

Strong Rainbow Connection in Digraphs

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December 9, 2017

Abstract

An arc-coloured digraph is strongly rainbow connected if for every pair of vertices (u, v) there exists a shortest path from u to v all of whose arcs have different colours. The strong rainbow connection number of a digraph is the minimum number of colours needed to make the graph strongly rainbow connected.

In this paper, we study the strong rainbow connection number of minimally strongly connected digraphs, non-Hamiltonian strong digraphs and strong tournaments.

1 Introduction

We consider finite and simple graphs only, and refer to [5] for terminology and notation not defined here.

An *edge-coloured* graph is a graph equipped with a (not necessarily proper) edge-colouring of its edges. In an edge-coloured graph, a path is *rainbow* if it does not contain two edges with the same colour. An edge-coloured graph is then *rainbow connected* if any two vertices are connected by a rainbow path. The *rainbow connection number* $rc(G)$ of a connected graph G is the smallest number of colours that are needed in order to make G rainbow connected. An edge-coloured graph is *strongly rainbow connected* if any two vertices are connected by a rainbow *shortest* path, also called a rainbow *geodesic*. The *strong rainbow connection number* $src(G)$ of a connected graph G is the smallest number of colours that are needed in order to make G strongly rainbow connected.

The concept of rainbow connection in graphs was introduced by Chartrand *et al.* in [7]. Additionally, Chartrand *et al.* computed the precise rainbow connection number of several graph classes including complete multipartite graphs. The rainbow connection number was studied for further graph classes in [8, 10, 11, 13, 14, 16] and for graphs with fixed minimum degree in [8, 12, 15, 17]. The computational complexity of rainbow connectivity was studied in [6], where it is proved that the computation of $rc(G)$ is NP-hard. In fact it is already NP-complete to decide if $rc(G) = k$ for any fixed $k \geq 2$ or to decide whether a given edge-coloured (with any number of colours) graph is rainbow connected. Also, different other parameters

similar to rainbow connection were introduced, such as rainbow k -connectivity, k -rainbow index and rainbow vertex connection. See [13] for a survey about these different parameters.

The notions of rainbow connection and strong rainbow connection readily extend to digraphs, using arc-colouring instead of edge-colourings and directed paths instead of paths. Note that in order to have bounded rainbow connection number, a digraph must be strongly connected. The study of rainbow connection in oriented graphs was initiated by Dorbec, Schiermeyer and the authors in [9] and then studied by Alva-Samos and Montellano-Ballesteros in [1, 2, 3]. Rainbow connectivity in digraphs was considered by Ananth, Nasre and Sarpatwar in [4] from the computational point of view. They proved in particular that the problem of deciding whether a directed graph can be rainbow connected using at most two colors is NP-hard. In [9] the characterization of all oriented graphs with rainbow connection number equal to their order has been given. Moreover, it has been proved that the rainbow connection number of a tournament can take almost any value in terms of its order, but is upper bounded by its diameter plus 2.

In this paper, we study the strong rainbow connection number of digraphs. We introduce in the next section the main definitions and notation, together with some basic results. In particular, we prove that the strong rainbow connection number of any strong digraph lies between its diameter and its order and that the difference between the strong rainbow connection number and the rainbow connection number of a digraph can be arbitrarily large. We consider minimally strongly connected digraphs in Section 3 and characterize those such graphs with strong rainbow connection number equal to their order. In Section 4, we prove that every strong non-Hamiltonian digraph with order n and diameter $n - 1$ has strong rainbow connection number at most $n - 1$ and conjecture that this relation holds for such digraphs with any smaller diameter. Finally, we study the strong rainbow connection number of strong tournaments in Section 5. We prove that every strong tournament of order n has strong rainbow connection number at most $n - 1$ and that the strong rainbow connection number of a strong tournament of diameter d can be arbitrarily large.

2 Definitions, notation and first results

For a given digraph G , we denote by $V(G)$ and $A(G)$ its sets of vertices and set of arcs, respectively. The number of vertices of G is the *order* of G . By an *oriented graph* we mean an antisymmetric digraph, that is a digraph G such that $yx \notin A(G)$ whenever $xy \in A(G)$. Given an arc xy in G , we say y is an *out-neighbour* of x while x is an *in-neighbour* of y .

We denote by $N_G^+(x)$ the set of out-neighbours of x and by $N_G^-(x)$ the set of in-neighbours of x in G . The *out-degree* of x is the order of its out-neighbourhood in G , $\deg_G^+(x) = |N_G^+(x)|$, and the *in-degree* of x is the order of its in-neighbourhood in G , $\deg_G^-(x) = |N_G^-(x)|$. The *degree* of x in G is then $\deg_G(x) = \deg_G^+(x) + \deg_G^-(x)$.

A *path* of length $k \geq 1$ in a digraph G is a sequence $x_0 \dots x_k$ of vertices such that $x_i x_{i+1} \in A(G)$ for every i , $0 \leq i \leq k - 1$. Such a path P , going from x_0 to x_k , is referred to as an $x_0 x_k$ -*path*. For every i, j , $0 \leq i < j \leq k$, we say that x_i *precedes* x_j in P . Any vertex in $V(P) \setminus \{x_0, x_k\}$ is an *internal vertex* of P . If X and Y are two subsets of $V(G)$, an XY -*path* is an xy -path linking a vertex $x \in X$ to a vertex $y \in Y$. If $X = \{x\}$, an XY -path will be simply called an xY -path. A path P is *elementary* if no vertex appears twice in P .

Given two paths $P_1 = x_1 \dots x_i$ and $P_2 = x_i \dots x_{i+j}$, we denote by $P_1 \cup P_2$ the path $x_1 \dots x_i \dots x_{i+j}$.

A *cycle* of length $k \geq 3$ in a digraph G is a sequence $x_0 \dots x_{k-1} x_0$ of vertices such that $x_i x_{i+1} \in A(G)$ for every i , $0 \leq i \leq k-2$, and $x_{k-1} x_0 \in A(G)$. For every i , $0 \leq i \leq k-1$, x_{i+1} is the *successor* of x_i in C and x_{i-1} is the *predecessor* of x_i in C (subscripts are taken modulo k).

An *ear* in a digraph G is either an xy -path Q such that $\deg_G(x) > 2$, $\deg_G(y) > 2$ and $\deg_G(z) = 2$ for every internal vertex z of Q or a cycle C such that $\deg_G(x) > 2$ for exactly one vertex of C and $\deg_G(z) = 2$ for every $z \in V(C) \setminus \{x\}$.

The *distance* from a vertex x to a vertex y in a digraph G , denoted by $\text{dist}_G(x, y)$, is the length of a shortest xy -path in G (if there is no such path, we let $\text{dist}_G(x, y) = \infty$). An *xy-geodesic* is a shortest xy -path in G . The *diameter* of G , denoted by $\text{diam}(G)$, is the maximum distance between any pair of vertices in G . A pair (u, v) is an *antipodal pair* of vertices if $\text{dist}_G(u, v) = \text{diam}(G)$.

A digraph G is *strongly connected* (*strong* for short) if there exists an xy -path in G for every two vertices x and y . The digraph G is *minimally strongly connected* (MSC for short) if G is strong and, for every arc xy in G , the digraph $G - xy$ is not strong.

Let G be a digraph. A k -*arc-colouring* of G , $k \geq 1$, is a mapping $\varphi : A(G) \rightarrow \{1, \dots, k\}$. Note that adjacent arcs may receive the same colour. An *arc-coloured* digraph is then a pair (G, φ) where G is a digraph and φ an arc-colouring of G . A path P in (G, φ) is *rainbow* if no two arcs of P are coloured with the same colour. An arc-coloured digraph (G, φ) is *rainbow connected* (or, equivalently, φ is a *rainbow arc-colouring* of G) if any two vertices in G are connected by a rainbow path. For any two vertices u, v of G , a *rainbow uv -geodesic* is a rainbow uv -path of length $\text{dist}_G(u, v)$. An arc-coloured digraph (G, φ) is *strongly rainbow connected* (or, equivalently, φ is a *strong rainbow arc-colouring* of G) if there exists a rainbow uv -geodesic for any two vertices u and v in G .

