

# Complete oriented colourings and the oriented achromatic number

ÉRIC SOPENA

Univ. Bordeaux, LaBRI, UMR5800, F-33400 Talence

CNRS, LaBRI, UMR5800, F-33400 Talence

`eric.sopena@labri.fr`

December 18, 2013

## Abstract

In this paper, we initiate the study of complete colourings of oriented graphs and the new associated notion of the oriented achromatic number of oriented and undirected graphs. In particular, we prove that for every integers  $a$  and  $b$  with  $2 \leq a \leq b$ , there exists an oriented graph  $\vec{G}_{a,b}$  with oriented chromatic number  $a$  and oriented achromatic number  $b$ . We also prove that adding a vertex, deleting a vertex or deleting an arc in an oriented graph may increase its oriented achromatic number by an arbitrarily large value, while adding an arc may increase its oriented achromatic number by at most 2.

Finally, we consider the behaviour of the oriented chromatic and achromatic numbers with respect to elementary homomorphisms and show in particular that, in contrast to the undirected case, there is no interpolation homomorphism theorem for oriented graphs.

Our notion of complete colouring of oriented graphs corresponds to the notion of complete homomorphisms of oriented graphs and, therefore, differs from the notion of complete colourings of directed graphs recently introduced by Edwards in [Harmonious chromatic number of directed graphs. *Discrete Appl. Math.* **161** (2013), 369–376.].

**Keywords:** Achromatic number; Complete colouring; Complete homomorphism; Elementary homomorphism; Oriented colouring; Oriented chromatic number.

**MSC2010 Classification.** 05C15 (primary), 05C20.

## 1 Introduction

All the graphs we consider in this paper are finite and simple. We denote by  $V(G)$  the set of vertices of an undirected graph  $G$  and by  $E(G)$  its set of edges.

A *complete colouring* of an undirected graph  $G$  is a proper vertex colouring of  $G$  such that for every pair of colours there is at least one edge in  $G$  whose endpoints are coloured with this pair of colours. The *achromatic number* of  $G$ , denoted  $\psi(G)$ , is then defined as the greatest number of colours in a complete colouring of  $G$ .

The achromatic number of a graph can be equivalently defined in terms of complete homomorphisms. A *homomorphism* of a graph  $G$  to a graph  $H$  is a mapping  $h$  from  $V(G)$  to  $V(H)$  such that  $h(u)h(v)$  is an edge in  $H$  whenever  $uv$  is an edge in  $G$ . The *homomorphic image* of  $G$  under  $h$ , denoted  $h(G)$ , is the subgraph of  $H$  given by  $V(h(G)) = h(V(G))$  and  $xy \in E(h(G))$  if and only if there exists an edge  $uv \in E(G)$  such that  $h(u) = x$  and  $h(v) = y$ . A homomorphism  $h$  of  $G$  to  $H$  is *complete* if and only if  $h(G) = H$ .

A proper vertex  $k$ -colouring of  $G$  can thus be viewed as a homomorphism of  $G$  to  $K_k$ , the complete graph of order  $k$ , and a complete  $k$ -colouring of  $G$  as a complete homomorphism of  $G$  to  $K_k$ . Moreover, the ordinary chromatic number  $\chi(G)$  of  $G$  corresponds to the smallest  $k$  for which there exists a complete homomorphism of  $G$  to  $K_k$ , while the achromatic number  $\psi(G)$  of  $G$  corresponds to the greatest such  $k$ .

The achromatic number was introduced by Harary, Hedetniemi and Prins [16] in 1967 and has attracted a lot of attention since then. The interested reader may consult the two surveys by Edwards [9] and by Hughes and MacGillivray [18], or Chapter 12 in the recent book *Chromatic Graph Theory* by Chartrand and Zhang [7] (whose reading clearly inspired our work), as well as the detailed online bibliography proposed by Edwards [10] which contains about 200 references on harmonious and complete colourings.

In this paper, we introduce the notions of complete colourings, complete homomorphisms and achromatic numbers of oriented graphs, that is antisymmetric digraphs. The notion of (oriented) colouring of oriented graphs was introduced by Courcelle [8] in 1994 and then studied by several authors (see [24] for a survey on oriented colourings). By taking an appropriate notion of an “oriented clique”, namely an oriented graph of order  $n$  for which every oriented colouring must use  $n$  colours, we get a natural extension to oriented graphs of complete colourings, complete homomorphisms, and the achromatic number.

Very recently, Edwards considered complete colourings of directed graphs [11], but our definition, based on oriented colourings, is quite different and exactly corresponds to complete homomorphisms of *oriented* graphs (roughly speaking, in [11], two arcs  $\vec{uv}$  and  $\vec{wx}$  may be coloured by a colouring function  $\gamma$  in such a way that  $\gamma(u) = \gamma(x)$  and  $\gamma(v) = \gamma(w)$ , while such a situation is forbidden in oriented colourings). In particular, every oriented graph admits a complete colouring in our sense, which is not the case for the directed version of Edwards (it is even NP-complete to decide whether an oriented graph admits a (directed) complete colouring or not).

Our paper is organised as follows. In Section 2 we recall the definitions of oriented colourings, homomorphisms of oriented graphs and oriented cliques and we introduce in Section 3 the new notions of complete oriented colourings and oriented achromatic numbers, together with some basic results. Our main results concerning the oriented achromatic number of graphs are then given in Section 4. Finally, we investigate in Section 5 the properties of elementary homomorphisms with respect to the oriented chromatic number and the oriented achromatic number of oriented graphs.

## 2 Oriented colourings, homomorphisms and oriented cliques

An *oriented graph*  $\vec{G} = (V(\vec{G}), E(\vec{G}))$  is an antisymmetric digraph obtained from an undirected (simple) graph  $G = (V(G), E(G))$  having the same set of vertices,  $V(G) = V(\vec{G})$ , by giving to each edge  $uv$  in  $E(G)$  one of its two possible orientations,  $\vec{uv}$  or  $\vec{vu}$ . Such an oriented graph  $\vec{G}$  is said to be an *orientation* of  $G$ . For every vertex  $u$  in  $V(\vec{G})$ , we denote by  $N_{\vec{G}}^+(u)$  the set of *out-neighbours* of  $u$ , that is  $N_{\vec{G}}^+(u) = \{v \mid \vec{uv} \in E(\vec{G})\}$ , and by  $N_{\vec{G}}^-(u)$  the set of *in-neighbours* of  $u$ , that is  $N_{\vec{G}}^-(u) = \{v \mid \vec{vu} \in E(\vec{G})\}$ . We will also denote by  $d_{\vec{G}}^+(u)$  the *outdegree* of  $u$ , that is  $d_{\vec{G}}^+(u) = |N_{\vec{G}}^+(u)|$  and by  $d_{\vec{G}}^-(u)$  the *indegree* of  $u$ , that is  $d_{\vec{G}}^-(u) = |N_{\vec{G}}^-(u)|$ . We will simply use the notation  $N^+(u)$ ,  $N^-(u)$ ,  $d^+(u)$  or  $d^-(u)$  whenever the graph  $\vec{G}$  is clear from the context. For every two non-adjacent vertices  $u$  and  $v$  in  $V(\vec{G})$ , we denote by  $\vec{G} + \vec{uv}$  the oriented graph  $(V(\vec{G}), E(\vec{G}) \cup \{\vec{uv}\})$ . For every arc  $\vec{uv}$  in  $E(\vec{G})$ , we denote by  $\vec{G} - \vec{uv}$  the oriented graph  $(V(\vec{G}), E(\vec{G}) \setminus \{\vec{uv}\})$ .

