

The acircuitic directed star arboricity of subcubic graphs is at most four

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Abstract

A directed star forest is a forest all of whose components are stars with arcs emanating from the center to the leaves. The acircuitic directed star arboricity of an oriented graph G (that is a digraph with no opposite arcs) is the minimum number of edge-disjoint directed star forests whose union covers all edges of G and such that the union of any two such forests is acircuitic. We show that every subcubic graph has acircuitic directed star arboricity at most four.

1 Introduction

We consider finite simple oriented graphs, that is digraphs with no opposite arcs. For an oriented graph G , we denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs.

In [1], Algor and Alon introduced the notion of the *directed star arboricity* of a digraph G , defined as the minimum number of edge-disjoint directed star forests needed to cover $A(G)$. (A directed star forest is a forest all of whose components are directed stars, that is stars with arcs emanating from the center.) In the same vein, we study here the new notion of the *acircuitic directed star arboricity* of an oriented graph G , defined as the minimum number of edge-disjoint directed star

forests needed to cover $A(G)$ in such a way that the union of any two such forests contains no circuit. In [5], Guiduli proved that every oriented graph with indegree and outdegree both less than D has directed star arboricity at most $D + 20 \log D + 84$ colors.

In this paper, we prove the following

Theorem 1 *Every graph with maximum degree at most 3 has acircuitic directed star arboricity at most 4.*

The notion of acircuitic directed star arboricity arises from the study of arc-coloring of oriented graphs. In [4], Courcelle introduced the notion of vertex-coloring of oriented graphs as follows: a k -vertex-coloring of an oriented graph G is a mapping f from $V(G)$ to a set of k colors such that (i) $f(u) \neq f(v)$ whenever \vec{uv} is an arc in G , and (ii) $f(u) \neq f(x)$ whenever \vec{uv} and \vec{wx} are two arcs in G with $f(v) = f(w)$. Vertex-coloring of oriented graphs have been studied by several authors in the last past years (see e.g. [2, 6] or [9] for an overview).

Recall that an *acyclic* coloring of an undirected graph U is a proper coloring of U such that every cycle in U uses at least three colors. Raspaud and Sopena proved in [8] that every orientation of an undirected graph that admits an acyclic k -coloring admits an oriented $(k \cdot 2^{k-1})$ -coloring.

One can define arc-colorings of oriented graphs in a natural way by saying that, as in the undirected case, an arc-coloring of an oriented graph G is a vertex-coloring of the line digraph of G . (Recall that the line digraph $L(G)$ of G is given by $V(L(G)) = A(G)$ and $(\vec{uv}, \vec{vw}) \in A(L(G))$ whenever $\vec{uv} \in A(G)$ and $\vec{vw} \in A(G)$.) It is not difficult to see that every oriented graph having a k -vertex-coloring admits a k -arc-coloring (from a k -vertex-coloring f , we obtain a k -arc-coloring g by setting $g(\vec{uv}) = f(u)$).

By adapting the proof of the above-mentioned result of Raspaud and Sopena, it is not difficult to prove that every oriented graph with acircuitic directed star arboricity at most k admits a $(k \cdot 2^{k-1})$ -arc-coloring.

This paper is organised as follows: we introduce the main definitions and notation in the next section and prove our main result in Section 3.

2 Definitions and notation

In the rest of the paper, oriented graphs will be simply called graphs. For a vertex v , we denote by $d^-(v)$ the indegree of v , by $d^+(v)$ its outdegree and by $d(v)$ its degree, that is $d(v) = d^+(v) + d^-(v)$. A *source vertex* is a vertex v with $d^-(v) = 0$. The *maximum degree* and *minimum degree* of a graph G are respectively denoted

by $\Delta(G)$ and $\delta(G)$. A graph G is said to be *cubic* if $\Delta(G) = \delta(G) = 3$ and *subcubic* if $\Delta(G) \leq 3$.

We denote by \vec{uv} the arc from u to v or simply uv whenever its orientation is not relevant (therefore $uv = \vec{uv}$ or $uv = \vec{vu}$). If $a = \vec{uv}$ is an arc, then u is the *tail* and v is the *head* of a .

For a graph G and a vertex v of $V(G)$, we denote by $G \setminus v$ the graph obtained from G by removing v together with the set of its incident arcs; similarly, for an arc a of $A(G)$, $G \setminus a$ denotes the graph obtained from G by removing a . These two notions are extended to sets in a standard way: for a set of vertices V' , $G \setminus V'$ denotes the graph obtained from G by successively removing all vertices of V' and their incident arcs, and for a set of arcs A' , $G \setminus A'$ denotes the graph obtained from G by removing all arcs of A' .

The notions of arboricity discussed in the previous section may be defined in terms of arc-colorings or partitions of the set of arcs. More precisely, a *k-directed-star-coloring* (or simply *k-dst-coloring*) of a graph G is a partition of $A(G)$ into k directed star forests $\{F_1, F_2, \dots, F_k\}$. Equivalently, a *k-dst-coloring* of G is a k -coloring f of $A(G)$ such that (i) $\vec{uv}, \vec{vw} \in A(G) \Rightarrow f(\vec{uv}) \neq f(\vec{vw})$, and (ii) $\vec{uv}, \vec{tv} \in A(G) \Rightarrow f(\vec{uv}) \neq f(\vec{tv})$. The *directed star arboricity* of G , denoted by $dst(G)$, is then the smallest k for which G admits a k -dst-coloring.