Note that in order to admit a rainbow arc-colouring, or a strong rainbow arc-colouring, a digraph must be strong.

The *rainbow connection number* of a strong digraph G , denoted by $\vec{rc}(G)$, is the smallest number k such that G admits a rainbow k -arc-colouring. The *strong rainbow connection number* of a strong digraph G , denoted $s\vec{rc}(G)$, is the smallest number k such that G admits a strong rainbow k -arc-colouring.

Proposition 1 *If G is a strong digraph of order n with diameter d , then $d \leq \vec{rc}(G) \leq s\vec{rc}(G) \leq n$.*

Proof. First observe that in a strong digraph G , any rainbow uv -path such that (u, v) is an antipodal pair of vertices must use at least $d = \text{diam}(G)$ colours, and thus $d \leq \vec{rc}(G)$. Furthermore, we clearly have $\vec{rc}(G) \leq s\vec{rc}(G)$. Let $V(G) = \{x_1, \dots, x_n\}$. We define an n -arc-colouring φ of G by setting $\varphi(x_i x_j) = j$ for every arc $x_i x_j$ in $A(G)$. Obviously, every elementary path in G is rainbow and, therefore, φ is a strong rainbow arc-colouring of G , which gives $s\vec{rc}(G) \leq n$. \square

Observe that the upper bound $s\vec{rc}(G) \leq n$ is tight since $\vec{rc}(C_n) = n$ for the cycle C_n of order n . On the other hand, the value of $s\vec{rc}(G) - \vec{rc}(G)$ can be arbitrarily large:

Theorem 2 *For every integer k there exists a strong digraph G_k with $s\vec{rc}(G) - \vec{rc}(G) = k$.*

Proof. Let G_k (see Figure 1) be the digraph given by $V(G_k) = V_k^1 \cup V_k^2 \cup V_k^3 \cup V_k^4$, with

$$V_k^1 = \{x_1, \dots, x_{3k+1}\}, V_k^2 = \{y_1, \dots, y_{3k+1}\}, V_k^3 = \{v_1, \dots, v_{k+1}\}, V_k^4 = \{u_3, \dots, u_{3k+2}\},$$

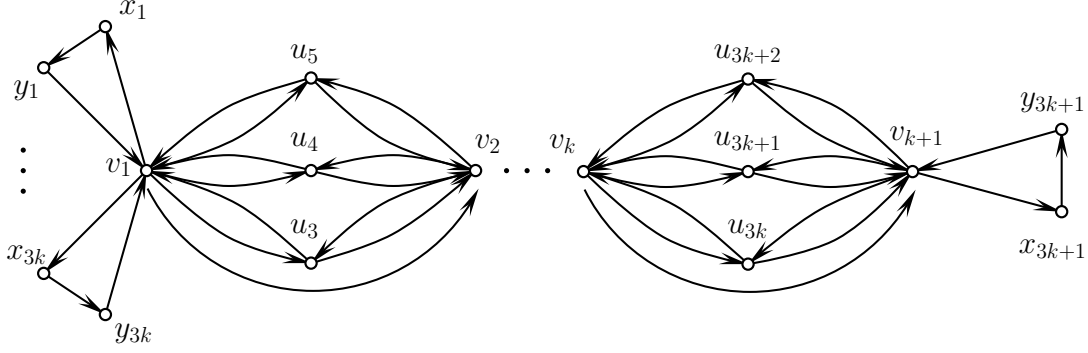


Figure 1: The digraph G_k .

and $A_k(G) = A_k^1 \cup A_k^2 \cup A_k^3 \cup A_k^4$, with

$$A_k^1 = \{x_i y_i : 1 \leq i \leq 3k + 1\},$$

$$A_k^2 = \{v_1 x_i, y_i v_1 : 1 \leq i \leq 3k\} \cup \{v_{k+1} x_{3k+1}, y_{3k+1} v_{k+1}\},$$

$$A_k^3 = \{v_i u_{3i+r}, u_{3i+r} v_i, v_{i+1} u_{3i+r}, u_{3i+r} v_{i+1} : 1 \leq i \leq k, 0 \leq r \leq 2\},$$

$$A_k^4 = \{v_i v_{i+1} : 1 \leq i \leq k + 1\}.$$

First observe that in any rainbow arc-colouring φ of G_k we have $\varphi(x_i y_i) \neq \varphi(x_j y_j)$ for every $i, j, i \neq j$, since each $x_i y_j$ -path contains both arcs $x_i y_i$ and $x_j y_j$. For the same reason we have $\varphi(v_1 x_j) \neq \varphi(x_i y_i)$, $\varphi(y_j v_1) \neq \varphi(x_i y_i)$ and $\varphi(v_1 x_j) \neq \varphi(y_j v_1)$ for every $i, j, 1 \leq i, j \leq 3k + 1$. This implies that $\overline{rc}(G) \geq 3k + 3$. Now, we show that G_k admits a rainbow arc-colouring which uses $3k + 3$ colours.

Let $\varphi : A_k(G) \rightarrow \{1, 2, \dots, 3k + 3\}$ be the arc-colouring of G_k given by:

- arcs of A_k^1 :

$$\varphi(x_i y_i) = i, \text{ for every } i, 1 \leq i \leq 3k + 1.$$

- arcs of A_k^2 :

$$\varphi(v_1 x_i) = 3k + 2 \text{ and } \varphi(y_i v_1) = 3k + 3, \text{ for every } i, 1 \leq i \leq 3k,$$

$$\varphi(v_{k+1} x_{3k+1}) = 3k + 2 \text{ and } \varphi(y_{3k+1} v_{k+1}) = 3k + 3.$$

- arcs of A_k^3 :

$$\varphi(v_i u_{3i+r}) = \varphi(u_{3i+r} v_i) = 3(i - 1) + r + 1, \text{ for every } i, r, 1 \leq i \leq k, 0 \leq r \leq 2,$$

$$\varphi(u_{3i+r} v_{i+1}) = \varphi(v_{i+1} u_{3i+r}) = \begin{cases} 3(i - 1) + r + 2, & \text{for } r \in \{0, 1\}, 1 \leq i \leq k, \\ 3(i - 1) + r - 1, & \text{for } r = 2, 1 \leq i \leq k. \end{cases}$$

- arcs of A_k^4 :

$$\varphi(v_i v_{i+1}) = 1, \text{ for every } i, 1 \leq i \leq k \text{ (note that the colour of these arcs can be chosen freely)}.$$

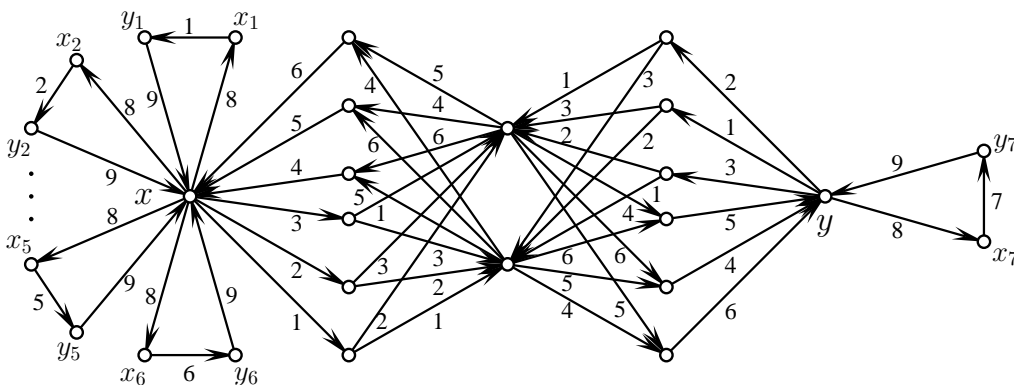


Figure 2: The oriented graph H .