Let  $\vec{G}$  be an oriented graph. A *directed  $k$ -path* (of length  $k \geq 0$ ) in  $\vec{G}$  is a sequence of  $k + 1$  distinct vertices  $u_0, \dots, u_k$  such that  $\vec{u_i u_{i+1}} \in E(\vec{G})$  for every  $i$ ,  $0 \leq i < k$ . Such a path is said to be directed from  $u_0$  to  $u_k$ . Let  $u$  and  $v$  be any two vertices in  $V(\vec{G})$ . The *directed distance*  $\vec{d}(\vec{G}, u, v)$  from  $u$  to  $v$  in  $\vec{G}$  is the minimal length of a directed path from  $u$  to  $v$  in  $\vec{G}$ . If no such path exists, then  $\vec{d}(\vec{G}, u, v) = \infty$ . Note that  $\vec{d}(\vec{G}, u, v)$  and  $\vec{d}(\vec{G}, v, u)$  may be distinct. Let  $d_o(\vec{G}, u, v) = \min \{ \vec{d}(\vec{G}, u, v), \vec{d}(\vec{G}, v, u) \}$ ; we thus have  $d_o(\vec{G}, u, v) = d_o(\vec{G}, v, u)$  for

every two vertices  $u$  and  $v$  in  $\vec{G}$ . The maximum value of  $d_o(\vec{G}, u, v)$ , taken over all pairs of vertices  $\{u, v\}$ , will be called the *weak directed diameter* of  $\vec{G}$ . Note that, in contrast to the ordinary directed diameter, the weak directed diameter of an oriented graph  $\vec{G}$  may be finite even when  $\vec{G}$  is not strongly connected. In the following, we shall simply write  $\vec{d}(u, v)$  or  $d_o(u, v)$  whenever the oriented graph  $\vec{G}$  is clear from the context. For an undirected graph  $G$ , we denote by  $d(G, u, v)$  the distance from  $u$  to  $v$  in  $G$ .

Let  $\vec{G}$  and  $\vec{H}$  be two oriented graphs. A *homomorphism* from  $\vec{G}$  to  $\vec{H}$  is a mapping  $h : V(\vec{G}) \rightarrow V(\vec{H})$  such that for every arc  $\vec{uv}$  in  $E(\vec{G})$ ,  $\vec{h(u)h(v)}$  is an arc in  $\vec{H}$ . We write  $\vec{G} \rightarrow \vec{H}$  whenever such a homomorphism exists. Let  $h$  be a homomorphism of  $\vec{G}$  to  $\vec{H}$ . The *homomorphic image*  $h(\vec{G})$  of  $\vec{G}$  under  $h$  is the oriented subgraph of  $\vec{H}$  given by

$$V(h(\vec{G})) = h(V(\vec{G})),$$

and

$$E(h(\vec{G})) = \{ \vec{ab} \mid \exists \vec{uv} \in E(\vec{G}) \text{ such that } h(u) = a \text{ and } h(v) = b \}.$$

An *oriented  $k$ -colouring* of an oriented graph  $\vec{G}$  is a partition of  $V(\vec{G})$  into  $k$  disjoint independent sets, such that all arcs linking any two of these sets have the same direction [23]. Such an oriented colouring is thus a mapping  $\gamma : V(\vec{G}) \rightarrow \{1, 2, \dots, k\}$  such that  $\gamma(u) \neq \gamma(v)$  for every arc  $\vec{uv}$  in  $E(\vec{G})$  and  $\gamma(u) \neq \gamma(x)$  whenever there exist two arcs  $\vec{uv}$  and  $\vec{wx}$  in  $E(\vec{G})$  with  $\gamma(v) = \gamma(w)$  ( $v$  and  $w$  are not necessarily distinct). Observe that in any oriented colouring of  $\vec{G}$ , every two vertices  $u$  and  $v$  with  $d_o(\vec{G}, u, v) \leq 2$  must be assigned distinct colours.

The *oriented chromatic number*  $\chi_o(\vec{G})$  of an oriented graph  $\vec{G}$  is then defined as the smallest  $k$  for which  $\vec{G}$  admits an oriented  $k$ -colouring. Observe that  $\chi_o(\vec{G})$  can equivalently be defined as the smallest order of an oriented graph  $\vec{T}$  for which  $\vec{G} \rightarrow \vec{T}$ . If  $G$  is an *undirected* graph, the oriented chromatic number  $\chi_o(G)$  of  $G$  is defined as the largest oriented chromatic number of its orientations:

$$\chi_o(G) = \max \{ \chi_o(\vec{G}) \mid \vec{G} \text{ is an orientation of } G \}.$$

An *oriented clique*, or simply an *o-clique*, is an oriented graph  $\vec{\mathcal{O}}$  such that  $\chi_o(\vec{\mathcal{O}}) = |V(\vec{\mathcal{O}})|$ . Hence, an oriented graph is an o-clique if and only if every two of its vertices are linked by a directed path of length 1 or 2 [19], that is, its weak directed diameter is at most 2. Figure 1 shows examples of o-cliques of order up to 7, each of them being minimal with respect to the number of arcs.

Let  $\vec{\mathcal{O}}_n$  be an o-clique of order  $n$ . The following construction [23] allows us to obtain from  $\vec{\mathcal{O}}_n$  a new o-clique  $\vec{\mathcal{O}}_{2n+1}$  of order  $2n+1$ : take two disjoint copies  $\vec{\mathcal{O}}'_n$  and  $\vec{\mathcal{O}}''_n$  of  $\vec{\mathcal{O}}_n$ , a new vertex  $u$  and add all arcs from  $u$  to every vertex of  $\vec{\mathcal{O}}'_n$  and from every vertex of  $\vec{\mathcal{O}}''_n$  to  $u$ . In Figure 1, the o-clique  $\vec{\mathcal{O}}_7$  has been obtained by applying this construction to the o-clique  $\vec{\mathcal{O}}_3$ .

Recall that the ordinary chromatic number of an undirected graph  $G$  may be defined as the smallest  $k$  such that  $G$  admits a homomorphism to the complete graph  $K_k$ . Similarly, the oriented chromatic number of an oriented graph  $\vec{G}$  corresponds to the smallest  $k$  for which there exists an oriented o-clique  $\vec{\mathcal{O}}_k$  of order  $k$  such that  $\vec{G}$  admits a homomorphism to  $\vec{\mathcal{O}}_k$ .

Let  $f(k)$  denote the minimum number of arcs in an o-clique of order  $k$ . The asymptotic behaviour of the function  $f$  has been independently studied in [13] and [20], where it was proved that  $\lim f(k)/(k \log k) = 1$  for  $k \rightarrow \infty$ . More precisely, Füredi *et al.* proved the following:

**Theorem 1 (Füredi, Horak, Pareek and Zhu [13])** *For every  $k \geq 9$ ,*

$$(1 - o(1))k \log k \leq f(k) \leq k \log k - \frac{3}{2}k.$$

In the following, we will say that an o-clique of order  $k$  is *optimal* whenever it has exactly  $f(k)$  arcs. Note that all the o-cliques depicted in Figure 1 are optimal.

