

# A NOTE ON THE ORIENTED CHROMATIC NUMBER OF GRAPHS WITH MAXIMUM DEGREE THREE

Éric Sopena and Laurence Vignal

LaBRI, Univ. Bordeaux I, 33405 Talence, France

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**Abstract.** A homomorphism of an oriented graph  $G$  to an oriented graph  $H$  is a mapping  $\varphi$  from the vertex set of  $G$  to the vertex set of  $H$  such that  $\varphi(u)\varphi(v)$  is an arc in  $H$  whenever  $uv$  is an arc in  $G$ . The oriented chromatic number of an oriented graph  $G$  is the minimum number of vertices in an oriented graph  $H$  such that there exists a homomorphism of  $G$  to  $H$ . We prove in this short note that every oriented graph with maximum degree 3 has oriented chromatic number at most 11.

**Keywords.** Oriented colorings, Bounded degree.

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## 1 Introduction

An *oriented graph* is a digraph having no opposite arcs. Let  $G$  and  $H$  be two oriented graphs. A homomorphism of  $G$  to  $H$  is a mapping  $\varphi$  from the vertex set of  $G$  to the vertex set of  $H$  such that  $\varphi(u)\varphi(v)$  is an arc in  $H$  whenever  $uv$  is an arc in  $G$ . If  $G$  has a homomorphism to  $H$  we say that  $G$  is  $H$ -colorable. Such a homomorphism is called a  $H$ -coloring of  $G$  and the vertices of  $H$  are called *colors*. The oriented chromatic number of an oriented graph  $G$  is the minimum number of vertices in an oriented graph  $H$  such that  $G$  is  $H$ -colorable.

Homomorphisms and oriented chromatic numbers of oriented graphs have been studied in [1, 3, 4, 5, 6]. In particular, it was proved in [6] that every oriented graph with maximum degree 3 has oriented chromatic number at most 16. It was also conjectured that the tight value of this bound is 7. The aim of this short note is to decrease this upper bound to 11. In Section 2 we introduce the tournament  $QR_{11}$  on 11 vertices and prove in Section 3 that every oriented graph with maximum degree three is  $QR_{11}$ -colorable.

## 2 The tournament $QR_{11}$

Let  $QR_{11}$  be the tournament build from the non-zero quadratic residues of 11, that is the tournament with vertex set  $\{0, 1, \dots, 10\}$  and such that  $ij$  is an arc if and only if  $j - i \equiv 1, 3, 4, 5$  or  $9 \pmod{11}$ . It is not difficult to check that this tournament is *arc-transitive* [2] (for every two arcs  $xy$  and  $zt$  there exists an automorphism of  $QR_{11}$  which maps  $xy$  to  $zt$ ).

Let  $G$  be an oriented graph. If  $xy$  is an arc in  $G$  we say that  $y$  is a *successor* of  $x$  and  $x$  a *predecessor* of  $y$ . Let us call an *orientation vector* of length  $n$  any sequence  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  in  $\{0, 1\}^n$ . Let  $X = (x_1, x_2, \dots, x_n)$  be a sequence of pairwise distinct vertices of  $G$ . A vertex  $y$  of  $G$  is said to be an  $\alpha$ -successor of  $X$  if for every  $i$ ,  $1 \leq i \leq n$ ,  $yx_i$  is an arc in  $G$  if  $\alpha_i = 1$  and  $x_iy$  is an arc in  $G$  otherwise. The following easy observations will be useful in the next Section.

**Observation 1** For every two distinct vertices  $x$  and  $y$  in  $QR_{11}$ ,  $(x, y)$  has two  $\alpha$ -successors for  $\alpha = (0, 0), (0, 1), (1, 1)$  and three  $\alpha$ -successors for  $\alpha = (1, 0)$ .

