}$ edges in G , or, equivalently, $f \leq \frac{2e}{g}$, thus $n - e + \frac{2e}{g} \geq 2$, using the fact that $\frac{2e}{n}$ is the average degree of G we conclude that the average degree of G is strictly less than $\frac{2g}{g-2}$. This is the proof of the following folklore fact:

Property 31. *If G is a 2-connected planar graph whose faces are all of length at least g , then G has average degree strictly less than $\frac{2g}{g-2}$.*

Thus we have the following corollary:

Corollary 32. *Let G be a planar graph where each face is of length at least 8 and let σ be any signature of G , then $(G, \sigma) \rightarrow (K_4, e)$.*

Furthermore, combined with Corollary 18, we have the followings.

Corollary 33. *Let G be a planar bipartite graph, let σ be a signature of G such that unbalanced-girth of (G, σ) is at least 8. Then $(G, \sigma) \rightarrow UC_4$.*

Corollary 34. *Let G be a planar bipartite graph, let σ be a signature of G such that unbalanced-girth of (G, σ) is at least $8k - 2$, then we have $(G, \sigma) \rightarrow UC_{2k}$.*

This Corollary is in support of Conjecture 14 where we conjecture that unbalance-girth at least $4k - 2$ is enough to map a signed bipartite planar graph to UC_{2k} . If true, then the condition of unbalanced-girth at least $4k - 2$ is the best possible as shown by the examples in Section 9.

For the special case of $k = 3$ Conjecture 14 claims that any planar signed bipartite graph of unbalanced-girth at least 10 admits a homomorphism to UC_6 . Further support for this case together with connection to well know Grötzsch theorem is as follows: Let G be a triangle-free planar graph and let $\bar{\bar{G}}$ be graph obtained from subdividing each edge exactly one. Observe that the result is indeed a planar bipartite graph. Let σ be a set of edges of $\bar{\bar{G}}$ such that for each original edge of G exactly one of the two corresponding edges in $\bar{\bar{G}}$ is in σ . Then the signed $(\bar{\bar{G}}, \sigma)$ is a signed planar bipartite graph which is of unbalanced-girth 10 because each unbalanced cycle of $(\bar{\bar{G}}, \sigma)$ is a subdivision of an odd-cycle of G and G is assumed to be triangle-free. A mapping of G to K_3 , provided by the Grötzsch theorem, can then be extended to a mapping of $(\bar{\bar{G}}, \sigma)$ to UC_6 . However, it is not clear if the case $k = 3$ of our conjecture would imply the Grötzsch theorem. We leave this as an open problem.

9 Tightness

Consider a K_4 and let M_1, M_2 and M_3 be partition of its edges into three perfect matchings. Subdivide edges in M_1 each one, noting that the resulting graph is bipartite, and then build a signed graph assigning two edges of M_2 the negative sign. Let Ω_4 be the resulted signed graph which is depicted in Figure 6, we will use labeling of vertices as