The arc-colouring φ is clearly rainbow and thus $\vec{rc}(G_k) = 3k + 3$.

For every j , $1 \leq i \leq j \leq 3k$, there is a unique $x_j y_{3k+1}$ -geodesic and this geodesic contains all arcs $v_i v_{i+1}$, $1 \leq i \leq k$. Thus, in any strong rainbow arc-colouring of G_k , no two arcs in $\{v_i v_{i+1} : 1 \leq i \leq k\}$ can share the same colour. Moreover, as already observed, all arcs $x_i y_i$, $1 \leq i \leq 3k + 3$, must be assigned distinct colours. We thus get $\vec{src}(G_k) \geq k + 3k + 1 = 4k + 3$. On the other hand, if we colour each arc $v_i v_{i+1}$, $1 \leq i \leq k$, with colour $3k + 3 + i$, we obtain a strong rainbow colouring of G_k and thus $\vec{src}(G_k) = 4k + 3$. \square

It follows from the definitions that if H is a strong spanning subdigraph of G (that is, $V(H) = V(G)$), then $\vec{rc}(G) \leq \vec{rc}(H)$. However, such an equality does not hold for the strong rainbow connection number of digraphs. Alva-Samos and Montellano-Ballesteros [1] showed that there exists digraphs G and H , such that H is a spanning subdigraph of G and $\vec{src}(G) > \vec{src}(H)$. The digraph G constructed in [1] contains pairs of opposite arcs. We can prove that this property also holds for oriented graphs:

Proposition 3 *There exist strong oriented graphs G and H , such that H is a spanning subdigraph of G and $\vec{src}(G) > \vec{src}(H)$.*

Proof. Let H be the oriented graph depicted in Figure 2. It is easy to check that the arc-colouring given in Figure 2 is a strong rainbow colouring of H . Hence, $\vec{src}(H) \leq 9$. Let G be the oriented graph obtained from H by adding the arc xy . Since for every i, j , $1 \leq i, j \leq 7$, $i \neq j$, each $x_i y_j$ -path contains both arcs $x_i y_i$ and $x_j y_j$, in any strong rainbow arc-colouring of G , no two arcs in $\{x_i y_i : 1 \leq i \leq 7\}$ can share the same colour. Similarly, the four arcs xx_i , $y_i x$, yx_7 and $y_7 y$ must get distinct colours. Finally, since for every i , $1 \leq i \leq 6$, the path $x_i y_i x y x_7 y_7$ is the unique $x_i y_7$ -geodesic, we cannot assign to xy any colour from $\{1, \dots, 9\}$, which implies $\vec{src}(G) \geq 10$. \square

3 Strong rainbow connection number of minimally strongly connected digraphs

In this section, we prove that the only MSC digraph of order n with strong rainbow connection number n is the cycle (Theorem 9). In [9], it was proved that the only MSC oriented graph

of order n with rainbow connection number n is the cycle. Since $\vec{rc}(G) \leq \vec{sc}(G) \leq n$ by Proposition 1, Theorem 9 immediately implies this result.

The next lemma, and its corollary, were proved in [9] for oriented graphs. Since their proofs readily extends to digraphs, we omit them.

Lemma 4 *Let G be an MSC digraph. If G is not a cycle, then there is a strong subdigraph H of G and a vertex $x \in V(G) \setminus V(H)$ such that $N_G^-(x) = \{y\}$ and $y \in V(H)$.*

Corollary 5 *Let G be an MSC digraph. If G is not a cycle, then there is a cycle C in G and a vertex $x \in V(G) \setminus V(C)$ such that $N_G^-(x) = \{y\}$ with $y \in V(C)$.*

The following Strong-path property, which generalizes the Path property introduced in [9] for oriented graphs, will be useful in the context of digraphs.

Definition 6 (Strong-path property) Let G be a digraph and let $x'x$ and $y'y$ be arcs in G . The pair $\{x'x, y'y\}$ is said to have the *Strong-path property* if there exists an $x'y$ -geodesic in G not containing $x'x$ and a $y'x$ -geodesic not containing $y'y$.

Lemma 7 *Let G be a strong digraph of order n , $x'x$ and $y'y$ be two arcs in G with $x \neq y$, $\deg_G^-(x) = 1$ and $\deg_G^-(y) = 1$. If $\{x'x, y'y\}$ has the Strong-path property, then $\vec{sc}(G) \leq n - 1$.*

Proof. Suppose we have two arcs $x'x$ and $y'y$ in G that satisfy the conditions of the lemma. Let P_1 be an $x'y$ -geodesic not containing $x'x$ and P_2 a $y'x$ -geodesic not containing $y'y$. Observe that by the in-degree condition on x and y , the geodesics P_1 and P_2 necessarily contain the arcs $y'y$ and $x'x$, respectively.

Let $c : V(G) \rightarrow \{1, 2, \dots, n - 1\}$ be a mapping such that $c(x) = c(y) = 1$, $c(v) \neq 1$ for $v \in V(G) \setminus \{x, y\}$, and $c(v) \neq c(w)$ for $v, w \in V(G) \setminus \{x, y\}$. Let now $\varphi : A(G) \rightarrow \{1, 2, \dots, n - 1\}$ be the arc-colouring of G given by $\varphi(uv) = c(v)$ for every arc $uv \in A(G)$. We claim that φ is a strong rainbow arc-colouring of G .

To see that, observe that the two arcs $x'x$ and $y'y$ are the only arcs sharing the same colour. Therefore, every elementary path containing at most one of these two arcs is rainbow. Assume that there exist two vertices u and v that are not linked by a rainbow geodesic and let P be any uv -geodesic. Without loss of generality, assume that the arc $x'x$ appears before the arc $y'y$ in P . Let Q be the uv -path obtained from P by replacing the subpath from x' to y by the geodesic P_1 . Since P is a uv -geodesic, the subpath from x' to y contained in P is also geodesic. Thus, Q is an elementary uv -path with the same number of edges as P and is clearly a rainbow uv -geodesic, a contradiction. The arc-colouring φ is thus a strong rainbow colouring of G . \square

By taking $x' = y'$ in Lemma 7 we get the following corollary:

Corollary 8 *Let G be a strong digraph of order n . If there is a vertex z in G having two distinct out-neighbours x and y with $\deg_G^-(x) = \deg_G^-(y) = 1$, then $\vec{sc}(G) \leq n - 1$.*

We are now ready to prove the main result of this section.

Theorem 9 *Let G be an MSC digraph of order n . If G is not a cycle then $\vec{sc}(G) \leq n - 1$.*

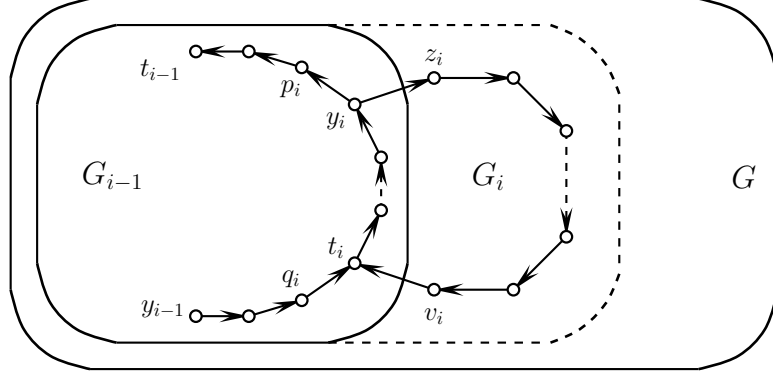


Figure 3: Construction of the pair (G_i, Q_i) from the pair (G_{i-1}, Q_{i-1}) .