Since  $QR_{11}$  is arc-transitive, it is sufficient to consider one arc, say 01. It is then easy to check that the (0,0)-successors are 4 and 5, the (0,1)-successors are 3 and 9, the (1,1)-successors are 7 and 8 and the (1,0)-successors are 2, 6 and 10. Observe that in this latter case vertices 0, 1 and 2, 6 or 10 induce a circuit in  $QR_{11}$ .

**Observation 2** *For every orientation vector  $\alpha$  of length 3, every circuit  $xyz$  in  $QR_{11}$  has an  $\alpha$ -successor.*

As before we may fix  $x = 0$  and  $y = 1$ . We then have three cases to consider, namely  $z = 2, 6$  or  $10$ . The corresponding  $\alpha$ -successors are given in the following table:

$\alpha$	$z = 2$	$z = 6$	$z = 10$
(0,0,0)	5	4	4
(0,0,1)	4	5	5
(0,1,0)	3	9	3
(0,1,1)	9	3	9
(1,0,0)	6	10	2
(1,0,1)	10	2	6
(1,1,0)	7	7	8
(1,1,1)	8	8	7

This notion of  $\alpha$ -successor will be used in the next Section as follows: suppose that an oriented graph  $G$  has a homomorphism  $\varphi$  to an oriented graph  $H$ . Let  $G'$  be the graph obtained from  $G$  by adding a new vertex  $x$  together with some arcs linking  $x$  to vertices  $x_1, x_2, \dots, x_n$ . Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  be the orientation vector defined by  $\alpha_i = 1$  (resp.  $\alpha_i = 0$ ) if  $x_i x$  (resp.  $xx_i$ ) is an arc in  $G'$  for every  $i$ ,  $1 \leq i \leq n$ . If the sequence  $(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n))$  has an  $\alpha$ -successor  $u$  in  $H$  then we can extend  $\varphi$  to a homomorphism of  $G'$  to  $H$  by assigning the color  $u$  to  $x$  (we say that the color  $u$  is *compatible* with the colors of  $x_1, x_2, \dots, x_n$ ).

### 3 $QR_{11}$ -coloring of graphs with maximum degree three

We prove in that section that every oriented graph with maximum degree three is  $QR_{11}$ -colorable and thus has oriented chromatic number at most 11.

**Theorem 3** *Every graph  $G$  with maximum degree three is  $QR_{11}$ -colorable.*

**Proof.** W.l.o.g. we may assume that  $G$  is connected. We proceed by induction on the number of vertices  $n$  of  $G$ . If  $n = 1$  the result is obvious. Otherwise, let  $x$  be a vertex with minimum degree in  $G$ . We have three cases to consider:

1.  $x$  has degree one in  $G$ .

Let  $G_1$  denotes the graph obtained from  $G$  by deleting the vertex  $x$ . Since every vertex in  $QR_{11}$  has 5 successors and 5 predecessors, every  $QR_{11}$ -coloring of  $G_1$  can be extended to a  $QR_{11}$ -coloring of  $G$ .

2.  $x$  has degree two in  $G$ .

Let  $y, z$  denote the neighbours of  $x$  in  $G$  and  $G_2$  be the graph obtained from  $G$  by deleting the vertex  $x$  and adding the arc  $yz$  if  $y$  and  $z$  are not already linked. Every  $QR_{11}$ -coloring of  $G_2$  assigns two distinct colors to  $y$  and  $z$ . By Observation 1, such a coloring can then be extended to a  $QR_{11}$ -coloring of  $G$ .

3.  $x$  has degree three in  $G$ .

Let  $y, z, t$  denote the neighbours of  $x$  in  $G$  and  $G_3$  be the graph obtained from  $G$  by deleting the vertices  $x, y, z, t$  and adding an arc between the two remaining neighbours of  $y$  (resp.  $z, t$ ) if they are not already linked. We know by Observation 1 that at least two distinct colors may be assigned to  $y, z, t$ . Let  $a_1, a_2$  (resp.  $b_1, b_2/c_1, c_2$ ) denote two colors that may be assigned to  $y$  (resp.  $z/t$ ) and such that  $a_1a_2$  (resp.  $b_1b_2/c_1c_2$ ) is an arc in  $QR_{11}$ .