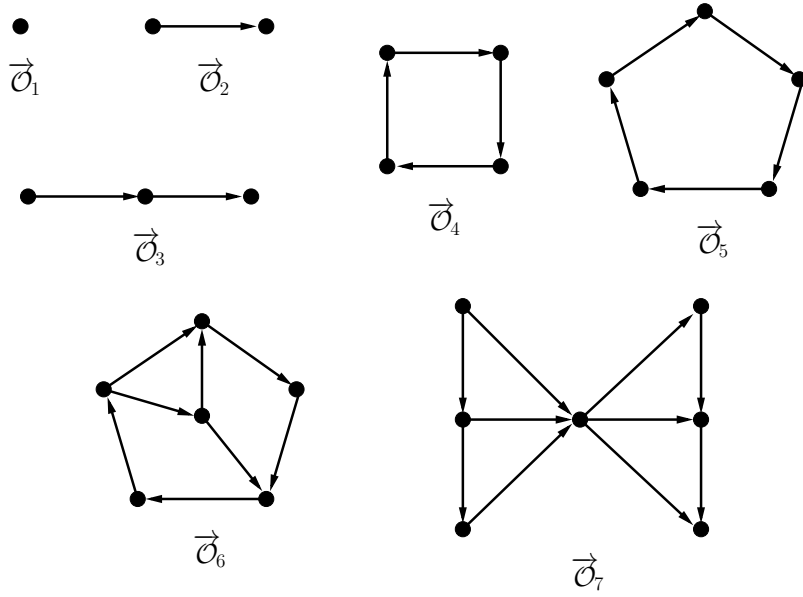


Figure 1: Sample o-cliques

### 3 Complete colourings, complete homomorphisms, and the oriented achromatic number

An oriented  $k$ -colouring  $\gamma : V(\vec{G}) \rightarrow \{1, 2, \dots, k\}$  of an oriented graph  $\vec{G}$  is *complete* if and only if for every two colours  $a, b \in \{1, 2, \dots, k\}$ ,  $a \neq b$ , at least one of the following two conditions hold:

1. there exists an arc  $\vec{uv}$  in  $E(\vec{G})$  such that either  $\gamma(u) = a$  and  $\gamma(v) = b$  or  $\gamma(u) = b$  and  $\gamma(v) = a$ , or
2. there exist two arcs  $\vec{uv}$  and  $\vec{wx}$  in  $E(\vec{G})$  such that either  $\gamma(u) = a$ ,  $\gamma(x) = b$  and  $\gamma(v) = \gamma(w)$  or  $\gamma(u) = b$ ,  $\gamma(x) = a$  and  $\gamma(v) = \gamma(w)$  (note here that we may have  $v = w$ ).

Let  $\vec{G}$  and  $\vec{H}$  be oriented graphs. A homomorphism  $h : \vec{G} \rightarrow \vec{H}$  is *complete* if and only if for every arc  $\vec{ab} \in E(\vec{H})$ , there exists an arc  $\vec{uv} \in E(\vec{G})$  such that  $h(u) = a$  and  $h(v) = b$ . In other words, the homomorphism  $h$  is complete whenever  $h(\vec{G}) = \vec{H}$ . We shall write  $\vec{G} \circ \rightarrow \vec{H}$  whenever there exists a complete homomorphism of  $\vec{G}$  to  $\vec{H}$ .

The following theorem shows that every complete oriented colouring can be equivalently viewed as a complete homomorphism to some oriented o-clique.

**Theorem 2** *An oriented graph  $\vec{G}$  admits a complete oriented  $k$ -colouring if and only if there exists an o-clique  $\vec{O}_k$  of order  $k$  such that  $\vec{G} \circ \rightarrow \vec{O}_k$ .*

**Proof.** Let  $\gamma$  be a complete oriented  $k$ -colouring of  $\vec{G}$ , and  $\vec{H}$  be the oriented graph defined by  $V(\vec{H}) = \{1, \dots, k\}$  and for every  $i, j$ ,  $1 \leq i, j \leq k$ ,  $\vec{ij} \in E(\vec{H})$  if and only if there exists an arc  $\vec{uv} \in E(\vec{G})$  such that  $\gamma(u) = i$  and  $\gamma(v) = j$ . We clearly have  $\vec{G} \circ \rightarrow \vec{H}$ . We claim that  $\vec{H}$  is an o-clique. To see that, let  $a$  and  $b$  be any two vertices in  $V(\vec{H})$ . Since  $\gamma$  is a complete  $k$ -colouring of  $\vec{G}$ , we get that (i) there exists an arc  $\vec{uv}$  in  $E(\vec{G})$  such that either  $\gamma(u) = a$  and  $\gamma(v) = b$ , and thus  $\vec{ab}$  is an arc in  $\vec{H}$ , or  $\gamma(u) = b$  and  $\gamma(v) = a$ , and thus  $\vec{ba}$  is an arc in  $\vec{H}$ , or (ii) there exist four vertices  $u, v, w$  and  $x$  in  $V(\vec{G})$  with  $\vec{uv}, \vec{wx} \in E(\vec{G})$  such that either  $\gamma(u) = a$ ,  $\gamma(x) = b$  and  $\gamma(v) = \gamma(w) = c$ , and thus  $\vec{acb}$  is a directed 2-path in  $\vec{H}$ , or  $\gamma(u) = b$ ,  $\gamma(x) = a$  and  $\gamma(v) = \gamma(w) = c$ , and thus  $\vec{bca}$  is a directed 2-path in  $\vec{H}$ .

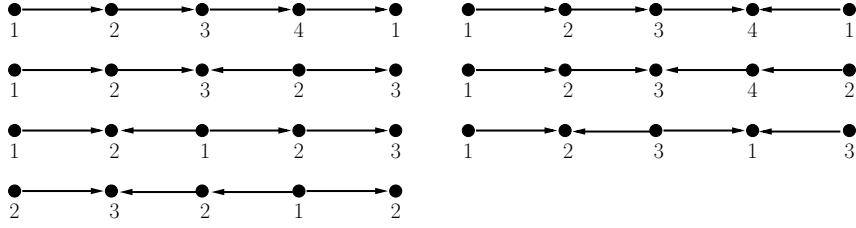


Figure 2: Complete oriented colouring of orientations of  $P_4$

Conversely, let  $\vec{\mathcal{O}}_k$  be an o-clique of order  $k$  with  $V(\vec{\mathcal{O}}_k) = \{1, \dots, k\}$ ,  $h$  a complete homomorphism of  $\vec{G}$  to  $\vec{\mathcal{O}}_k$  and  $a, b$  any two colours from  $\{1, \dots, k\}$ . If  $\vec{ab} \in E(\vec{\mathcal{O}}_k)$  then there exists an arc  $\vec{uv}$  in  $E(\vec{G})$  such that  $h(u) = a$  and  $h(v) = b$ . Otherwise, there exists a directed 2-path  $acb$  in  $\vec{\mathcal{O}}_k$  and, therefore, there exist four vertices  $u, v, w$  and  $x$  in  $V(\vec{G})$  with  $\vec{uv}, \vec{wx} \in E(\vec{G})$  such that  $h(u) = a, h(x) = b$  and  $h(v) = h(w)$ . The mapping  $h$  is thus a complete oriented  $k$ -colouring of  $\vec{G}$ .  $\square$

The *oriented achromatic number* of an oriented graph  $\vec{G}$ , denoted  $\psi_o(\vec{G})$ , is the largest  $k$  such that  $\vec{G}$  admits a complete oriented  $k$ -colouring or, equivalently, the largest order of an o-clique  $\vec{\mathcal{O}}$  such that  $\vec{G} \circ \rightarrow \vec{\mathcal{O}}$ . We will say that a complete oriented colouring of  $\vec{G}$  is *optimal* whenever it uses  $\psi_o(\vec{G})$  colours.