A graph G is *acircuitic* if it does not contain any circuit. A *k-acircuitic-directed-star-coloring* (or simply *k-adst-coloring*) of a graph G is a partition of $A(G)$ into k directed star forests $\{F_1, F_2, \dots, F_k\}$ such that for all $i, j \in [1, k]$, $F_i \cup F_j$ is acircuitic. Equivalently, a *k-adst-coloring* of G is a k -dst-coloring of G such that no circuit in G is bichromatic. The *acircuitic directed star arboricity* of G , denoted by $adst(G)$, is the smallest k for which G admits a k -adst-coloring.

Note that from the above definitions we get that every edge-coloring of an undirected graph H is a dst-coloring of any orientation of H . Similarly, every acyclic edge-coloring of H is an adst-coloring of any orientation of H .

The following notation will be extensively used in the rest of the paper. Consider a graph G and let $A' = \{a_1, a_2, \dots, a_n\}$ be a subset of $A(G)$. We denote by $C_G(a_1, a_2, \dots, a_n)$, or simply $C_G(A')$, the set of circuits of G that contain all the arcs a_1, a_2, \dots, a_n .

Drawing conventions. In all the figures, we shall use the following convention: a vertex whose neighbors are totally specified will be black, whereas a vertex whose neighbors are partially specified will be white. Moreover, an edge will represent an arc with any of its two possible orientations.

3 Proof of Theorem 1

Suppose that Theorem 1 is false and consider a minimal counter-example G . We prove a series of lemmas. In each of them, we reduce G to a smaller graph G' (that is $|A(G)| > |A(G')|$) which admits a 4-adst-coloring f' which is also a *partial* adst-coloring of G (that is an adst-coloring only defined on some subset A' of $A(G)$). We extend such a partial adst-coloring f' to an adst-coloring f of G . In this case, it should be understood that we set $f(a) = f'(a)$ for every arc $a \in A(G')$. We then explain how to set $f(a)$ for every uncolored $a \in A(G)$. The existence of f proves that G does not contain some specific configurations. This set of configurations will finally lead to a contradiction.

Consider a circuit C and let $u, v \in V(C)$. We denote by $P_C(u, v)$ the directed path from u to v in C .

The following observation will be extensively used in the sequel:

Observation 2 *Let C be a circuit, f an adst-coloring of C , and C' the circuit obtained from C by replacing $P_C(u, v)$ by a directed path $P_{C'}(u, v)$. If f' is a dst-coloring of C' such that $f'(a) = f(a)$ for every $a \notin P_{C'}(u, v)$ and $\{f(a); a \in P_C(u, v)\} \subseteq \{f'(a'); a' \in P_{C'}(u, v)\}$ then f' is an adst-coloring of C' .*

This directly follows from the fact that $|f'(C')| \geq |f(C)| \geq 3$.

We first show that a minimal counter-example to Theorem 1 is necessarily a cubic graph.

Lemma 3 *If G is a minimal counter-example to Theorem 1, then $\delta(G) \geq 3$.*

Proof. Let $v \in V(G)$ with $d(v) \leq 2$. We consider two cases:

Case 1 : $d_G(v) = 1$.

Consider the dangling arc uv in G and let f' be any 4-adst-coloring of the graph $G' = G \setminus \{v\}$. We extend f' to a 4-adst-coloring f of G by setting $f(uv) = a$ for some color a distinct from the colors of the at most two arcs incident to uv .

Case 2 : $d_G(v) = 2$.

Consider the two arcs uv and wv in G and let f' be any 4-adst-coloring of the graph G' obtained from G by contracting uv in a single vertex x . We extend f' to a 4-adst-coloring f of G by setting $f(wv) = f'(wx)$ and $f(uv) = a$ for any a distinct from the colors of the three arcs incident to uv . (By Observation 2, no circuit in G can be bichromatic.)

In both cases we thus obtain a 4-adst-coloring f of G , a contradiction. ■

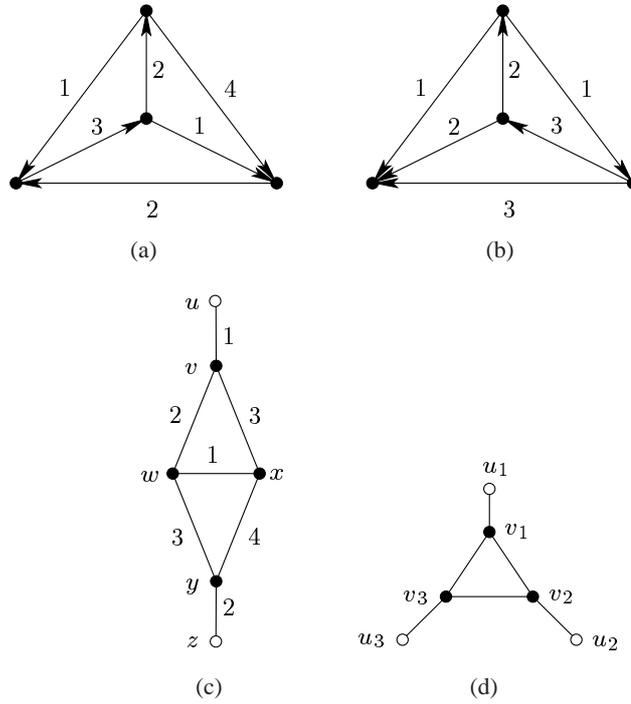


Figure 1: Configurations of Lemma 5

Lemma 4 *If G is a minimal counter-example to Theorem 1, then G does not contain any source vertex.*