Proof. Let G be an MSC digraph not reduced to a cycle. By Corollary 5 there is a cycle C in G and a vertex $x \in V(G) \setminus C$ such that $N_G^-(x) = \{y\}$ and $y \in V(C)$. Let z be the successor of y in C . Let (G_0, Q_0) be the pair constructed as follows. Let P be a shortest xC -path and $t \in V(C)$ its last vertex (it may happen that $t = y$). We then set $G_0 = C \cup yx \cup P$ and Q_0 as the yt -path contained in C if $t \neq y$ or $Q_0 = C$, otherwise. If $\deg_G^-(z) = 1$, then the two arcs yz and yx satisfy the assumptions of Corollary 8 and we are done.

Otherwise, we construct a sequence of pairs $\{(G_i, Q_i)\}_{i \geq 1}$ (see Figure 3) satisfying the following conditions for every $i \geq 1$:

- (i) G_i is a strong subdigraph of G and G_0 is a subdigraph of G_i ,
- (ii) Q_i is an ear in G_i such that both ends of Q_i are internal vertices of Q_{i-1} ; denoting by y_i the first vertex of Q_i and by t_i the last vertex of Q_i , t_i precedes y_i in Q_{i-1} (it may happen that $t_i = y_i$),
- (iii) G_{i-1} is a proper subdigraph of G_i (more precisely, $G_i = G_{i-1} \cup Q_i$),
- (iv) let q_i be the in-neighbour of t_i belonging to Q_{i-1} ; there exist a $q_i y$ -path in G not containing $q_i t_i$ and a $y q_i$ -geodesic in G not containing yx ,
- (v) denoting by $y_i z_i$ the first arc of Q_i , there exists either a $y_i x$ -geodesic in G not containing the arc $y_i z_i$ or an out-neighbour p_i of y_i in Q_{i-1} with $d_G^-(p_i) = 1$.

Observe that if we reach a pair with $d_G^-(z_i) = 1$, then either the arcs yx and $y_i z_i$ satisfy the assumptions of Lemma 7 or the arcs $y_i z_i$ and $y_i p_i$ satisfy the assumptions of Corollary 8, which in both cases allows us to conclude.

Suppose that we have constructed a sequence of pairs $\{(G_i, Q_i)\}$, $i = 1, \dots, k$ that satisfy items (i)–(v). We first prove two claims concerning the pairs (G_i, Q_i) .

Claim 10 For every vertex $v \in V(Q_i)$, each yv -path contains the arc $q_i t_i$.

Proof. Suppose that there is a yv -path P not containing the arc $q_i t_i$. Item (iv) implies that there is a $q_i y$ -path P' not containing the arc $q_i t_i$. Furthermore, there is a vt_i -path P'' not containing the arc $q_i t_i$, going through Q_i . Thus, the path $P' \cup P \cup P''$ is a $q_i t_i$ -path not containing the arc $q_i t_i$. This implies that $G - q_i t_i$ is strong, contradicting the assumption that G is an MSC digraph. \square

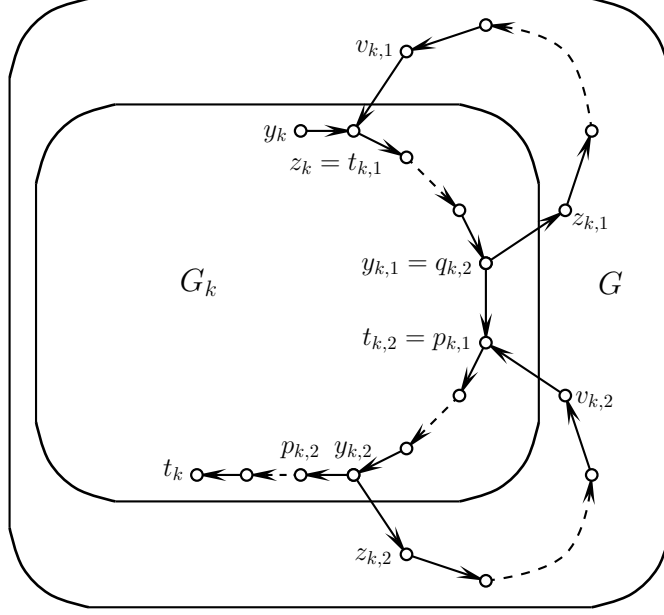


Figure 4: Construction of the sequence of vertex disjoint ears $\{Q_{k,j}\}_{j \geq 0}$.

Claim 11 For every vertex $v \in V(Q_i)$ there is a yv -geodesic in G not containing the arc yx .

Proof. Let P be a yv -geodesic. Suppose that P contains the arc yx , so that yx is the first arc of P . By Claim 10, each yv -path contains the arc $q_i t_i$, and thus $P = yx \dots q_i t_i \dots v$. Item (iv) implies that there is a yq_i -geodesic P' not containing the arc yx . Let P'' be the yv -path obtained from P by replacing the subpath $yx \dots q_i$ by the geodesic P' . The path P'' has the same length as P , and thus P'' is a yv -geodesic not containing the arc yx , as required. \square

If we have constructed a sequence of pairs $\{(G_i, Q_i)\}$, $i = 1, \dots, k$, and in the last pair (G_k, Q_k) we still have $d_G^-(z_k) > 1$, then we must construct the next pair (G_{k+1}, Q_{k+1}) .

To find an ear Q_{k+1} satisfying items (ii)–(v) we use one more construction. We construct a sequence of vertex disjoint ears $\{Q_{k,j}\}_{j \geq 1}$ (see Figure 4) with first vertex $y_{k,j}$ and last vertex $t_{k,j}$ (it may happen that $t_{k,j} = y_{k,j}$) satisfying the following conditions:

- (i') $V(G_k) \cap V(Q_{k,j}) = \{y_{k,j}, t_{k,j}\}$,
- (ii') $y_{k,j}, t_{k,j}$ are interior vertices of Q_k ,
- (iii') $t_{k,j}$ precedes $y_{k,j}$ in Q_k ,
- (iv') $t_{k,1}$ is the out-neighbour of y_k in Q_k and $t_{k,j+1}$ is the out-neighbour of $y_{k,j}$ in Q_k ,
- (v') denoting by $q_{k,j}$ the in-neighbour of $t_{k,j}$ in Q_k , there exist a $q_{k,j}y$ -path in G not containing the arc $q_{k,j}t_{k,j}$, and a $yq_{k,j}$ -geodesic in G not containing the arc xy .

If we reach an ear $Q_{k,j}$ such that there is either a $y_{k,j}x$ -geodesic not containing the first arc of $Q_{k,j}$, or an out-neighbour $p_{k,j}$ of $y_{k,j}$ in Q_k with $\deg_G^-(p_{k,j}) = 1$, then we set $Q_{k+1} := Q_{k,j}$. The ear Q_{k+1} thus obtained satisfies item (v).

We first construct $Q_{k,1}$. Since $\deg_G^-(z_k) \geq 2$, there is an in-neighbour $v_{k,1}$ of z_k , distinct from y_k . Since G is an MSC digraph and the subdigraph G_k is strong, $v_{k,1} \notin V(G_k)$. Let $P_{k,1} =$

$y_{k,1}z_{k,1} \dots v_{k,1}$ be a $G_k v_{k,1}$ -geodesic in G (it may happen that $y_{k,1} = z_{k,1}$), and set $Q = P_{k,1} \cup v_{k,1}z_{k,1}$. Let $t_{k,1} := z_{k,1}$. Since $P_{k,1}$ is a $G_k v_{k,1}$ -geodesic in G , we have $V(G_k) \cap V(Q) = \{y_{k,1}, t_{k,1}\}$. Observe that $y_{k,1}$ is an internal vertex of $V(Q_k)$, since otherwise $G - y_{k,1}z_{k,1}$ would be strong, contradicting the fact that G is an MSC digraph. Moreover, $t_{k,1}$ precedes $y_{k,1}$ in Q_k and $t_{k,1}$ is the out-neighbour of $y_{k,1}$. Thus, items (i')–(iv') hold for the path Q . We have to show that item (v') also holds. The vertex y_k is the in-neighbour of $t_{k,1}$ in Q_k . If $k \geq 1$, then we consider G_{k-1} . Since G_{k-1} is strong, there is a $y_k y$ -path not containing $y_k t_{k,1}$. If $k = 0$, then $y_k = y$ and hence such a path obviously exists. Finally, by Claim 11, there exists a yy_k -geodesic not containing the arc yx and we can thus set $Q_{k,1} := Q$.