Since the composition of two (complete) homomorphisms is clearly a (complete) homomorphism, we have:

**Proposition 3** *For every oriented graph  $\vec{G}$ , if  $\vec{H}$  is a homomorphic image of  $\vec{G}$ , then  $\chi_o(\vec{G}) \leq \chi_o(\vec{H})$  and  $\psi_o(\vec{G}) \geq \psi_o(\vec{H})$ .*

If  $G$  is an undirected graph, we define the *oriented achromatic number*  $\psi_o(G)$  of  $G$  as the largest oriented achromatic number of its orientations:

$$\psi_o(G) = \max \{ \psi_o(\vec{G}) \mid \vec{G} \text{ is an orientation of } G \}.$$

Consider the undirected path  $P_4$  of order 5; up to symmetries, this path admits seven distinct orientations. The corresponding oriented paths, together with an optimal complete oriented colouring, are depicted in Figure 2. We thus get that  $\psi_o(P_4) = 4$  (recall that the smallest o-clique of order 5 has five arcs).

It is not difficult to observe that if  $G$  is an undirected complete bipartite graph then  $\psi(G) = 2$ . Our next theorem shows that the oriented achromatic number of the family of undirected complete bipartite graphs is not bounded. We first need the following easy result:

**Proposition 4** *Let  $\vec{G}$  be an oriented graph and  $u, v$  two vertices of  $\vec{G}$ . If  $N^+(u) \subseteq N^+(v)$  and  $N^-(u) \subseteq N^-(v)$  then  $\chi_o(\vec{G}) = \chi_o(\vec{G} - u)$ .*

**Proof.** Since  $\vec{G} - u$  is a subgraph of  $\vec{G}$ , we clearly have  $\chi_o(\vec{G} - u) \leq \chi_o(\vec{G})$ . On the other hand, every oriented coloring  $\gamma$  of  $\vec{G} - u$  can be extended to an oriented coloring  $\gamma_u$  of  $\vec{G}$  by setting  $\gamma_u(u) = \gamma(v)$  and  $\gamma_u(w) = \gamma(w)$  for every vertex  $w \neq u$ . Therefore  $\chi_o(\vec{G}) \leq \chi_o(\vec{G} - u)$  and the result follows.  $\square$

**Theorem 5** *For every two integers  $m$  and  $n$  with  $1 \leq m \leq n$ ,  $\psi_o(K_{m,n}) = \min\{m+n, m+2^m\}$ .*

**Proof.** Let  $(X, Y)$  denote the bipartition of  $V(K_{m,n})$  with  $X = \{x_1, \dots, x_m\}$  and  $Y = \{y_1, \dots, y_n\}$ , and let  $\nu$  be any bijection from the set  $\{1, \dots, 2^m\}$  to the set of subsets of  $\{1, \dots, m\}$  such that  $\nu(1) = \emptyset$  and  $\nu(i) = \{i-1\}$  for every  $i$ ,  $2 \leq i \leq m+1$ .

Suppose first that  $n \leq 2^m$  and let  $\vec{K}_{m,n}$  be the orientation of  $K_{m,n}$  defined by  $\overrightarrow{x_i y_j} \in E(\vec{K}_{m,n})$  if  $i \in \nu(j)$  and  $\overrightarrow{y_j x_i} \in E(\vec{K}_{m,n})$  otherwise. Since  $m \leq n$ , for every  $i$  and  $j$  with  $1 \leq i < j \leq m$ , we have  $i \in \{i\} = \nu(i+1)$  and  $j \notin \nu(i+1)$ , so that  $x_i y_{i+1} x_j$  is a directed path. Moreover, for every  $k$  and  $\ell$  with  $1 \leq k < \ell \leq n$ , there exists some  $\alpha \in \{1, \dots, m\}$  such that  $\alpha$  belongs to exactly one of the subsets  $\nu(k)$  or  $\nu(\ell)$ , so that either  $y_k x_\alpha y_\ell$  or  $y_\ell x_\alpha y_k$  is a directed path. Hence, every two vertices of  $\vec{K}_{m,n}$  are linked by a directed path of length at most 2. The oriented graph  $\vec{K}_{m,n}$  is thus an o-clique and, therefore,  $\psi_o(K_{m,n}) = m + n$ .

Suppose now that  $n > 2^m$ . Assume  $\psi_o(K_{m,n}) = k$  and let  $\vec{K}_{m,n}$  be an orientation of  $K_{m,n}$  that admits a complete homomorphism to an o-clique  $\mathcal{O}_k$  of order  $k$ . Since  $K_{m,n}$  is a complete bipartite graph,  $\mathcal{O}_k$  must also be bipartite, with two parts of order  $m' \leq m$  and  $k - m'$ , respectively. By Proposition 4 we thus have  $k - m' \leq 2^{m'}$ , since otherwise two vertices in the second part would have the same sets of in-neighbours and out-neighbours. It follows that  $\psi_o(K_{m,n}) \leq m' + 2^{m'} \leq m + 2^m$ . The orientation of  $K_{m,n}$  introduced above clearly admits a complete homomorphism to the bipartite o-clique with parts of size  $m$  and  $2^m$ , so that  $\psi_o(K_{m,n}) = m + 2^m$  and the result follows.  $\square$

In particular, the oriented achromatic number of every star  $K_{1,n}$ ,  $n \geq 2$ , equals 3 (every such star admits an orientation containing a directed 2-path).

If  $G$  is an undirected bipartite graph, then it admits  $K_2$  as a homomorphic image. If  $G$  contains a 2-path, then  $\chi_o(G) \geq 3$ . Therefore, since  $\chi_o(K_2) = 2$ , the first inequality of Proposition 3 does not hold for undirected graphs with respect to the oriented chromatic number.

On the other hand, every orientation of a homomorphic image  $H$  of an undirected graph  $G$  is a homomorphic image of some orientation of  $G$  so that every complete oriented colouring of any orientation of  $H$  can be extended to a complete oriented colouring of some orientation of  $G$  using the same number of colours. Hence, the second inequality of Proposition 3 still holds for undirected graphs with respect to the oriented achromatic number:

**Proposition 6** *For every undirected graph  $G$ , if  $H$  is a homomorphic image of  $G$ , then  $\psi_o(G) \geq \psi_o(H)$ .*

Let  $G$  be an undirected graph and  $\gamma$  be a complete (ordinary)  $k$ -colouring of  $G$ . Consider the orientation  $\vec{G}$  of  $G$  given by  $\overrightarrow{uv} \in E(\vec{G})$  if and only if  $uv \in E(G)$  and  $\gamma(u) < \gamma(v)$ . It is not difficult to check that  $\gamma$  is indeed a complete oriented colouring of  $\vec{G}$ , corresponding to a complete homomorphism of  $\vec{G}$  to the transitive tournament of order  $k$ . Hence we have:

**Proposition 7** *For every undirected graph  $G$ ,  $\psi_o(G) \geq \psi(G)$ .*

Since every complete oriented colouring is an oriented colouring, we get  $\chi_o(\vec{G}) \leq \psi_o(\vec{G})$  for every oriented graph  $\vec{G}$  and, therefore,  $\chi_o(G) \leq \psi_o(G)$  for every undirected graph  $G$ . On the other hand, we clearly have  $\psi_o(G) \leq n$  for every undirected graph  $G$  of order  $n$ . Therefore:

**Proposition 8** *For every undirected graph  $G$  of order  $n$ ,  $\chi_o(G) \leq \psi_o(G) \leq n$ .*

If  $G$  is the underlying graph of an o-clique of order  $n$ , we clearly have  $\chi_o(G) = \psi_o(G) = n$ . A graph  $G$  may satisfy the equality  $\chi_o(G) = \psi_o(G)$  without being the underlying graph of an o-clique. Consider for instance the path  $P_3$  on four vertices. Since the minimum number of edges of an o-clique of order 4 is four,  $P_3$  is not the underlying graph of an o-clique and  $\psi_o(P_3) < 4$ . Now, considering the directed path  $\vec{P}_3$ , we get  $\chi_o(P_3) \geq \chi_o(\vec{P}_3) = 3$  and, therefore,  $\chi_o(P_3) = \psi_o(P_3)$ .