Proof. Let $v \in V(G)$ be a source vertex, u_1 , u_2 and u_3 be the three neighbors of v and f' be any 4-adst-coloring of the graph $G' = G \setminus v$. By Lemma 3, we know that $d^+(v) = 3$. Each of the arcs $\overrightarrow{vu_1}$, $\overrightarrow{vu_2}$ and $\overrightarrow{vu_3}$ has at least two available colors. Since they can get the same color, we can extend f' to a 4-adst-coloring f of G , a contradiction. ■

We now prove that a minimal counter-example to Theorem 1 contains no triangle.

Lemma 5 *If G is a minimal counter-example to Theorem 1, then G is triangle-free.*

Proof. If G contains three pairwise adjacent triangles, then G is an orientation of the complete graph K_4 . By Lemma 4 we only have to consider the two orientations of K_4 depicted on Figures 1(a) and 1(b) that both admit a 4-adst-coloring.

If G contains two adjacent triangles, then G contains the configuration of Figure 1(c). Consider the graph $G' = G \setminus \{w, x\}$ and let f' be a 4-adst-coloring of G' such that $f'(uv) \neq f'(yz)$ (this can be done since we have two possible choices for coloring each of uv and uz). Suppose without loss of generality that $f'(uv) = 1$ and $f'(yz) = 2$. In this case, we can produce an acyclic 4-edge-coloring as depicted on Figure 1(c). Indeed, this coloring is a proper edge-coloring and no path linking u and z is bichromatic. Hence, for all possible orientations of the arcs of the configuration, this coloring gives a 4-adst-coloring f of G .

Suppose finally that G contains the configuration of Figure 1(d), and let f' be any 4-adst-coloring of the graph G' obtained from G by contracting the triangle $v_1v_2v_3$ in a single vertex v . Therefore, every circuit $C \in C_G(\overrightarrow{u_i v_i}, \overrightarrow{v_j u_j})$ corresponds to a circuit $C' \in C_{G'}(\overrightarrow{u_i v}, \overrightarrow{v u_j})$.

We now extend the partial adst-coloring f' to a 4-adst-coloring f of G as follows. We distinguish two cases:

Case 1 : $f'(vu_1) \neq f'(vu_2) \neq f'(vu_3) \neq f'(vu_1)$.

Without loss of generality, suppose that $f'(v_1u_1) = 1$, $f'(v_2u_2) = 2$ and $f'(v_3u_3) = 3$. We then set $f(v_3v_1) = 2$, $f(v_1v_2) = 3$ and $f(v_2v_3) = 1$.

Case 2 : $\exists i, j \in \{1, 2, 3\}, i \neq j, f'(v_iu_i) = f'(v_ju_j) = a$.

In this case we necessarily have $\overrightarrow{v_iu_i}, \overrightarrow{v_ju_j} \in A(G)$. Let $k \in \{1, 2, 3\}, k \neq i, j$. We then set $f(v_iv_k)$, $f(v_jv_k)$ and $f(v_iv_j)$ as follows:

1. $f(v_iv_k) = b$ for any $b \notin \{a, f'(u_kv_k)\}$,
2. $f(v_jv_k) = c$ for any $c \notin \{a, b, f'(u_kv_k)\}$,
3. $f(v_iv_j) = d$ for any $d \notin \{a, b, c\}$.

This can be done since we have four available colors.

In both cases, thanks to Observation 2, we obtain a 4-adst-coloring f of G , a contradiction. ■

Let G be a graph and C a circuit in G . An arc having exactly one of its endpoints in C is said to be *incident* to C . Moreover, two such incident arcs are *neighboring* if their endpoints in C are linked by an arc of C .

The four next lemmas will allow us to prove that a minimal counter-example G to Theorem 1 is necessarily acircuic.

Lemma 6 *If G is a minimal counter-example to Theorem 1, then G does not contain a circuit all of whose vertices have indegree one and outdegree two.*

Proof. Suppose that there exists a circuit $C = \{\overrightarrow{v_0v_1}, \overrightarrow{v_1v_2}, \dots, \overrightarrow{v_{k-2}v_{k-1}}, \overrightarrow{v_{k-1}v_0}\}$ in G such that $d^+(v_i) = 1$ and $d^-(v_i) = 2$ for $i \in [0, k-1]$ and let f' be any 4-adst-coloring of the graph $G' = G \setminus C$. Let $\{\overrightarrow{v_iu_i} \mid i \in [0, k-1]\}$ be the set of arcs incident to C .