Assume now that an ear $Q_{k,j}$, $j \geq 1$, satisfying items (i')–(v') has been constructed. If there is an $y_{k,j}x$ -geodesic not containing the first arc of $Q_{k,j}$ or there is an out-neighbour $p_{k,j}$ of $y_{k,j}$ in Q_k with $\deg_G^-(p_{k,j}) = 1$, then we set $Q_{k,j+1} := Q_{k,j}$. Otherwise, all $y_{k,j}x$ -geodesics contain the first arc of $Q_{k,j}$ (i.e., the arc $y_{k,j}z_{k,j}$) and $\deg_G^-(p_{k,j}) \geq 2$. In this case we construct the next ear $Q_{k,j+1}$. Let $v_{k,j+1}$ be an in-neighbour of $p_{k,j}$ distinct from $y_{k,j}$. Since G is an MSC digraph and the subdigraph $V(G_k) \cup V(Q_{k,1}) \cup \dots \cup V(Q_{k,j})$ is strong, it follows that $v_{k,j+1} \notin V(G_k) \cup V(Q_{k,1}) \cup \dots \cup V(Q_{k,j})$.

Now, let $P_{k,j+1} = y_{k,j+1}z_{k,j+1} \dots v_{k,j+1}$ be a $G_k v_{k,j+1}$ -geodesic in G (it may happen that $y_{k,j+1} = p_{k,j}$), and set $Q = P_{k,j+1} \cup v_{k,j+1}p_{k,j}$. Let $t_{k,j+1} := p_{k,j}$. Since $P_{k,j+1}$ is a $G_k v_{k,j+1}$ -geodesic in G , we have $V(G_k) \cap V(Q) = \{y_{k,j+1}, t_{k,j+1}\}$. Let $Q'_k = y_{k,j} \dots t_k$ be the $y_{k,j}t_k$ -subpath of Q_k and $\text{Int}(Q'_k)$ be the set of internal vertices of Q'_k . We show that $y_{k,j+1} \in \text{Int}(Q'_k)$.

First observe the following:

Claim 12 *For every vertex $v \in (V(G_k) \cup V(Q_{k,1}) \cup \dots \cup V(Q_{k,j})) \setminus \text{Int}(Q'_k)$ there exists a yv -path in G not containing the arc $y_{k,j}p_{k,j}$.*

Proof. If $v \in G_{k-1}$ then, since G_{k-1} is strong, there is a yv -path in G_{k-1} and, obviously, it does not contain $y_{k,j}p_{k,j}$. Let P be a yy_k -path in G_k not containing $y_{k,j}p_{k,j}$. If $v \in \text{Int}(Q_k)$ (where $\text{Int}(Q_k)$ stands for the set of internal vertices of Q_k), then the yv -path $P \cup y_k \dots v$ (where $y_k \dots v$ is a subpath of Q_k) does not contain $y_{k,j}p_{k,j}$.

Suppose now that $v \in V(Q_{k,i})$ for some i , $1 \leq i \leq j$. Then we can find a $y_k v$ -path P' going through Q_k and $Q_{k,i}$ not containing $y_{k,j}p_{k,j}$. Thus, $P \cup P'$ is a yv -path not containing $y_{k,j}p_{k,j}$, as required. \square

Suppose that $y_{k,j+1} \in (V(G_k) \cup V(Q_{k,1}) \cup \dots \cup V(Q_{k,j})) \setminus \text{Int}(Q'_k)$. We claim that there is a $y_{k,j}y$ -path in G not containing the arc $y_{k,j}t_{k,j+1}$. By our assumption, each $y_{k,j}x$ -geodesic goes through the arc $y_{k,j}z_{k,j}$ and hence it does not contain the arc $y_{k,j}t_{k,j+1}$. Since $\deg_G^-(x) = 1$, each $y_{k,j}x$ -geodesic goes through y and hence each $y_{k,j}y$ -geodesic does not contain the arc $y_{k,j}t_{k,j+1}$, and thus each $y_{k,j}y$ -geodesic is such a path. By Claim 12 and our assumption on $y_{k,j+1}$, there exists a $yy_{k,j+1}$ -path not containing the arc $y_{k,j}t_{k,j+1}$. Thus, together with Q , we obtain a $y_{k,j}t_{k,j+1}$ -path not containing the arc $y_{k,j}t_{k,j+1}$, which contradicts the fact that G is an MSC digraph. This implies that $y_{k,j+1}$ must belong to $\text{Int}(Q'_k)$, and hence $y_{k,j+1}$ and $t_{k,j+1}$ are internal vertices of Q_k , and $t_{k,j+1}$ precedes $y_{k,j+1}$ in Q_k . Thus, items (i')–(iii') hold for Q . It is easy to see that item (iv') also holds. As we observed above, there is a $y_{k,j}y$ -path not containing the arc $y_{k,j}t_{k,j+1}$. Moreover, by Claim 11, there exists a $yy_{k,j}$ -geodesic in G not containing the arc xy . These arguments imply that item (v') also holds. Therefore, we can set $Q_{k,j+1} := Q$.

Since the graph G is finite, we eventually construct an ear Q_i such that either there is a $y_i x$ -geodesic (y_i is the first vertex of Q_i) not containing the first arc of Q_i or there is an out-neighbour p_i of y_i in Q_{i-1} with $\deg_G^-(p_i) = 1$.

Now, we are ready to construct the whole sequence of pairs $\{(G_i, Q_i)\}_{i \geq 0}$. We already constructed the pair (G_0, Q_0) , so we start with (G_1, Q_1) . Consider the subdigraph G_0 . Since $\deg_G^-(z) \geq 2$, we construct the sequence of ears $Q_{0,1}, Q_{0,2}, \dots, Q_{0,j}$ with the last ear $Q_{0,j} = Q_1$. The ear Q_1 with first vertex y_1 and last vertex t_1 satisfies items (i')–(v') and either there is an y_1x -geodesic not containing the first arc of Q_1 or there is an out-neighbour p_1 of y_1 in Q_0 with $\deg_G^-(p_1) = 1$. We then set $G_1 = G_0 \cup Q_1$. Obviously, the pair (G_1, Q_1) satisfies items (i)–(v).

Assume now that a pair (G_i, Q_i) satisfying items (i)–(v) has been constructed. If there is a $y_i x$ -geodesic in G not containing the arc $y_i z_i$ and $\deg_G^-(z_i) = 1$, then the two arcs yx and $y_i z_i$ satisfy the assumptions of Lemma 7 and we are done. Otherwise, $d_G^-(p_i) = 1$. If $\deg_G^-(z_i) = 1$, then the arcs $y_i z_i$ and $y_i p_i$ satisfy the assumptions of Corollary 8 and we are done. If it is not the case, then $\deg_G^-(z_i) \geq 2$. We construct the next pair (G_{i+1}, Q_{i+1}) as follows. Since $\deg_G^-(z_i) \geq 2$, we construct the sequence of ears $Q_{i,1}, Q_{i,2}, \dots, Q_{i,j}$ with the last ear $Q_{i,j} = Q_{i+1}$. The ear Q_{i+1} with first vertex y_{i+1} and last vertex t_{i+1} satisfies items (i')–(v') and either there is a $y_{i+1}x$ -geodesic not containing the first arc of Q_i or there is an out-neighbour p_{i+1} of y_{i+1} in Q_i with $\deg_G^-(p_{i+1}) = 1$. We then set $G_{i+1} = G_i \cup Q_{i+1}$.