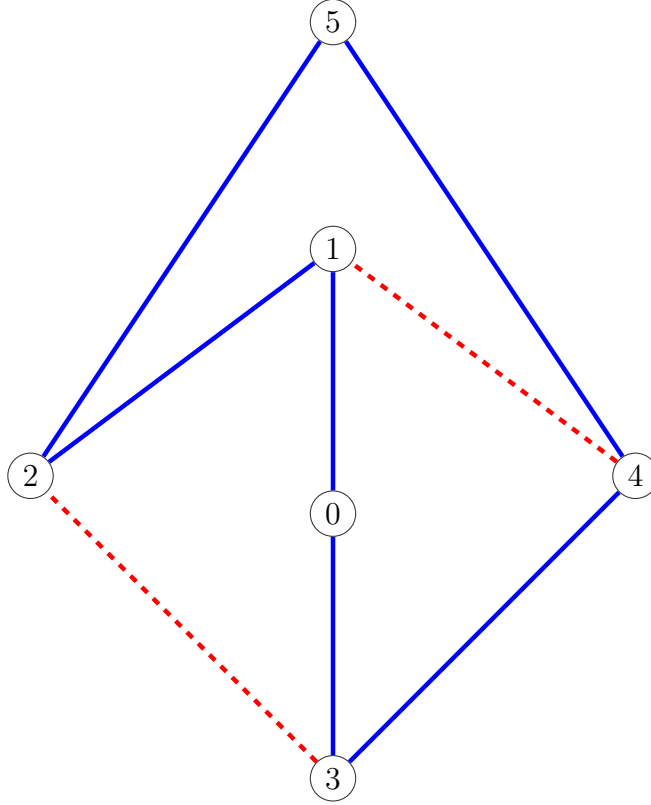


Figure 6: Graph Ω_4 (Lemma ??)

given in this figure. We prove here that this signed graph is of maximum average degree $\frac{8}{3}$ which admits no homomorphism to neither (K_4, e) nor (C_4, e) , proving both Theorem 38 and Theorem ?? are tight.

Proofs of claims are based on following simple lemma:

Lemma 35 ([?]). *Given unbalanced cycles UC_l and UC_k we have $UC_l \rightarrow UC_k$ if and only if we have*

- $l = k \pmod{2}$
- $l \geq k$

Lemma 36. *The signed graph Ω_4 has no homomorphism to UC_4 .*

Proof. We consider UC_4 as a 4-cycles with exactly one negative edge. The graph Ω_4 is planar and has four faces each isomorphic to UC_4 . By Lemma 35, any mapping of each of these faces to UC_4 must be onto and thus must have exactly one negative and three positive edges. Toward a contradiction assume there is a homomorphism of Ω_4 to UC_4 under an equivalent signature σ . Then σ must contain exactly one edges of each facial cycle. The only choices for that are either two edges of M_2 , or two edges of M_3 or two non adjacent edges obtained from subdivision of M_1 . But in any of these cases two end vertices of negative edges are adjacent by a positive edges and form an obstacle given in Theorem 15 for mapping to UC_4 . \square

Lemma 37. *Graph Ω_4 is of maximum average degree $\frac{8}{3}$.*

Proof. Our graph has six vertices and eight edges, thus the average degree of the whole graph is $\frac{8}{3}$. Using the labeling of vertices as in Figure 6 if one of vertices x or y is deleted, the remaining graph has average degree $\frac{12}{5}$. We claim that any other proper subgraph has average degree at most 2. To this end note that a vertex of degree 1 cannot contribute for exceeding the maximum average degree past 2. One can then easily check that if by the process of removing vertices of degree one from a proper subgraph which is not one of the two mentioned subgraphs one either gets a cycle or single vertex. \square

Corollary 38. *There exists a signed bipartite graph of maximum average degree $\frac{8}{3}$ which does not map $(C_4, \{e\})$.*

Since Ω_4 is a signed bipartite graph, and by Theorem 16 we have:

Corollary 39. *There exists a signed graph of maximum average degree $\frac{8}{3}$ which does not map $(K_4, \{e\})$.*

Next we show that the claim of Conjecture 14, if true, is also tight.

Theorem 40. *There exists signed planar bipartite graph of unbalanced-girth $4k - 4$ which does not map to UC_{2k} .*