We extend the partial coloring f' to a 4-adst-coloring f of G as follows. Due to the orientation of G , C is the only circuit of G that does not belong to G' . Therefore, we only need to color the arcs of C in such a way that C is not bichromatic. We distinguish two cases depending on the colors of the arcs incident to C .

1. All arcs incident to C are colored with the same color. In this case, we color the arcs of C using the three other remaining colors.
2. Two neighboring arcs incident to C have distinct colors. Suppose without loss of generality that $f'(\overrightarrow{v_0u_0}) = c_0$ and $f'(\overrightarrow{v_1u_1}) \neq c_0$. In this case, we set
 - (a) $f(\overrightarrow{v_0v_1}) = c_0$,
 - (b) $\forall i \in [1, k-2], f(\overrightarrow{v_iv_{i+1}}) = c_i$ for any $c_i \notin \{c_{i-1}, f(\overrightarrow{v_{i+1}u_{i+1}})\}$,
 - (c) $f(\overrightarrow{v_{k-1}v_0}) = c_{k-1}$ for any $c_{k-1} \notin \{c_0, c_1, c_{k-2}\}$.

The circuit C is clearly not bichromatic since $c_0 \neq c_1 \neq c_{k-1} \neq c_0$.

In both cases, we obtain a 4-adst-coloring f of G , a contradiction. ■

Lemma 7 *If G is a minimal counter-example to Theorem 1, then G does not contain a circuit all of whose vertices have indegree two and outdegree one.*

Proof. Suppose that there exists a circuit $C = \{\overrightarrow{v_0v_1}, \overrightarrow{v_1v_2}, \dots, \overrightarrow{v_{k-2}v_{k-1}}, \overrightarrow{v_{k-1}v_0}\}$ in G (see Figures 2(a) or 3(a)) such that $d^+(v_i) = 2$ and $d^-(v_i) = 1$ for $i \in [0, k-1]$. Let $\{\overrightarrow{u_iv_i} \mid i \in [0, k-1]\}$ be the set of arcs incident to C . By Lemma 5, the tails of two neighboring arcs incident to C are necessarily distinct.

We consider two cases depending on whether the vertices u_0 and u_2 are distinct or not. We first show that in both cases there exists a reduction G' of G (see Figures 2(b) and 3(b)) which admits a 4-adst-coloring f' such that $f'(\overrightarrow{u_0v_0}) \neq f'(\overrightarrow{u_1v_1}) \neq f'(\overrightarrow{u_2v_2}) \neq f'(\overrightarrow{u_0v_0})$.

Case 1 : $u_0 \neq u_2$ (see Figure 2(a)).

Let f' be any 4-adst-coloring of the graph G' obtained from $G \setminus C$ by identifying v_0, v_1 and v_2 in a single vertex v (see Figure 2(b)). We clearly have $f(\overrightarrow{u_0v}) \neq f(\overrightarrow{u_1v}) \neq f(\overrightarrow{u_2v}) \neq f(\overrightarrow{u_0v})$.

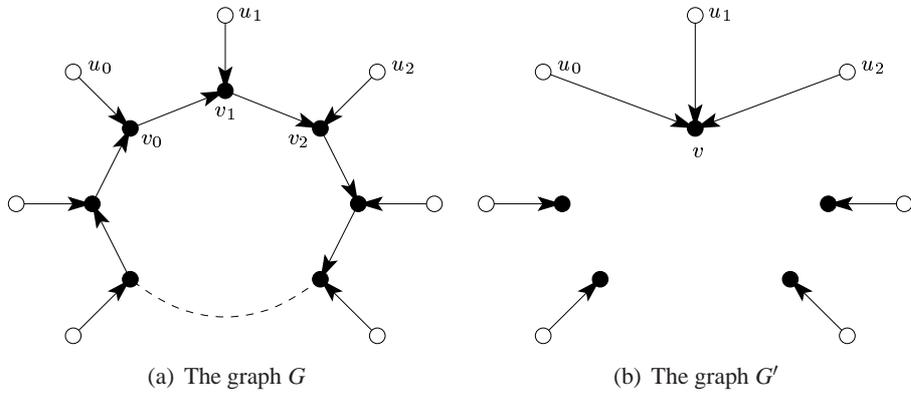


Figure 2: The configuration of Case 1 of Lemma 7 and its reduction.

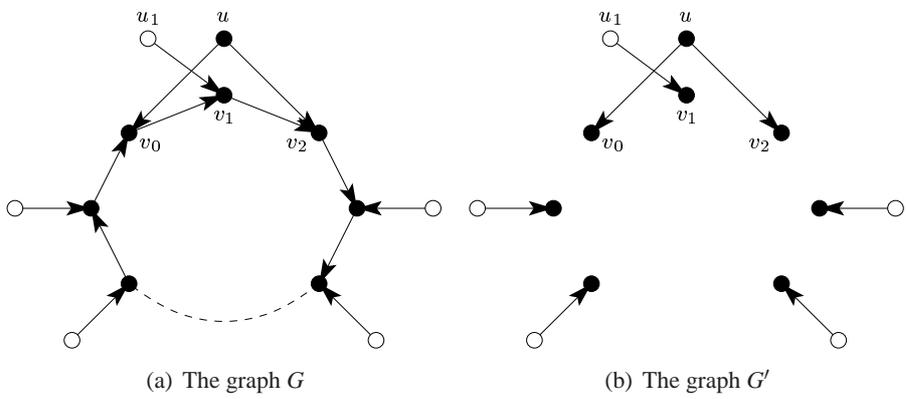


Figure 3: The configuration of Case 2 of Lemma 7 and its reduction.