On the other hand, considering the definitions of o-cliques and complete oriented colourings, we easily get that being the underlying graph of an o-clique is a necessary and sufficient condition for a graph  $G$  of order  $n$  to satisfy the equality  $\psi_o(G) = n$ :

**Proposition 9** *If  $\vec{G}$  is an oriented graph of order  $n$ , then  $\psi_o(\vec{G}) = n$  if and only if  $\vec{G}$  is an o-clique.*

Hence, an undirected graph  $G$  of order  $n$  has oriented achromatic number  $n$  if and only if it admits an orientation with weak directed diameter at most 2 (in particular,  $\psi_o(K_n) = n$  for every complete graph  $K_n$ ). Chvátal and Thomassen proved in [6] that the problem of determining whether an undirected graph admits an orientation with *directed diameter* 2 is  $\mathcal{NP}$ -complete. Recently, Bensmail, Duval and Kirgizov proved that the problem of determining whether an undirected graph admits an orientation with *weak diameter*  $k$  is  $\mathcal{NP}$ -complete for every  $k \geq 2$  [1]. Hence we have:

**Theorem 10** *The problem of determining whether  $\psi_o(G) = n$  for an undirected graph  $G$  of order  $n$  is  $\mathcal{NP}$ -complete.*

Note that this is in contrast to the undirected case, since an undirected graph of order  $n$  has achromatic number  $n$  if and only if it is a complete graph.

## 4 General results

In case of undirected graphs, Bhave proved in [2] that for every two integers  $a$  and  $b$ , with  $2 \leq a \leq b$ , there exists a graph  $G_{a,b}$  with  $\chi(G_{a,b}) = a$  and  $\psi(G_{a,b}) = b$ . The same result holds for oriented chromatic and achromatic numbers of oriented graphs.

We first need two easy lemmas. For every oriented graph  $\vec{G}$ , we denote by  $\vec{G}^{(1)}$  the oriented graph obtained from  $\vec{G}$  by adding a universal source vertex, that is  $V(\vec{G}^{(1)}) = V(\vec{G}) \cup \{x\}$  and  $E(\vec{G}^{(1)}) = E(\vec{G}) \cup \{xu \mid u \in V(\vec{G})\}$ . Now, for every integer  $k \geq 2$ , we inductively define  $\vec{G}^{(k)} = (\vec{G}^{(k-1)})^{(1)}$ .

Observe that in every oriented colouring of  $\vec{G}^{(1)}$ , complete or not complete, the new universal source vertex  $x$  must be assigned a colour distinct from the colours of the other vertices.

Therefore we get:

**Lemma 11** *For every oriented graph  $\vec{G}$  and every integer  $k \geq 1$ ,  $\chi_o(\vec{G}^{(k)}) = \chi_o(\vec{G}) + k$  and  $\psi_o(\vec{G}^{(k)}) = \psi_o(\vec{G}) + k$ .*

Since no directed path can have a source or a sink vertex as an internal vertex, we also have:

**Lemma 12** *If  $\vec{O}_k$  is an o-clique of order  $k$  and  $u$  is a source or a sink vertex of  $\vec{O}_k$ , then  $\vec{O}_k - u$  is an o-clique of order  $k - 1$ .*

We can now prove the following:

**Theorem 13** *For every two integers  $a$  and  $b$ , with  $2 \leq a \leq b$ , there exists an oriented graph  $\vec{G}_{a,b}$  with  $\chi_o(\vec{G}_{a,b}) = a$  and  $\psi_o(\vec{G}_{a,b}) = b$ .*

**Proof.** We first consider the case when  $a = 2$ . For  $b = 2$  we let  $\vec{G}_{2,2} = \vec{P}_1$  and, for  $b = 3$ ,  $\vec{G}_{2,3} = 2\vec{P}_1$  (two disjoint arcs). Clearly,  $\chi_o(\vec{G}_{2,2}) = \chi_o(\vec{G}_{2,3}) = 2$ ,  $\psi_o(\vec{G}_{2,2}) = 2$  and  $\psi_o(\vec{G}_{2,3}) = 3$ . Suppose now that  $b \geq 4$  and let  $\vec{O}_b$  be an optimal o-clique of order  $b$  with  $V(\vec{O}_b) = \{v_1, v_2, \dots, v_b\}$ .

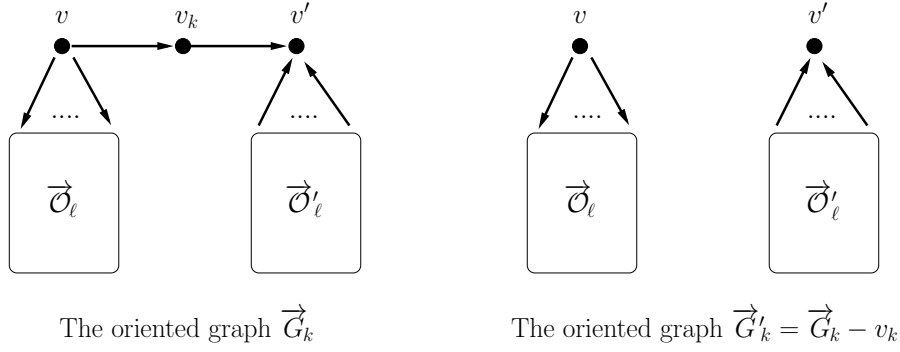


Figure 3: Construction for Theorem 14(1)

For every  $p \geq 1$ , let  $\vec{S}_p$  denote the orientation of  $K_{1,p}$  obtained by orienting all the edges from the central vertex towards the leaves ( $\vec{S}_p$  is thus an *outgoing star*). Let now  $\vec{G}_{2,b} = \bigsqcup_{i=1}^b \vec{S}_{d^+(v_i)}$  (disjoint union of outgoing stars). Since  $\chi_o(\vec{S}_p) = 2$  for every  $p \geq 1$ , we clearly have  $\chi_o(\vec{G}_{2,b}) = 2$ . Let  $h$  be the mapping from  $V(\vec{G}_{2,b})$  to  $V(\vec{\mathcal{O}}_b)$  that assigns the central vertex of every outgoing star  $\vec{S}_{d^+(v_i)}$  to the corresponding vertex  $v_i$  and the leaves of  $\vec{S}_{d^+(v_i)}$  to the out-neighbours of  $v_i$ . Clearly,  $h$  is a complete homomorphism of  $\vec{G}_{2,b}$  to  $\vec{\mathcal{O}}_b$  and, therefore,  $\psi_o(\vec{G}_{2,b}) \geq b$ .

Suppose now that there exists a complete homomorphism  $h'$  of  $\vec{G}_{2,b}$  to  $\vec{\mathcal{O}}_{b'}$ , where  $\vec{\mathcal{O}}_{b'}$  is an o-clique of order  $b' > b$ . Note that  $|E(\vec{\mathcal{O}}_{b'})| \leq |E(\vec{G}_{2,b})| = |E(\vec{\mathcal{O}}_b)|$ . Due to the special structure of  $\vec{G}_{2,b}$ , the o-clique  $\vec{\mathcal{O}}_{b'}$  has at least  $b' - b$  sink vertices (namely, the vertices that are not the image of a central vertex of some outgoing star) and, according to Lemma 12, by deleting these vertices we get an o-clique of order  $b$  having strictly less arcs than  $\vec{\mathcal{O}}_b$ , which contradicts the optimality of  $\vec{\mathcal{O}}_b$ . Hence,  $\psi_o(\vec{G}_{2,b}) = b$  as required.