Proof. Label the vertices of the unbalanced cycle UC_{4k-4} by $v_1, v_2, v_3, \dots, v_{4k-4}$ and consider a circular drawing on the plane. Add a vertex u in the center, then for each vertex v_{2j} , $j = 1, 2, \dots, 2k - 2$, (vertices of even index) join it to u by two internally disjoint paths of length $2k - 2$ and assign to one of the $4k - 4$ resulting edges a negative sign. Call the resulting signed graph W_k , see Figure 7 for depiction of W_2 . It is easily observed that W_k is a signed bipartite planar graph of unbalanced-girth $4k - 4$, in fact the girth of the underlying graph is $4k - 4$. We claim that it does not admits a homomorphism to UC_{2k} . By contradiction suppose ϕ is a mapping of W_k to UC_{2k} . First observe that ϕ must preserve bipartite as the image is bipartite. Thus among vertices $v_1, v_2, v_3, \dots, v_{4k-4}$ those of even index map to a same part of UC_{2k} and those of odd index map to the other part. Furthermore, as u is of even distance vertices of even index it must map to the same part as vertices of even index. Second observation is that since UC_{4k-4} is an unbalanced cycle, the restriction of ϕ onto this cycle must be surjective. Thus the image of u , $\phi(u)$ must be also the image of a vertex v_{2j} for some j , $1 \leq j \leq 2k - 2$. Then in the image of unbalanced $(4k - 4)$ -cycle build on u and v_j there must be an unbalanced cycle of length at most $2k - 2$. But this contradicts Lemma 35. \square

10 Concluding remarks

This is a first work in the study of homomorphisms of sparse signed graphs. A main result here was to show that given any signed graph (H, π) there exist a positive ϵ such that any signed graph satisfying $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ and $MAD(G) \leq 2 + \epsilon$ maps to (H, π) . We determined a best value of epsilon for (H, π) being (K_4, e) and for (C_4, e) . We believe geometric condition such as planarity can help proving parallel results.

The study generalize various questions from graphs coloring. For example one can define (H, π) -critical graph to be a graph satisfying $g_{ij}(G, \sigma) \geq g_{ij}(H, \pi)$ such that (G, σ)

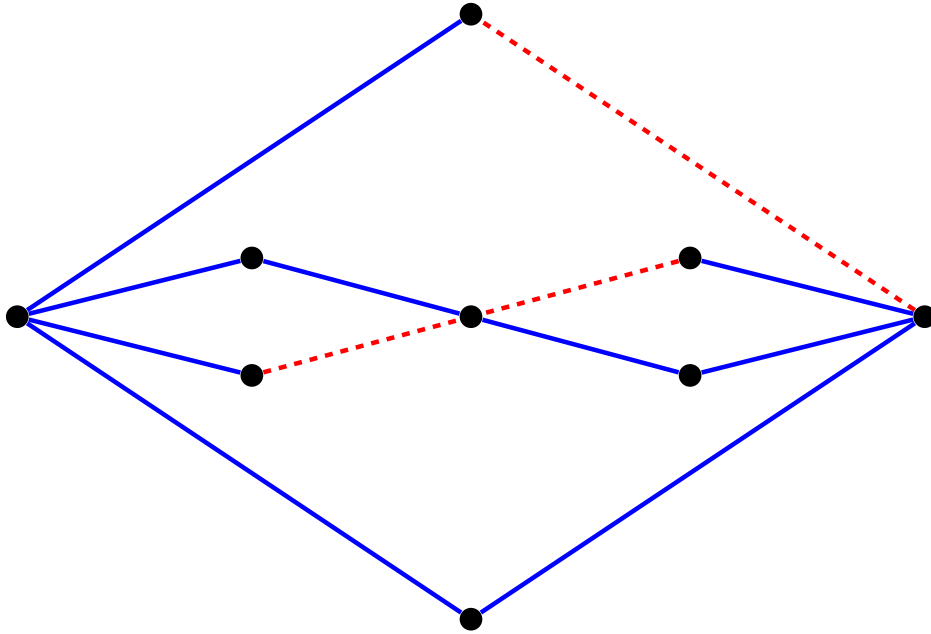


Figure 7: Graph W_2 of Theorem 40.

does not admit a homomorphism to (H, Π) but any subgraph of it maps to (H, π) . Extending the notion of *excess* of k -critical graphs, a natural question then is to give a lower bound on the number of edges of an (H, π) -critical graph. In particular, considering Theorem 6.2 of [13], the case when (H, π) is a complete bipartite graph where a perfect matching is negative is of special interest.