Case 2 : $u_0 = u_2 = u$ (see Figure 3(a)).

Note that by Lemma 4 we have $u_1 \neq u$. Let f' be any 4-adst-coloring of the graph $G' = G \setminus C$ (see Figure 3(b)). Since we have at least three available colors for the arcs $\overrightarrow{uv_0}$ and $\overrightarrow{uv_2}$, we can choose f' in such a way that $f'(\overrightarrow{uv_0}) \neq f'(\overrightarrow{u_1v_1}) \neq f'(\overrightarrow{uv_2}) \neq f'(\overrightarrow{uv_0})$.

Assume now that $f(\overrightarrow{u_0v_0}) = c_1$, $f(\overrightarrow{u_1v_1}) \neq c_1$ and $f(\overrightarrow{u_2v_2}) \neq c_1$. As in the previous lemma, C is the only circuit of G that does not belong to G' . Therefore, we only need to color the arcs of C in such a way that C is not bichromatic. We then set f as follows:

1. $f(\overrightarrow{v_1v_2}) = c_1$,
2. $\forall i \in [2, k-1]$, $j = i+1 \pmod{k}$, $f(\overrightarrow{v_iv_j}) = c_i$
for any $c_i \notin \{c_{i-1}, f(\overrightarrow{u_iv_i}), f(\overrightarrow{u_jv_j})\}$,
3. $f(\overrightarrow{v_0v_1}) = c_0$ for any $c_0 \notin \{c_{k-1}, c_1, f(\overrightarrow{u_1v_1})\}$.

Note that $c_{k-1} \neq f(\overrightarrow{u_0v_0}) = c_1$. Therefore, $c_{k-1} \neq c_0 \neq c_1 \neq c_{k-1}$ and C is not bichromatic. We thus obtain a 4-adst-coloring f of G , a contradiction. ■

From the two previous lemmas, we get that if C is a circuit in a minimal counter-example to Theorem 1, there exist two neighboring arcs incident with C having opposite directions (with respect to C). The next two lemmas will show that this situation is also not possible.

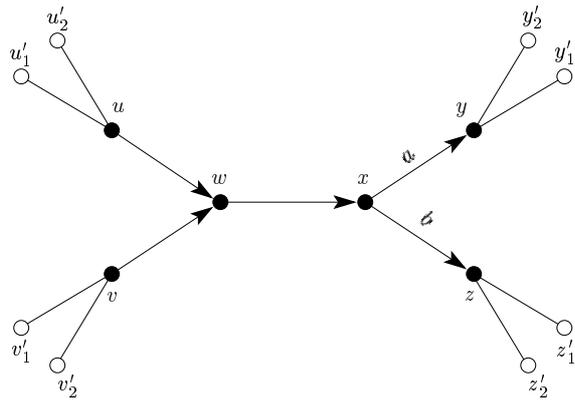
Lemma 8 *If G is a minimal counter-example to Theorem 1, then G does not contain the configuration depicted on Figure 4(a).*

Proof. Suppose that the graph G contains the configuration of Figure 4(a), with the arcs u'_1u , u'_2u , y'_1y , y'_2y , v'_1v , v'_2v , z'_1z and z'_2z being pairwise distinct, and let f' be any 4-adst-coloring of the graph $G' = G \setminus \{w, x\}$ by adding the arcs \overrightarrow{uy} and \overrightarrow{vz} (see Figure 4(b)). Suppose that $f'(\overrightarrow{uy}) = a$ and $f'(\overrightarrow{vz}) = b$.

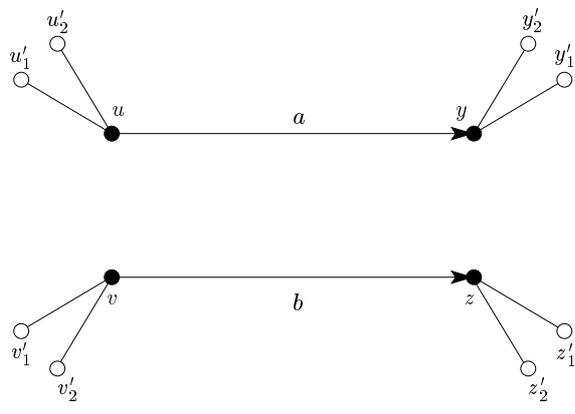
We extend the partial 4-adst-coloring f' to a 4-adst-coloring f of G as follows.

Let $E = C_G(\overrightarrow{uw}, \overrightarrow{wx}, \overrightarrow{xy}) \cup C_G(\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xz})$ and $F = C_G(\overrightarrow{uw}, \overrightarrow{wx}, \overrightarrow{xz}) \cup C_G(\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xy})$. We first set $f(\overrightarrow{xy}) = a$ and $f(\overrightarrow{xz}) = b$. Clearly, all circuits in G not belonging to $E \cup F$ also belong to G' , and thus are already not bichromatic. Moreover, by Observation 2, the circuits in E will not be bichromatic. Therefore, we only have to pay attention to the circuits in F .