We finally consider the case when  $a > 2$ . If  $a = b$ , we let  $\vec{G}_{a,b} = \vec{G}_{a,a}$  be any o-clique of order  $a$ . Otherwise, according to Lemma 11, the oriented graph  $\vec{G}_{a,b} = \vec{G}_{2,b-a+2}^{(a-2)}$  has the required property.  $\square$

A natural question is to ask how the (oriented) achromatic number of a graph can be affected by the removal of a single vertex or a single edge. In [14], Geller and Kronk proved that if  $G$  is a nontrivial undirected graph then, for every vertex  $v$  of  $V(G)$ ,  $\psi(G) - 1 \leq \psi(G - v) \leq \psi(G)$ . The situation is rather different in the oriented case, since deleting a vertex in an oriented graph may increase or decrease its oriented achromatic number by an arbitrarily large value. In particular, this proves that, unlike the undirected case, having oriented achromatic number at most  $k$  is not a hereditary property with respect to induced subgraphs. More precisely, we have:

**Theorem 14** For every  $k \geq 1$ ,

- (1) there exists an oriented graph  $\vec{G}_k$  with  $\psi_o(\vec{G}_k - v_k) \geq \psi_o(\vec{G}_k) + k$  for some vertex  $v_k \in V(\vec{G}_k)$ ,
- (2) there exists an oriented graph  $\vec{H}_k$  with  $\psi_o(\vec{H}_k - w_k) + k \leq \psi_o(\vec{H}_k)$  for some vertex  $w_k \in V(\vec{H}_k)$ .

**Proof.** (1) Let  $\vec{\mathcal{O}}_\ell$  be an o-clique such that  $2\psi_o(\vec{\mathcal{O}}_\ell) \geq \psi_o(2\vec{\mathcal{O}}_\ell) + k + 2$ , where  $2\vec{\mathcal{O}}_\ell$  denotes the disjoint union of two copies of  $\vec{\mathcal{O}}_\ell$ . Note that such an o-clique exists thanks to Theorem 1 (since the number of arcs in an optimal o-clique of order  $\ell$  tends to  $\ell \log \ell$  when  $\ell$  tends to infinity, by taking a large enough  $\ell$  and an optimal o-clique  $\vec{\mathcal{O}}_\ell$ , the difference  $2\psi_o(\vec{\mathcal{O}}_\ell) - \psi_o(2\vec{\mathcal{O}}_\ell)$  can be



made arbitrarily large). Let  $p = \psi_o(\vec{\mathcal{O}}_\ell) = |V(\vec{\mathcal{O}}_\ell)|$  and  $q = \psi_o(2\vec{\mathcal{O}}_\ell)$ . Hence,  $2p - q \geq k + 2$ . Let now  $\vec{G}_k$  be the oriented graph obtained from two disjoint copies of  $\vec{\mathcal{O}}_\ell$ , denoted  $\vec{\mathcal{O}}_\ell$  and  $\vec{\mathcal{O}}'_\ell$ , and a directed 2-path  $vv_kv'$ , by adding all arcs from  $v$  to every vertex of  $\vec{\mathcal{O}}_\ell$ , and from every vertex of  $\vec{\mathcal{O}}'_\ell$  to  $v'$  (see Figure 3).

We first claim that  $\psi_o(\vec{G}_k) \leq q + 3$ . Note that since  $v$  and  $v'$  are forced to get distinct colours in every complete oriented colouring of  $\vec{G}_k$ , no additional constraint can arise between vertices of  $\vec{\mathcal{O}}_\ell$  and vertices of  $\vec{\mathcal{O}}'_\ell$ . Therefore, since  $\vec{G}_k$  contains in addition the three vertices  $v$ ,  $v_k$  and  $v'$ ,  $\psi_o(\vec{G}_k) \leq \psi_o(2\vec{\mathcal{O}}_\ell) + 3 = q + 3$ .

Let now  $\vec{G}'_k = \vec{G}_k - v_k$ . We claim that  $\psi_o(\vec{G}'_k) \geq 2p + 1$ . Let  $\gamma : V(\vec{G}'_k) \rightarrow \{1, \dots, 2p + 1\}$  be any mapping such that  $\gamma(v) = \gamma(v') = 2p + 1$ ,  $\gamma(\vec{\mathcal{O}}_\ell) = \{1, \dots, p\}$  and  $\gamma(\vec{\mathcal{O}}'_\ell) = \{p + 1, \dots, 2p\}$ . Since for every vertex  $x$  in  $\vec{\mathcal{O}}_\ell$  and every vertex  $x'$  in  $\vec{\mathcal{O}}'_\ell$  the two arcs  $\vec{vx}$  and  $\vec{x'v'}$  force  $x$  and  $x'$  to be assigned distinct colours, every such mapping  $\gamma$  is a complete oriented  $(2p + 1)$ -colouring of  $\vec{G}'_k$ .

Finally,  $\psi_o(\vec{G}'_k) - \psi_o(\vec{G}_k) \geq 2p + 1 - (q + 3) = 2p - q - 2 \geq k$  and the result follows.

(2) Similarly as above, let now  $\vec{\mathcal{O}}_\ell$  be an o-clique such that  $2\psi_o(\vec{\mathcal{O}}_\ell) \geq \psi_o(2\vec{\mathcal{O}}_\ell) + k - 1$ ,  $p = \psi_o(\vec{\mathcal{O}}_\ell)$  and  $q = \psi_o(2\vec{\mathcal{O}}_\ell)$ , so that  $2p - q \geq k - 1$ . Let then  $\vec{H}_k$  be the oriented graph obtained from two disjoint copies of  $\vec{\mathcal{O}}_\ell$ , denoted  $\vec{\mathcal{O}}_\ell$  and  $\vec{\mathcal{O}}'_\ell$ , and an isolated vertex  $w_k$ , by adding all arcs from  $w_k$  to every vertex of  $\vec{\mathcal{O}}_\ell$  and from every vertex of  $\vec{\mathcal{O}}'_\ell$  to  $w_k$ .

The oriented graph  $\vec{H}_k$  is clearly an o-clique and, therefore,  $\psi_o(\vec{H}_k) = 2p + 1$ . On the other hand,  $\psi_o(\vec{H}_k - w_k) = q$  and the result follows.  $\square$

Geller and Kronk also proved that if  $G$  is a nontrivial undirected graph then, for every edge  $uv$  of  $E(G)$ ,  $\psi(G) - 1 \leq \psi(G - uv) \leq \psi(G)$  [14]. Using a proof similar to the proof of Theorem 14(1) and taking  $\vec{G}'_k = \vec{G}_k - \vec{vv}_k$ , we also get:

**Theorem 15** *For every  $k \geq 1$ , there exists an oriented graph  $\vec{G}_k$  with  $\psi_o(\vec{G}_k - \vec{uv}) \geq \psi_o(\vec{G}_k) + k$  for some arc  $\vec{uv} \in E(\vec{G}_k)$ .*

On the other hand, deleting an arc in an oriented graph can decrease its oriented achromatic number by at most 2:

**Theorem 16** *For every non-empty oriented graph  $\vec{G}$  and every arc  $\vec{uv} \in E(\vec{G})$ ,  $\psi_o(\vec{G}) - 2 \leq \psi_o(\vec{G} - \vec{uv})$ .*