On the other hand, when (H, π) is a signed projective cube, as discussed in [12], the condition of planarity in place of maximum average degree conjectured to be a sufficient. This conjecture directly generalizes the four-color theorem and is in connection with the study of various notions of coloring of planar graphs as discussed in [11].

Acknowledgement. This work is supported by the French ANR project HOSIGRA (ANR-17-CE40-0022).

References

- [AM98] N. Alon and T.H. Marshall. Homomorphisms of edge-colored graphs and Coxeter groups. *J. Algebraic Combin.* **8(1)** (1998), 5–13.
- [1] J.A. Bondy and U.S.R. Murty. *Graph Theory*. Graduate Texts in Mathematics, Vol. 244, 3rd Ed., Springer, London (2011).
- [2] R.C. Brewster. Vertex colourings of edge-coloured graphs. Ph.D. thesis (1993), Simon Fraser University, Canada.
- [3] R.C. Brewster, F. Foucaud, P. Hell and R. Naserasr. The complexity of signed graph and edge-coloured graph homomorphisms. *Discrete Math.* 340 (2017), no. 2, 223–235.

- [4] C. Charpentier, R. Naserasr, E. Sopena and T. Zaslavsky. Homomorphisms of signed graphs: An update, *in preparation*.
- [5] D.W. Cranston, D.B. West. An introduction to the discharging method via graph coloring. *Discrete Math.* 340 (2017), no. 4, 766–793.
- [6] T. Feder and M.Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: a study through datalog and group theory. *SIAM J. Comput.* 28 (1998), no. 1, 57–104.
- [7] F. Foucaud and R. Naserasr. The complexity of homomorphisms of signed graphs and signed constraint satisfaction. In: Proc. 11th Latin American Symposium on Theoretical Informatics 2014, LATIN 2014, *Lecture Notes Comput. Sci.* 8392 (2014), 526–537.
- [8] F. Jaeger. On circular flows in graphs. In: *Finite and Infinite Sets* (Eger, 1981), Colloquia Mathematica Societatis Janos Bolyai, Vol. 37, North-Holland, Amsterdam, 1984, pp. 391–402.
- [9] W. Klostermeyer and C.Q. Zhang. $(2 + \varepsilon)$ -coloring of planar graphs with large odd girth. *J. Graph Theory* 33 (2000), no. 2, 109–119.
- [10] D. König. *Theory of Finite and Infinite Graphs*. Birkhauser, Boston, 1990. English translation with commentary by W.T. Tutte and a biographical sketch of the author by T. Gallai. (Originally published as “*Theorie der endlichen und unendlichen Graphen*”, Akademische Verlagsgesellschaft, Leipzig, 1936.)
- [11] R. Naserasr, Mapping planar graphs into projective cubes. *J. of Graph theory*, **74(3)** (2013), 249–259.
- [12] R. Naserasr, E. Rollová and É. Sopena. Homomorphisms of planar signed graphs to signed projective cubes. *Discrete Math. Theoret. Comput. Sci.* 15 (2013), no. 3, 1–12.
- [13] R. Naserasr, E. Rollová and É. Sopena. Homomorphisms of signed graphs. *J. Graph Theory* 79 (2015), no. 3, 178–212.
- [14] R. Naserasr and Z. Wang, Cores of subdivisions of K_4 . In preparation.
- [15] P. Ochem, A. Pinlou and S. Sen. Homomorphisms of 2-Edge-Colored Triangle-Free Planar Graphs. *J. Graph Theory* 85 (2017), no. 1, 258–277.
- [RS94] A. Raspaud and E. Sopena. Good and semi-strong colorings of oriented planar graphs. *Inform. Proc. Letters* **51** (1994), 171–174.
- [16] C. Thomassen. Embeddings of graphs with no short noncontractible cycles. *J. Combin. Theory, Ser. B*, 48 (1990), no. 2, 155–177.
- [17] T. Zaslavsky. Signed graphs. *Discrete Applied Math.* 4 (1982), no. 1, 47–74.