We consider two cases depending on the colors a and b :



(a) The graph G



(b) The reduction G'

Figure 4: The configuration of Lemma 8 and its reduction.

Case 1 : $a \neq b$.

We set $f(\overrightarrow{uw}) = a$, $f(\overrightarrow{vw}) = b$ and $f(\overrightarrow{wx}) = c$ for any $c \notin \{a, b\}$. Since $|\{f(\overrightarrow{uw}), f(\overrightarrow{wx}), f(\overrightarrow{xz})\}| = |\{a, c, b\}| = 3$ and $|\{f(\overrightarrow{vw}), f(\overrightarrow{wx}), f(\overrightarrow{xy})\}| = |\{b, c, a\}| = 3$, no circuit in F is bichromatic.

Case 2 : $a = b$.

We consider three subcases.

1. $\{\overrightarrow{uu'_1}, \overrightarrow{uu'_2}, \overrightarrow{vv'_1}, \overrightarrow{vv'_2}\} \cap A(G) \neq \emptyset$.

We assume without loss of generality that $\overrightarrow{vv'_1} \in A(G)$. In this case, we first set $f(\overrightarrow{uw}) = c$ for any $c \notin \{a, f(\overrightarrow{uu'_1}), f(\overrightarrow{uu'_2})\}$ and $f(\overrightarrow{vw}) = d$ for any $d \notin \{a, c, f(\overrightarrow{vv'_2})\}$. Now, we can color the arc \overrightarrow{wx} with the fourth color $e \notin \{a, c, d\}$. Therefore, $|\{f(\overrightarrow{uw}), f(\overrightarrow{wx}), f(\overrightarrow{xz})\}| = |\{c, e, a\}| = 3$, $|\{f(\overrightarrow{vw}), f(\overrightarrow{wx}), f(\overrightarrow{xy})\}| = |\{d, e, a\}| = 3$, and so no circuit in F is bichromatic.

2. $\overrightarrow{u'_1u}, \overrightarrow{u'_2u}, \overrightarrow{v'_1v}, \overrightarrow{v'_2v} \in A(G)$

and $\{f(\overrightarrow{u'_1u}), f(\overrightarrow{u'_2u})\} \neq \{f(\overrightarrow{v'_1v}), f(\overrightarrow{v'_2v})\}$.

Note that since $a \notin \{f(\overrightarrow{u'_1u}), f(\overrightarrow{u'_2u}), f(\overrightarrow{v'_1v}), f(\overrightarrow{v'_2v})\}$, we necessarily have $\{f(\overrightarrow{u'_1u}), f(\overrightarrow{u'_2u})\} \cap \{f(\overrightarrow{v'_1v}), f(\overrightarrow{v'_2v})\} \neq \emptyset$. Therefore, we can assume without loss of generality that $f(\overrightarrow{u'_1u}) = f(\overrightarrow{v'_1v}) = c$, $f(\overrightarrow{u'_2u}) = d$ and $f(\overrightarrow{v'_2v}) = e$, with a, c, d, e being pairwise distinct. In this case, we set $f(\overrightarrow{uw}) = e$ and $f(\overrightarrow{vw}) = d$. Now, we can color the arc \overrightarrow{wx} with the color c . Therefore, $|\{f(\overrightarrow{uw}), f(\overrightarrow{wx}), f(\overrightarrow{xz})\}| = |\{e, c, a\}| = 3$, $|\{f(\overrightarrow{vw}), f(\overrightarrow{wx}), f(\overrightarrow{xy})\}| = |\{d, c, a\}| = 3$, and so no circuit in F is bichromatic.

3. $\overrightarrow{u'_1u}, \overrightarrow{u'_2u}, \overrightarrow{v'_1v}, \overrightarrow{v'_2v} \in A(G)$

and $\{f(\overrightarrow{u'_1u}), f(\overrightarrow{u'_2u})\} = \{f(\overrightarrow{v'_1v}), f(\overrightarrow{v'_2v})\}$.

We assume without loss of generality that $f(\overrightarrow{u'_1u}) = f(\overrightarrow{v'_1v}) = c$ and $f(\overrightarrow{u'_2u}) = f(\overrightarrow{v'_2v}) = d$, $a \neq c \neq d \neq a$. We then set $f(\overrightarrow{vw}) = a$ and $f(\overrightarrow{uw}) = e$ with $e \notin \{a, c, d\}$.

Since \overrightarrow{uw} and \overrightarrow{xz} are colored with distinct colors, no circuit in $C_G(\overrightarrow{uw}, \overrightarrow{wx}, \overrightarrow{xz})$ is bichromatic.

We still have to set the color of the arc \overrightarrow{wx} . We consider three subcases.