If there exist  $i_1, i_2, i_3$  such that  $a_{i_1}, b_{i_2}, c_{i_3}$  induce a circuit in  $QR_{11}$ , we know by Observation 2 that if we assign those colors to  $y, z, t$  then we can extend the  $QR_{11}$ -coloring of  $G_3$  to a  $QR_{11}$ -coloring of  $G$ .

Otherwise it is not difficult to check that no more than four distinct colors can appear in  $\{a_1, a_2, b_1, b_2, c_1, c_2\}$ . We then have two cases to consider:

(a) *All the arcs linking  $x$  to  $y, z, t$  have the same direction.*

We can then color  $y, z, t$  by using at most two distinct colors. By Observation 1 we can then find a color for  $x$  and thus extend the  $QR_{11}$ -coloring of  $G_3$  to a  $QR_{11}$ -coloring of  $G$ .

(b) *Only two arcs linking  $x$  to  $y, z, t$  have the same direction.*

W.l.o.g. we can suppose that the arcs linking  $x$  to  $y, z$  have the same direction. We then have the two following sub-cases:

i.  $\{a_1, a_2\} \cap \{b_1, b_2\} \neq \emptyset$ .

Let  $d$  be any color in  $\{a_1, a_2\} \cap \{b_1, b_2\}$ . We color  $y, z$  with color  $d$  and  $t$  with a color distinct from  $d$ . By Observation 1 we can then find a color for  $x$  and thus extend the  $QR_{11}$ -coloring of  $G_3$  to a  $QR_{11}$ -coloring of  $G$ .

ii.  $\{a_1, a_2\} \cap \{b_1, b_2\} = \emptyset$ .

In that case we necessarily have  $\{c_1, c_2\} \subseteq \{a_1, a_2\} \cup \{b_1, b_2\}$ . Suppose that  $xt$  is an arc in  $G$ . We color  $t$  and  $y$  (or  $t$  and  $z$ ) in such a way that there is an arc in  $QR_{11}$  from the color assigned to  $t$  to the color assigned to  $y$  (or  $z$ ). It is not difficult to check that this is always possible. By Observation 1 there exist 3 colors that can be assigned to  $x$  in a way that is compatible with the colors of  $t$  and  $y$  (or  $z$ ). By Observation 1 at most two of them are not compatible with  $b_1, b_2$  (or  $a_1, a_2$ ). Thus we can color  $x$  and  $z$  (or  $y$ ) and extend the  $QR_{11}$ -coloring of  $G_3$  to a  $QR_{11}$ -coloring of  $G$ .

□

## References

- [1] Borodin O.V., A.V. Kostochka, J. Nešetřil, A. Raspaud and E. Sopena. On the maximum average degree and the oriented chromatic number of a graph. Preprint, 1995.
- [2] Fried E. On homogeneous tournaments. In: *Combinatorial Theory and its Applications*, Vol. II., P. Erdős *et al.* (Eds), North Holland, Amsterdam, 467–476, 1970.
- [3] Kostochka A.V., E. Sopena and X. Zhu. Acyclic and oriented chromatic numbers of graphs. Preprint 95-087, Univ. Bielefeld, 1995.
- [4] Nešetřil J., A. Raspaud and E. Sopena. Colorings and girth of oriented planar graphs. *Discrete Math.*, to appear, 1997.
- [5] Raspaud A. and E. Sopena. Good and semi-strong colorings of oriented planar graphs. *Inf. Processing Letters* **51**, 171–174, 1994.
- [6] Sopena E. The chromatic number of oriented graphs. Preprint 1083-95, Univ. Bordeaux I, 1995.