Since the digraph G is finite, we eventually construct a pair (G_k, Q_k) such that $\deg_G^-(z_k) = 1$, and then either the two arcs yx and $y_k z_k$ satisfy the assumptions of Lemma 7 or the two arcs $y_k z_k$ and $y_k p_k$ satisfy the assumptions of Corollary 8. In both cases, we get $s\vec{r}c(G) \leq n - 1$.

This concludes the proof of Theorem 9. \square

4 Strong rainbow connection number of non-Hamiltonian digraphs

Since $r\vec{c}(G) \leq r\vec{c}(H)$ whenever H is a spanning subdigraph of an oriented graph G , it follows from Theorem 9 that $r\vec{c}(G) \leq n - 1$ for every non-Hamiltonian oriented graph G of order n (this was already observed in [9]). Although the inequality $r\vec{c}(G) \leq r\vec{c}(H)$ does not hold in general for digraphs (see Proposition 3), we can prove a similar result for non-Hamiltonian digraphs, with an additional assumption on the diameter. However, we conjecture that this diameter condition can be dropped.

Theorem 13 *Let G be a strong digraph of order n . If G is non-Hamiltonian and $\text{diam}(G) = n - 1$, then $s\vec{r}c(G) \leq n - 1$.*

Proof. Let G be a strong digraph that is non-Hamiltonian and such that $\text{diam}(G) = n - 1$. Let (v_1, v_n) be an antipodal pair of vertices and $P = v_1 v_2 \dots v_n$ be a $v_1 v_n$ -geodesic. Since P is a geodesic, all arcs of G that are not in P are of the form $v_i v_j$ with $1 \leq j < i \leq n$. Furthermore, $v_n v_1 \notin A(G)$, since G is non-Hamiltonian.

Let φ be the partial $(n - 1)$ -arc-colouring of G defined by $\varphi(v_i v_j) = i$ for every i, j , $1 \leq i \leq n - 1$, $1 \leq j \leq n$ (the uncoloured arcs are those arcs of the form $v_n v_i$, $i < n$). Clearly, all paths not going through the vertex v_n are rainbow. We will show that we can colour every arc of the form $v_n v_i$ with a colour in $\{1, \dots, n - 1\}$ in such a way that we obtain a strong rainbow arc-colouring of G .

Let Q be a $v_n v_1$ -geodesic and $v_n v_i$ be the first arc in Q . Let v_k be the first vertex in the sequence v_2, \dots, v_i such that there is a $v_n v_k$ -geodesic that does not contain the arc $v_1 v_2$. Observe that by our assumption on v_k we have $k - 1 < i$ and there is no $v_n v_1$ -geodesic containing an arc coloured with $k - 1$. We then colour the arcs of the form $v_n v_i$ in the following way:

- we colour the arc $v_n v_i$ with colour $k - 1$ (it may happen that $k - 1 = 1$),
- we colour any other arc $v_n v_j$, $j \neq i$, with colour 1.

Since all geodesics whose arcs were already coloured by φ are rainbow, it is enough to prove that for every two vertices v_a and v_b such that every $v_a v_b$ -geodesic contains an arc of the form $v_n v_c$, there exists a rainbow $v_a v_b$ -geodesic.

Suppose first that the colour 1 is used twice in a $v_a v_b$ -geodesic R , which implies that R is of the form $v_a \dots v_n v_c \dots v_1 v_2 \dots v_b$, with either $v_c \neq v_i$ or $v_c = v_i$ and $k = 2$. In the former case, replacing the subpath $v_n v_c \dots v_1$ of R by the $v_n v_1$ -geodesic Q gives a $v_a v_b$ -geodesic using exactly once the colour 1. In the latter case, we can replace the subpath $v_n v_c \dots v_1 v_2$ of R by the $v_n v_2$ -geodesic that does not contain the arc $v_1 v_2$, which gives a $v_a v_b$ -geodesic using exactly once the colour 1.

Suppose now that the colour $k - 1$ is used twice in a $v_a v_b$ -geodesic R , which implies that R is of the form $v_a \dots v_n v_i \dots v_{k-1} v_j \dots v_b$, and that the colour 1 is used at most once in R . Since v_k is the first vertex in the sequence $\{v_2, \dots, v_i\}$ such that there is a $v_n v_k$ -geodesic not containing the arc $v_1 v_2$, the subpath $v_n v_i \dots v_{k-1}$ of R necessarily contains the vertex v_1 , so that R is of the form $v_a \dots v_n v_i \dots v_1 \dots v_{k-1} v_j \dots v_b$. Moreover, there is a $v_n v_k$ -geodesic Q' not containing v_{k-1} . If $j = k$, then replacing the subpath $v_n v_i \dots v_1 \dots v_{k-1} v_k$ of R by Q' gives a $v_a v_b$ -geodesic using at most once colour $k - 1$ and at most once colour 1. Thus, $j < k - 1$. But in that case, the path $v_1 v_2 \dots v_j$, of length $j - 1$, is shorter than the path $v_1 \dots v_{k-1} v_j$, of length $k - 1$, a contradiction.

Therefore, every two vertices are joined by a rainbow geodesic. This concludes the proof. \square

We then propose the two following conjectures:

Conjecture 14 *Let G be a strong digraph of order n . If G is non-Hamiltonian, then $s\vec{r}c(G) \leq n - 1$.*

Conjecture 15 *Let G be a strong digraph of order n . If $\text{diam}(G) \leq n - 2$, then $s\vec{r}c(G) \leq n - 1$.*

Note that if Conjecture 15 is true, it follows from Theorem 9 that Conjecture 14 is also true.

5 Strong rainbow connection number of tournaments

It was proved in [9] that the rainbow connection number of every strong tournament of order $n \geq 5$ is at most $n - 1$. We first prove that a similar result holds for the strong rainbow connection number. Again, by Proposition 1, our result implies the result of [9].

Theorem 16 *Let T be a strong tournament of order $n \geq 4$. Then $s\vec{r}c(T) \leq n - 1$.*

Proof. Let $\text{diam}(T) = d$, (s, t) be an antipodal pair of vertices and $P = s s_1 \dots s_{d-1} t$ be an st -geodesic. Note that we necessarily have $tv \in A(T)$ for every $v \in V(P) \setminus \{s_{d-1}, t\}$. Let $c : V(T) \rightarrow \{1, 2, \dots, n\}$ be a mapping such that $c(t) = n$ and $c(v) \neq c(w)$ for every two distinct vertices v and w in $V(T)$.

We will construct a strong rainbow arc-colouring φ of T using at most $n - 1$ colours. We consider two cases, depending on the value of $\text{diam}(T)$.

1. $\text{diam}(T) \geq 3$.

Let $\varphi : A(T) \rightarrow \{1, 2, \dots, n-1\}$ be the arc-colouring given by

- $\varphi(vu) = c(v)$ for every arc vu with $v \neq t$,
- $\varphi(ts) = c(s_1)$,
- $\varphi(tv) = c(s)$ for every $v \in V(T) \setminus \{s, t\}$.

The only colours that can appear more than once on an elementary path Q are $c(s)$ (on two arcs su_1 and tu_2 , $u_2 \neq s$) and $c(s_1)$ (on two arcs s_1v_1 and ts).

In the former case, either s precedes t in Q , in which case Q cannot be a geodesic since $|Q| > \text{dist}_T(s, t) = \text{diam}(T)$ (where $|Q|$ denotes the length of Q), or t precedes s in Q , in which case Q cannot be a geodesic since $ts \in A(T)$ and thus the path ts is shorter than the subpath $tu_2 \dots s$ of Q .

Consider the latter case. If s_1 precedes t in Q , Q cannot be a geodesic since otherwise the subpath $s_1v_1Q'ts$ of Q would be a geodesic of length d (since $\text{dist}_T(s_1, t) = d - 1$), which would imply $sv_1 \in A(T)$ and thus the path $sv_1Q't$ would be shorter than P , a contradiction. On the other hand, if t precedes s_1 in Q then Q cannot be a geodesic since $ts_1 \in A(T)$ and thus the path ts_1 is shorter than the subpath $ts \dots s_1$ of Q .