**Proof.** Let  $\gamma$  be any optimal complete colouring of  $\vec{G}$  using  $\psi_o(\vec{G})$  colours and let  $\gamma(u) = a$  and  $\gamma(v) = b$ . If  $\gamma$  is not a complete colouring of  $\vec{G} - \vec{uv}$  then this is necessary due to the colours  $a$  and  $b$ , since the arc  $\vec{uv}$  is not involved in the constraints between colours other than  $a$  and  $b$ . We may then recolour vertices with colour  $a$  or  $b$  (with colours used on  $V(\vec{G}) \setminus \{u, v\}$ ) in order to get a complete colouring of  $\vec{G} - \vec{uv}$ , so that  $\psi_o(\vec{G} - \vec{uv}) \geq \psi_o(\vec{G}) - 2$  as required.  $\square$

**Corollary 17** *For every non-empty undirected graph  $G$  and every edge  $uv \in E(G)$ ,  $\psi_o(G) - 2 \leq \psi_o(G - uv)$ .*

**Proof.** Let  $\vec{G}$  be any orientation of  $G$  with  $\psi_o(\vec{G}) = \psi_o(G)$  and, without loss of generality, suppose that  $\vec{uv} \in E(\vec{G})$ . By Theorem 16, we then have  $\psi_o(G - uv) \geq \psi_o(\vec{G} - \vec{uv}) \geq \psi_o(\vec{G}) - 2 = \psi_o(G) - 2$  as required.  $\square$

Note that the results given by Theorems 14(2) and 16 are similar to the results on the oriented chromatic number of Borodin *et al.* [3] who proved that deleting a vertex in an oriented graph may decrease its oriented chromatic number by an unbounded value, while deleting an arc may decrease its oriented chromatic number by at most 2.

However, it is not difficult to check that the relation  $\psi_o(\vec{G} - v) \leq \psi_o(\vec{G})$  still holds whenever  $v$  is a source or a sink vertex in  $\vec{G}$ . More generally, such a relation holds for every subgraph  $\vec{H}$  of  $\vec{G}$  such that for every two vertices  $u$  and  $v$  with  $d_o(\vec{H}, u, v) > 2$ , we have  $d_o(\vec{G}, u, v) > 2$ :

**Theorem 18** *If  $\vec{H}$  is a subgraph of an oriented graph  $\vec{G}$  such that for every two vertices  $u$  and  $v$  in  $\vec{H}$ ,  $d_o(\vec{G}, u, v) > 2$  whenever  $d_o(\vec{H}, u, v) > 2$ , then  $\psi_o(\vec{H}) \leq \psi_o(\vec{G})$ .*

**Proof.** It suffices to observe that, under the assumption of the theorem, every complete oriented colouring of  $\vec{H}$  can be extended (for instance in a greedy way) to a complete oriented colouring of  $\vec{G}$ , since the vertices of  $\vec{G} \setminus \vec{H}$  cannot contradict the precolouring of the vertices of  $\vec{H}$ .  $\square$

**Corollary 19** *If  $H$  is a subgraph of an undirected graph  $G$  such that for every two vertices  $u$  and  $v$  in  $H$ ,  $d(H, u, v) \leq 2$  whenever  $d(G, u, v) \leq 2$ , then  $\psi_o(H) \leq \psi_o(G)$ .*

## 5 Elementary homomorphisms

Let  $\vec{G}$  be an oriented graph and  $u$  and  $v$  be two vertices in  $V(\vec{G})$  such that  $d_o(\vec{G}, u, v) > 2$ . The oriented graph  $\vec{G}'$  obtained by *identifying* the vertices  $u$  and  $v$  into  $u$  is defined by

$$V(\vec{G}') = V(\vec{G}) \setminus \{v\},$$

and

$$E(\vec{G}') = (E(\vec{G}) \cap V(\vec{G}') \times V(\vec{G}')) \cup \{\vec{uv} \mid \vec{vb} \in E(\vec{G})\} \cup \{\vec{wu} \mid \vec{wb} \in E(\vec{G})\}.$$

The mapping  $\epsilon : V(\vec{G}) \rightarrow V(\vec{G}')$  defined by  $\epsilon(u) = \epsilon(v) = x$  and  $\epsilon(w) = w$  for every  $w \in V(\vec{G}) \setminus \{u, v\}$  is an *elementary homomorphism* of  $\vec{G}$  to  $\vec{G}'$ . The only effect of such an elementary homomorphism is thus to identify two vertices whose directed distance is at least 3 (which ensures that the so-obtained homomorphic image is an oriented graph). Such an elementary homomorphism will be denoted by  $\epsilon_{u=v}$  whenever we need to specify which vertices are identified and the corresponding obtained oriented graph will be denoted by  $\vec{G}_{u=v}$ .

Let  $h$  be a homomorphism of  $\vec{G}$  to  $\vec{H}$ . For every vertex  $x$  in  $V(h(\vec{G}))$  let  $h^{-1}(x)$  denote the set of vertices of  $V(\vec{G})$  which are mapped to  $x$  by  $h$  and let  $u_x$  be any fixed vertex in  $h^{-1}(x)$ . Let  $\Upsilon(\vec{G}, h)$  be the set of elementary homomorphisms defined by

$$\Upsilon(\vec{G}, h) = \{ \epsilon_{u_x=v_x} \mid x \in V(h(\vec{G})), v_x \in h^{-1}(x), v_x \neq u_x \}.$$

It is easy to check that successively applying all the elementary homomorphisms of  $\Upsilon(\vec{G}, h)$  to  $\vec{G}$  gives the homomorphic image  $h(\vec{G})$  of  $\vec{G}$ .

Hence, an oriented graph  $\vec{G}$  has oriented achromatic number at least  $k$  if and only if there exists a set of elementary homomorphisms such that applying successively all the homomorphisms of this set gives an o-clique of order  $k$ . The oriented achromatic number of  $\vec{G}$  is then the maximum order of such an o-clique.

We now give some basic results on elementary homomorphisms and oriented chromatic numbers of oriented graphs. If  $G$  is an undirected graph, it is not difficult to check that  $\chi(G) \leq \chi(\epsilon(G)) \leq \chi(G) + 1$  for every elementary homomorphism  $\epsilon$  of  $G$  [7, Chapter 12]. Here again, the situation is quite different for the oriented chromatic number of oriented graphs, since the difference between  $\chi_o(\vec{G})$  and  $\chi_o(\epsilon(\vec{G}))$  can be made arbitrarily large:

**Theorem 20** *If  $\epsilon$  is an elementary homomorphism of an oriented graph  $\vec{G}$ , then  $\chi_o(\vec{G}) \leq \chi_o(\epsilon(\vec{G}))$ . On the other hand, for every  $k \geq 0$ , there exists an oriented graph  $\vec{G}_k$  and an elementary homomorphism  $\epsilon_k$  of  $\vec{G}_k$  such that  $\chi_o(\epsilon_k(\vec{G}_k)) \geq \chi_o(\vec{G}_k) + k$ .*