- (a) $\{y'_1y, y'_2y\} \cap A(G) \neq \emptyset$

We assume without loss of generality that $y'_1y \in A(G)$. So, if $f(yy'_2) = c$ (resp. $f(yy'_2) = d$), we set $f(\overrightarrow{wx}) = d$ (resp. $f(\overrightarrow{wx}) = c$), otherwise ($f(yy'_2) = e$), we use either c or d . Therefore, $|\{f(\overrightarrow{wx}), f(\overrightarrow{xy}), f(\overrightarrow{yy'_2})\}| = 3$, and thus no circuit in $C_G(\overrightarrow{vw}, \overrightarrow{wx}, \overrightarrow{xy})$ is bichromatic.

(b) $\overrightarrow{yy'_1}, \overrightarrow{yy'_2} \in A(G)$ and $\{f(\overrightarrow{v'_1v}), f(\overrightarrow{v'_2v})\} \neq \{f(\overrightarrow{yy'_1}), f(\overrightarrow{yy'_2})\}$.

We assume without loss of generality that $f(\overrightarrow{yy'_2}) = e$. Now, if $f(\overrightarrow{yy'_1}) = c$ (resp. $f(\overrightarrow{yy'_1}) = d$) we set $f(\overrightarrow{w\bar{x}}) = d$ (resp. $f(\overrightarrow{w\bar{x}}) = c$). This implies that for any $i \in \{1, 2\}$, $|\{f(\overrightarrow{w\bar{x}}), f(\overrightarrow{x\bar{y}}), f(\overrightarrow{yy'_i})\}| = 3$, and thus no circuit in $C_G(\overrightarrow{v\bar{w}}, \overrightarrow{w\bar{x}}, \overrightarrow{x\bar{y}})$ is bichromatic.

(c) $\overrightarrow{yy'_1}, \overrightarrow{yy'_2} \in A(G)$ and $\{f(\overrightarrow{v'_1v}), f(\overrightarrow{v'_2v})\} = \{f(\overrightarrow{yy'_1}), f(\overrightarrow{yy'_2})\}$.

We can suppose without loss of generality that $f(\overrightarrow{yy'_1}) = c$ and $f(\overrightarrow{yy'_2}) = d$. We then set $f(\overrightarrow{w\bar{x}}) = c$. If there is no arc emanating from y'_1 and colored with a , no circuit in $C_G(\overrightarrow{v\bar{w}}, \overrightarrow{w\bar{x}}, \overrightarrow{x\bar{y}})$ is bichromatic. If there exists an arc emanating from y'_1 and colored with a , then there exists at least one available color distinct from c that can be used to recolor the arc $\overrightarrow{yy'_1}$ in such a way that we forbid bichromatic circuits in $C_G(\overrightarrow{v\bar{w}}, \overrightarrow{w\bar{x}}, \overrightarrow{x\bar{y}})$.

In all cases, we obtain a 4-adst-coloring f of G , a contradiction. \blacksquare

In the configuration of the previous Lemma, the arcs uu'_i and vv'_j on one hand, yy'_i and zz'_j on the other hand, are necessarily distinct since, by Lemma 5, a minimal counter-example to Theorem 1 contains no triangle. The next lemma deals with the case where two arcs uu'_i (or vv'_i) and yy'_j (or zz'_j) are the same. Without loss of generality, we will suppose that the arcs v'_1v and z'_1z are the same.

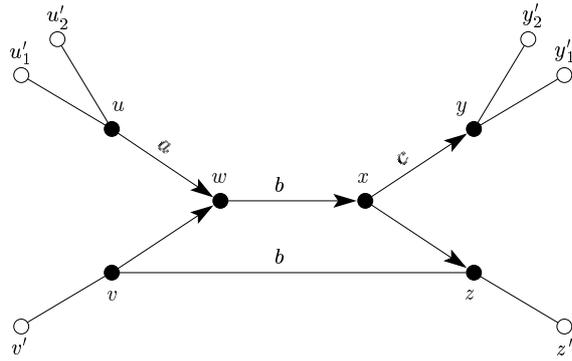
Lemma 9 *If G is a minimal counter-example to Theorem 1, then G does not contain the configuration depicted on Figure 5(a).*

Proof. Suppose that the graph G contains the configuration of Figure 5(a) (in this configuration, two arcs linking a black and a white vertex may be the same provided it does not produce a triangle). We consider two cases depending on the orientation of the arc vz .

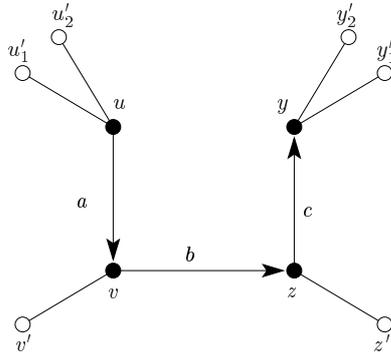
Case 1 : $\overrightarrow{vz} \in A(G)$.

Consider the graph G'_1 (see Figure 5(b)) obtained from $G \setminus \{w, x\}$ by adding the arcs $\overrightarrow{u\bar{v}}$ and $\overrightarrow{z\bar{y}}$ (see Figure 5(b)) and let f'_1 be any 4-adst-coloring of G'_1 . Assume that $f'_1(\overrightarrow{u\bar{v}}) = a$, $f'_1(\overrightarrow{v\bar{z}}) = b$ and $f'_1(\overrightarrow{z\bar{y}}) = c$ (see Figure 5(b)). We extend the partial 4-adst-coloring f'_1 to a 4-adst-coloring f of G as follows.