Therefore, every geodesic is rainbow and we are done.

2. $\text{diam}(T) = 2$.

Observe first that since T is strong and $n \geq 4$, the vertex s can be chosen in such a way that there exists a vertex $t' \neq t$ with $\text{dist}_T(s, t') = \text{dist}_T(s, t) = 2$. For that, it suffices to choose a vertex s with $\text{deg}_T^-(s) \geq 2$, so that $ts, t's \in A(T)$.

Let $\varphi : A(T) \rightarrow \{1, 2, \dots, n-1\}$ be the arc-colouring given by

- $\varphi(vu) = c(v)$ for every arc vu with $v \neq t$,
- $\varphi(ts) = c(t')$,
- $\varphi(tv) = c(s)$ for $v \in V(T) \setminus \{s, t\}$.

Every geodesic is of length 1 or 2 and we claim that no geodesic of length 2 can use twice the same colour. Clearly, there is no path of length 2 using twice the colour $c(s)$. Moreover, the only path using twice the colour $c(t')$ is $t'ts$, which is not a geodesic since $t's \in A(T)$. Therefore, every geodesic is rainbow and we are done.

This completes the proof. □

It was proved in [9] that for every two integers n and k with $3 \leq k \leq n-1$, there exists a strong tournament of order n with rainbow connection number k . We can adapt the construction and the proof given in [9] to get a similar result for the strong rainbow connection number.

Given two digraphs G and H and a vertex $u \in V(G)$, we call *digraph obtained from G by expanding u into H* the digraph G' obtained from G by replacing the vertex u by a copy of H , and replacing every arc xu (resp. ux) in G by all the arcs xv (resp. vx) for $v \in V(H)$.

Lemma 17 *Let G be a strong digraph, H be a transitive digraph, $u \in V(G)$. Let G' be the digraph obtained from G by expanding u into H . If there is a strong rainbow arc-colouring φ of G with $s\vec{r}c(G)$ colours such that u belongs to some rainbow shortest cycle C , then $s\vec{r}c(G') \leq s\vec{r}c(G)$.*

Proof. To prove this lemma, it suffices to prove that the strong rainbow colouring φ of G can be extended to a strong rainbow colouring of G' . Let φ' be the extension of the arc-colouring φ to G' given by $\varphi'(xv) = \varphi(xu)$ (resp. $\varphi'(vx) = \varphi(ux)$) for every arc xv or vx with $x \in V(G) \setminus \{u\}$ and $v \in V(H)$. The colours of any other arc in H are chosen freely among the $s\vec{r}c(G)$ available colours.

Observe that every rainbow geodesic in G (under φ) that does not contain u remains a rainbow geodesic in G' (under φ').

If for $x, y \notin V(H)$ the rainbow geodesic P in G contains u , then we replace the vertex u of P by any vertex of H to obtain a rainbow geodesic in G' . A rainbow geodesic between any pair of vertices v and v' in H is either an arc of H or can be found using the shortest rainbow cycle C of G : writing $C = ux_1 \dots x_\ell u$ in G , the path $vx_1 \dots x_\ell v'$ is rainbow in G' . Therefore, φ' is a strong rainbow arc-colouring of G' using $s\vec{r}c(G)$ colours. \square

Theorem 18 *For every two integers n and k , $3 \leq k \leq n - 1$, there exists a strong tournament $T_{n,k}$ of order n with $s\vec{r}c(T_{n,k}) = k$.*

Proof. We first define $T_{k+1,k}$, which will be used later on to build the tournament for other values of n . For $k \geq 3$, let $T_{k+1,k}$ be the tournament given by $V(T_{k+1,k}) = \{v_1, \dots, v_{k+1}\}$ and $A(T_{k+1,k}) = \{v_i v_{i+1}, 1 \leq i \leq k\} \cup \{v_j v_i, 1 \leq i \leq j - 2 \leq k - 1\}$. The tournament $T_{k+1,k}$ is thus a directed path of length k with all other arcs directed backward.

Since the only directed path from v_1 to v_{k+1} is $v_1 v_2 \dots v_{k+1}$, we get $\text{diam}(T_{k+1,k}) = k$ and thus $s\vec{r}c(T_{k+1,k}) \geq k$ by Proposition 1. Let now φ be the k -arc-colouring of $T_{k+1,k}$ defined by

- $\varphi(v_i v_{i+1}) = i$ for every i , $1 \leq i \leq k$, and
- $\varphi(v_j v_i) = 1$ for every i, j , $1 \leq i \leq j - 2 \leq k - 1$, and
- $\varphi(v_3 v_1) = 3$.

We claim that that every two vertices are linked by a rainbow geodesic:

- if $i < j$, then $v_i v_{i+1} \dots v_j$ is a rainbow geodesic,
- $v_j v_i$ is a rainbow geodesic for every i, j , $i \leq j - 2$, and
- either $v_i v_{i+1} v_{i-1}$ or $v_i v_{i-2} v_{i-1}$ is a rainbow geodesic otherwise.

Hence, φ is a strong rainbow arc-colouring of $T_{k+1,k}$ which gives $s\vec{r}c(T_{k+1,k}) = k$, as required.

Let us now define $T_{n,k}$ for $n \geq k + 2$. Consider any family T_1, \dots, T_{k+1} of non-empty transitive tournaments such that $|V(T_1)| + \dots + |V(T_{k+1})| = n$. Let $T_{n,k}$ be the tournament of order n obtained from $T_{k+1,k}$ by expanding each vertex v_i into T_i , $1 \leq i \leq k + 1$, and extending the colouring φ as in the proof of Lemma 17.

Since every vertex of $T_{k+1,k}$ belongs to a shortest rainbow cycle (of length three), we get $s\vec{r}c(T_{n,k}) \leq k$ by Lemma 17. Moreover, since the diameter of $T_{n,k}$ is still at least k , we also have $s\vec{r}c(T_{n,k}) \geq k$ by Proposition 1. \square

In [9], it has been proved that the rainbow connection number of every tournament is bounded by a function of its diameter, namely $\vec{rc}(T) \leq \text{diam}(T) + 2$. However, such a result does not hold for the strong rainbow connection number, as shown by the next theorem.

Theorem 19 *For every two integers d and k , $3 \leq d \leq k$, there exists a strong tournament $F_{d,k}$ with $\text{diam}(F_{d,k}) = d$ and $s\vec{rc}(F_{d,k}) = k$.*

Proof. For every $k \geq d \geq 3$ we construct a tournament with diameter d and strong rainbow connection number k . We consider two cases.

1. $d = 3$.

Let $F_{k,3}$, $k \geq 3$, be tournament given by $V(F_{k,3}) = \{x_1, \dots, x_k\} \cup \{y_1, \dots, y_k\}$ and $A(F_{k,3}) = A_{k,3}^1 \cup A_{k,3}^2 \cup A_{k,3}^3 \cup A_{k,3}^4$, with

$$\begin{aligned} A_{k,3}^1 &= \{x_i x_j : 1 \leq i < j \leq k\}, \\ A_{k,3}^2 &= \{y_i y_j : 1 \leq i < j \leq k\}, \\ A_{k,3}^3 &= \{y_i x_j : 1 \leq i, j \leq k, i \neq j\}, \\ A_{k,3}^4 &= \{x_i y_i : 1 \leq i \leq k\}. \end{aligned}$$

Observe that $\text{dist}_{F_{k,3}}(x_i, y_j) = 3$ for $i > j$ and any two other vertices are at distance either 1 or 2 from each other. Thus, $\text{diam}(F_{k,3}) = 3$. Since there is exactly one $x_i y_j$ -geodesic for any $i > j$, which contains both arcs $x_i y_i$ and $x_j y_j$, no two arcs in $A_{k,3}^4$ can be assigned the same colour in any strong rainbow colouring of $F_{k,3}$. Therefore, $s\vec{rc}(F_{k,3}) \geq k$.