11 Appendix

Proof of Lemma 19 First of all, we may assume that $L_{xy} \leq 2g - 1$ for every $x, y \in \{a, b, c, d\}$, as otherwise, we may delete internal vertices of P_{xy} in order to get a signed graph (G, σ) which is of unbalanced-girth $2g$. Applying Corollary 18, we get that the unbalanced cycle of length $2g$ is a homomorphic image of (G, σ) . A G is a connected graph, is any such a homomorphism x and y are mapped to matching parts of the bipartite graph C_{2g} . Subject to this condition, and regardless of which vertices of UC_{2g} these two end vertices of P_{xy} are mapped to, assuming $L_{xy} \geq 2g$, we can extend this mapping to the internal vertices of P_{xy} and we are done.

Considering a planar embedding of $\overline{K_4}$, since $(\overline{K_4}, \sigma)$ is not balanced, we have two cases to consider.

1. *Exactly two faces are unbalanced.*

Using a repeated application of our folding lemma (Lemma 17) on the two balanced faces, we can map the whole graph to C_{abc} which is isomorphic to UC_{2g} .

2. *All four faces are unbalanced.*

Thanks to Lemma 3, we may assume, up to resigning, that σ consists of exactly one edge on each path P_{xy} , $x, y \in \{a, b, c, d\}$.

We first claim that for at least one face incident to d , say C_{abd} , we have $L_{ad} + L_{bd} \geq L_{ab} + 2g$. To see that, assume to the contrary that $L_{ad} + L_{bd} < L_{ab} + 2g$, $L_{ad} + L_{cd} < L_{ac} + 2g$, and $L_{bd} + L_{cd} < L_{cb} + 2g$. This implies

$$2(L_{ad} + L_{bd} + L_{cd}) < L_{ab} + L_{bc} + L_{ac} + 6g = 8g,$$

contradicting the assumption of the Lemma.

As $L_{ad} + L_{bd} \geq L_{ab} + 2g$, and since $L_{ad} \leq 2g - 1$, we have $L_{bd} > L_{ab}$. Let x_a be a vertex of C_{abc} such that the path P_{abx_a} (a subgraph of C_{abc} , connecting a to x_a through b) is of length L_{da} . Similarly, it follows that $L_{bd} > L_{ab}$, and thus we can choose a vertex x_b on C_{abc} such that P_{bax_b} is of length L_{db} .

Observe that $I = P_{bax_b} \cap P_{abx_a} - P_{ab} \neq \emptyset$ and, furthermore, that there is a vertex $y \in I$ which is in a same part of the bipartition as d . We claim that it is possible to choose this vertex y , in such a way that $d_{P_{acb}}(y, c) \leq L_{cd}$. If $c \in I$, then we choose y to be either c or a neighbour of c (depending on whether d belongs to the same part as c or not). If $c \notin I$, then either I is a subset of P_{ac} or a subset of P_{bc} . By symmetry, suppose I is a subset of P_{ac} . In this case, we choose y to be the vertex of I that belongs to the same part as d and is the closest to c such vertex in P_{ac} . This means in particular that $|P_{bay}|$ is of length exactly L_{db} . Then, since

$$|P_{cy}| + |P_{bay}| = |P_{bac}| \leq |P_{cd} \cup P_{db}| = L_{cd} + L_{db},$$

we get $|P_{cy}| \leq L_{cd}$.

Finally, observing that the negative edges of P_{ad} and P_{bd} can be chosen to be anywhere in these paths, by resigning at their internal vertices, we conclude that the mapping of d to y can be extended to a mapping of $\overline{K_4}$ to C_{abc} , which is isomorphic to UC_{2g} .

This completes the proof.