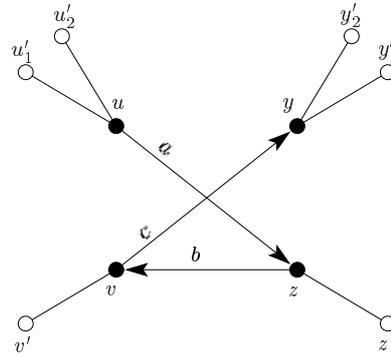
We first set $f(\overrightarrow{u\bar{w}}) = a$, $f(\overrightarrow{w\bar{x}}) = f(\overrightarrow{v\bar{z}}) = b$ and $f(\overrightarrow{x\bar{y}}) = c$ (see Figure 5(a)). By Observation 2, no circuit in G is thus bichromatic. We then color the arcs $\overrightarrow{v\bar{w}}$ and $\overrightarrow{x\bar{z}}$ so that $f(\overrightarrow{v\bar{w}}) \notin \{a, b, f(\overrightarrow{v'v})\}$ and $f(\overrightarrow{x\bar{z}}) \notin \{b, f(\overrightarrow{z'z})\}$.



(a) The graph G



(b) The reduction G'_1



(c) The reduction G'_2

Figure 5: The configuration of Lemma 9 and its reductions.

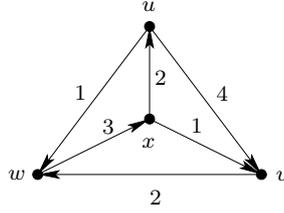


Figure 6: The orientation \vec{K}_4 such that $adst(\vec{K}_4) = 4$

Case 2 : $\vec{zv} \in A(G)$.

Consider the graph G'_2 obtained from $G \setminus \{w, x\}$ by adding the arcs \vec{uz} and \vec{vy} and let f'_2 be any 4-adst-coloring of G'_2 . Assume that $f'_2(\vec{uz}) = a$, $f'_2(\vec{zv}) = b$ and $f'_2(\vec{vy}) = c$ (see Figure 5(c)). We extend the partial 4-adst-coloring f' to a 4-adst-coloring f of G as follows.

As in the previous case, we set $f(\vec{uw}) = a$, $f(\vec{wx}) = f(\vec{vz}) = b$ and $f(\vec{xy}) = c$ (see Figure 5(a)). By Observation 2, we only have to pay attention to the circuits in $C_G(\vec{vw}, \vec{wx}, \vec{xz})$. We then color the arcs \vec{vw} and \vec{xz} in such a way that $f(\vec{vw}) \notin \{a, b, f(v'v)\}$ and $f(\vec{xz}) = a$ (this can be done since $f'(zz') \neq a$). Since $a \neq b \neq f(\vec{vw}) \neq a$, no circuit in $C_G(\vec{vw}, \vec{wx}, \vec{xz})$ is bichromatic.

In both cases we obtain a 4-adst-coloring f of G , a contradiction. ■

Using the previous lemmas, we can now prove our main result.

Proof of Theorem 1. By Lemmas 6, 7, 8 and 9, a minimal counter-example G to Theorem 1 does not contain any circuit. Therefore, any 4-dst-coloring of G is a 4-adst-coloring of G . Moreover, it follows from the definitions that any k -edge-coloring of the underlying undirected graph of G is a k -dst-coloring of G . Therefore, by Vizing's theorem [11], the graph G admits a 4-edge-coloring and thus a 4-adst-coloring, a contradiction. ■

The bound given in Theorem 1 is optimal. To see that, consider the orientation \vec{K}_4 of the complete graph K_4 given on Figure 6. If we want to color this graph with three colors, the only way to color the arcs \vec{xu} , \vec{uw} , \vec{wx} , \vec{xv} and \vec{vw} is clearly the one depicted on Figure 6. But in this case, we need one more color for the arc \vec{uv} and thus, $adst(\vec{K}_4) = 4$.

4 Discussion

In [3] Burnstein proved that every graph with maximum degree 4 admits an acyclic 5-vertex-coloring. Since the line graph of a subcubic graph has maximum degree at most 4, we get that every subcubic graph admits an acyclic 5-edge-coloring and thus a 5-adst-coloring. Our result shows that this bound can be decreased to 4 when considering oriented graphs and acircuitic arc-colorings.

We also provided an oriented cubic graph with acircuitic directed star arboricity 4. However, we do not know any other example of a cubic oriented graph that does not admit a 3-adst-coloring.

From our result, we get that every oriented graph with maximum degree three admits a $4 \cdot 2^{4-1} = 32$ -arc-coloring. However, every such graph admits an 11-vertex-coloring [10] and thus an 11-arc-coloring.

In a companion paper [7] we show that every K_4 -minor free oriented graph G has acircuitic directed star arboricity at most $\min\{\Delta(G), \Delta^-(G) + 2\}$, where $\Delta^-(G)$ stands for the maximum indegree of G . This class of graphs contains in particular outerplanar graphs. It would thus be interesting to determine the acircuitic directed star arboricity of planar graphs.

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