Let $\varphi : A(F_{k,3}) \rightarrow \{1, 2, \dots, k\}$ be the arc-colouring of $F_{k,3}$ defined by

- Arcs of $A_{k,3}^1$:
For every i, j , $1 \leq i < j \leq k$, $\varphi(x_i x_j) = 1$.
- Arcs of $A_{k,3}^2$:
For every i, j , $1 \leq i < j \leq k$, $\varphi(y_i y_j) = \begin{cases} 2 & \text{if } i = 1, \\ 1 & \text{otherwise.} \end{cases}$
- Arcs of $A_{k,3}^3$:
For every i, j , $1 \leq i, j \leq k$, $i \neq j$, $\varphi(y_i x_j) = \begin{cases} 2 & \text{if } i = k \text{ and } j = 1, \\ 1 & \text{if } i = k \text{ and } j \neq 1, \\ i + 1 & \text{otherwise.} \end{cases}$
- Arcs of $A_{k,3}^4$:
For every i , $1 \leq i \leq k$, $\varphi(x_i y_i) = i$.

Let us first consider pairs of vertices (x_i, y_j) , with $i > j$. The only $x_i y_j$ -geodesic is $x_i y_i x_j y_j$, which uses colours i , $i + 1$ and j if $i \neq k$, or k , 1 and j if $i = k$ and $j \neq 1$, or k , 2 and 1 if $i = k$ and $j = 1$. In all cases, this geodesic is thus rainbow.

The other pairs of vertices that are joined by a geodesic of length 2 are either of the form (x_i, y_j) with $j > i$, or of the form (y_i, x_i) . In the former case, the geodesic $x_i y_i y_j$ uses colours i and 1 if $i \neq 1$, or 1 and 2 if $i = 1$, and is thus rainbow. In the latter case, if $i \neq 1$ then the geodesic $y_i x_i x_i$ uses colours $i + 1$ and 1 if $i \neq k$, or 2 and 1 if $i = k$, and is thus rainbow; if $i = 1$, the geodesic $y_1 y_2 x_1$ uses colours 2 and 3 and is thus rainbow. Again, all these pairs of vertices are linked by a rainbow geodesic.

2. $d \geq 4$.

Let $F_{k-d+1,3}$ be the tournament from Case 1 and $T_{d-1,d-2}$ the tournament considered in the proof of Theorem 18, that is $V(T_{d-1,d-2}) = \{v_1, \dots, v_{d-1}\}$ and $A(T_{d-1,d-2}) = \{v_i v_{i+1}, 1 \leq i \leq d-2\} \cup \{v_j v_i, 1 \leq i \leq j-2 \leq d-3\}$.

Let $F_{d,k}$ be the tournament given by $V(F_{d,k}) = V(F_{k-d+1,3}) \cup V(T_{d-1,d-2})$ and $A(F_{d,k}) = A(F_{k-d+1,3}) \cup A(T_{d-1,d-2}) \cup A'_{d,k} \cup A''_{d,k}$, with:

$$A'_{d,k} = \{v_1 x_i, y_i v_1 : 1 \leq i \leq k-d+1\},$$

$$A''_{d,k} = \{v_i x_j, v_i y_j : 2 \leq i \leq d-1, 1 \leq j \leq k-d+1\}.$$

Since $\text{dist}_{F_{d,k}}(x_i, v_{d-1}) = d$ for every i , $1 \leq i \leq k-d+1$, and any other two vertices are at distance less than d from each other, it follows that $\text{diam}(F_{d,k}) = d$.

Observe that the path $x_i y_i v_1 v_2 \dots v_{d-1}$ is the unique $x_i v_{d-1}$ -geodesic for every i , $1 \leq i \leq k-d+1$. Therefore, in any strong rainbow colouring of $F_{d,k}$ no two arcs in this path can share the same colour. Moreover, for every i, j , $1 \leq j < i \leq k-d+1$, $\text{dist}_{F_{d,k}}(x_i, y_j) = 3$ and the unique $x_i y_j$ -geodesic is $x_i y_i x_j y_j$, which contains both arcs $x_i y_i$ and $x_j y_j$. Therefore, in any strong rainbow colouring of $F_{d,k}$ no two arcs in the set $\{x_i y_i, 1 \leq i \leq k-d+1\}$ can share the same colour. We thus get $s\vec{r}c(F_{d,k}) \geq d-1 + k-d+1 = k$.

We will now extend the strong rainbow arc-colouring φ of $F_{k-d+1,3}$ defined in Case 1 in order to get a strong rainbow arc-colouring of $F_{d,k}$. Let φ' be the arc-colouring of $F_{d,k}$ defined by

- Arcs of $F_{2(k-d+1),3}$:
 $\varphi'(e) = \varphi(e)$ for every $e \in A(F_{k-d+1,3})$,
- Arcs of $T_{d-1,d-2}$:
 $\varphi'(v_i v_{i+1}) = k-d+2+i$ for every i , $1 \leq i \leq d-2$,
 $\varphi'(v_j v_i) = k-d+3$ for every i, j , $1 \leq i \leq j-2 \leq d-3$,
 $\varphi'(v_3 v_1) = k-d+5$.
- Arcs of A' :
 $\varphi'(y_i v_1) = k-d+2$ for every i , $1 \leq i \leq k-d+1$,
 $\varphi'(v_1 x_i) = k-d+3$ for every i , $1 \leq i \leq k-d+1$.
- Arcs of A'' :
 $\varphi'(e) = 1$ for $e \in E''$ (those edges can be coloured with any colour).

Observe that φ' and φ are identical of the subdigraph $F_{k-d+1,3}$, and that the colouring of the subdigraph $T_{d-1,d-2}$ induced by φ' is the same as in the proof of Theorem 18, shifted by $k-d+2$. We will prove that any two vertices a and b in $F_{d,k}$ are joined by a rainbow ab -geodesic, so that φ' is a strong rainbow arc-colouring of $F_{d,k}$.

- If $a, b \in V(F_{k-d+1,3})$, then any ab -geodesic in the subdigraph $F_{k-d+1,3}$ is also an ab -geodesic in $F_{d,k}$. Since φ is a strong rainbow arc-colouring of $F_{k-d+1,3}$, there exists a rainbow ab -geodesic.
- If $a, b \in V(T_{d-1,d-2})$, then any ab -geodesic in the subdigraph $T_{d-1,d-2}$ is also an ab -geodesic in $F_{d,k}$. Thus, there exists a rainbow ab -geodesic.

- If $a \in V(F_{k-d+1,3})$ and $b \in V(T_{d-1,d-2})$, then a subpath of a path of the form $x_i y_i v_1 v_2 \dots v_{d-1}$ is an ab -geodesic, which is thus rainbow.
- If $a \in V(T_{d-1,d-2}) \setminus \{v_1\}$ and $b \in V(F_{k-d+1,3})$, then $ab \in A(F_{d,k})$.
- If $a = v_1$ and $b \in V(F_{k-d+1,3})$, then either $v_1 b \in A(T)$ (in this case $b = x_i$) or $\text{dist}_{F_{d,k}}(v_1, b) = 2$ (in this case $b = y_i$) and the path $v_1 x_i y_i$ is a rainbow geodesic.

Therefore, $s\vec{r}c(T) = k$, as required.

This concludes the proof. □

Theorem 19 states that the strong rainbow connection number of a tournament with diameter at least 3 can be arbitrarily large. We believe that this is no longer true for tournaments with diameter 2 and thus propose the following conjecture:

Conjecture 20 *There exists a constant t such that for every strong digraph G with $\text{diam}(G) = 2$, $s\vec{r}c(G) \leq t$.*

Acknowledgement

Research contained in this paper were supported by the joint program PICS CNRS 6367 “Graph partitions”. The research of Elżbieta Sidorowicz were also partly supported by the French Government and the French Embassy in Poland.

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