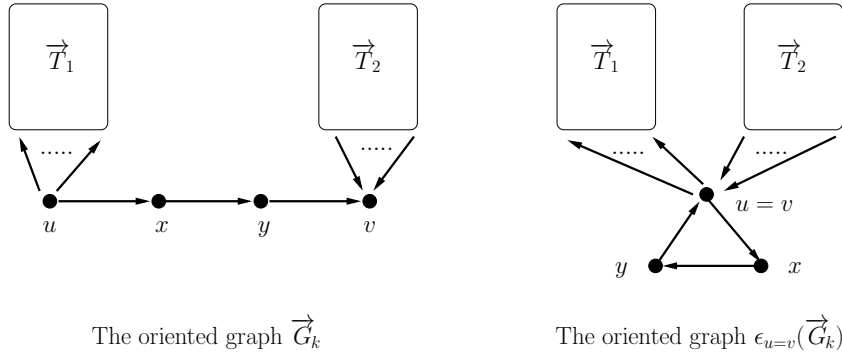


Figure 4: Construction for Theorem 20

**Proof.** The first claim directly follows from Proposition 3.

Let now  $\vec{T}$  be any tournament on  $k+3$  vertices and  $\vec{G}_k$  be the oriented graph obtained from two disjoint copies  $\vec{T}_1$  and  $\vec{T}_2$  of  $\vec{T}$  and a directed path  $uxyv$  of length 3 by adding all arcs from  $u$  to every vertex of  $\vec{T}_1$  and from every vertex of  $\vec{T}_2$  to  $v$  (see Figure 4). Let then  $\gamma$  be any mapping assigning distinct colours from  $\{1, \dots, k+3\}$  to vertices of  $\vec{T}_1$  and  $\vec{T}_2$  — twin vertices in  $\vec{T}_1$  and  $\vec{T}_2$  having the same colour — and colours  $k+4, \dots, k+7$  to vertices  $u, x, y$  and  $v$ , respectively. The mapping  $\gamma$  is clearly an oriented colouring of  $\vec{G}_k$  and, therefore,  $\chi_o(\vec{G}_k) \leq k+7$ . Let now  $\epsilon_k = \epsilon_{u=v}$ . The obtained oriented graph  $\epsilon_k(\vec{G}_k)$  then contains an oriented o-clique subgraph of order  $2k+7$ , made of vertex  $u$  together with vertices of  $\vec{T}_1$  and  $\vec{T}_2$ . Hence,  $\chi_o(\epsilon_k(\vec{G}_k)) \geq 2k+7$  and the result follows.  $\square$

In [7, Chapter 12], it was proved that for every undirected graph  $G$  of order  $n$ , the achromatic number of  $G$  cannot be closer to  $n$  than to its chromatic number. More precisely, the inequality

$$\psi(G) \leq \frac{\chi(G) + n}{2}$$

holds for every undirected graph  $G$ . The oriented graph  $\vec{G}_k$  used in the proof of Theorem 20 proves that such an inequality does not hold in the oriented case. Indeed, we have  $n = |V(\vec{G}_k)| = 2k+10$ ,  $\chi_o(\vec{G}_k) \leq k+7$  and, by Proposition 3,  $\psi_o(\vec{G}_k) \geq \psi_o(\epsilon_k(\vec{G}_k)) \geq \chi_o(\epsilon_k(\vec{G}_k)) \geq 2k+7$  (for  $\epsilon_k$  being the elementary homomorphism used in the proof of Theorem 20). Hence, for  $k > 3$ ,

$$\frac{\chi_o(\vec{G}_k) + n}{2} \leq \frac{k+7+2k+10}{2} = \frac{3}{2}k + \frac{17}{2} < 2k+7 \leq \psi_o(\vec{G}_k).$$

The relation  $\chi(\epsilon(G)) \leq \chi(G) + 1$  was used by Harary, Hedetniemi and Prins [16] to prove the “Interpolation Homomorphism Theorem” in the undirected case, which states that for every undirected graph  $G$  and every integer  $\ell$  with  $\chi(G) \leq \ell \leq \psi(G)$ , there is a homomorphic image  $H$  of  $G$  with  $\chi(H) = \ell$  (and thus a complete  $\ell$ -colouring of  $G$ ). We do not have such an interpolation theorem in the oriented case, as shown by the following result:

**Theorem 21** *For every integer  $k$ ,  $k \geq 3$ , there exists an oriented graph  $\vec{G}_k$  with  $\chi_o(\vec{G}_k) = k$ ,  $\psi_o(\vec{G}_k) \geq k+2$  and, for every homomorphic image  $\vec{H}_k$  of  $\vec{G}_k$ ,  $\chi_o(\vec{H}_k) \neq k+1$ . Therefore,  $\vec{G}_k$  does not admit any complete oriented  $(k+1)$ -colouring.*

**Proof.** Consider the directed cycle  $\vec{C}_9$  of order 9. As shown by the oriented 3-colouring  $\gamma$  and the complete oriented 5-colouring  $\gamma^*$  depicted in Figure 5(a) and Figure 5(b), respectively, we have  $\chi_o(\vec{C}_9) = 3$  and  $\psi_o(\vec{C}_9) \geq 5$ . Obviously, every homomorphic image  $\vec{H}$  of  $\vec{C}_9$  obtained by identifying only vertices that are assigned the same colour by  $\gamma$  is such that  $\chi_o(\vec{H}) = \chi_o(\vec{C}_9)$ .

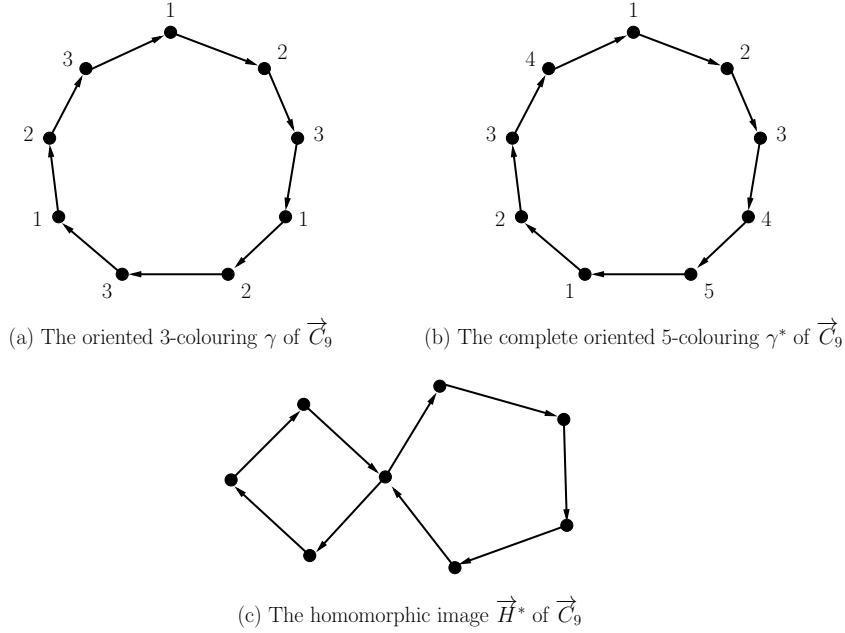


Figure 5: An oriented graph which does not satisfy the “Interpolation Homomorphism Theorem”

Up to symmetry, there is only one homomorphic image  $\vec{H}^*$  of  $\vec{C}_9$  (depicted in Figure 5(c)) that identifies vertices that are assigned distinct colours by  $\gamma$  and, since  $\vec{H}^*$  contains a directed cycle of order 5 (which is an o-clique) as a subgraph, we have  $\chi_o(\vec{H}^*) = 5$ . Hence, no homomorphic image of  $\vec{C}_9$  has oriented chromatic number 4.

We therefore set  $\vec{G}_3 = \vec{C}_9$ . Now, for every  $k \geq 4$ , let  $\vec{G}_k$  be the oriented graph obtained from  $\vec{C}_9$  and any o-clique  $\vec{O}_{k-3}$  of order  $k-3$  by adding an arc from every vertex of  $\vec{C}_9$  to every vertex of  $\vec{O}_{k-3}$ . From the discussion above, we clearly have  $\chi_o(\vec{G}_k) = k-3+3 = k$  and  $\psi_o(\vec{G}_k) \geq k-3+5 = k+2$ . Moreover, since only vertices of  $\vec{C}_9$  may be identified in a homomorphic image of  $\vec{G}_k$ , we get that the only homomorphic image  $\vec{H}_k^*$  of  $\vec{G}_k$  with  $\chi_o(\vec{H}_k^*) \neq k$  is such that  $\chi_o(\vec{H}_k^*) = k+2$ .  $\square$

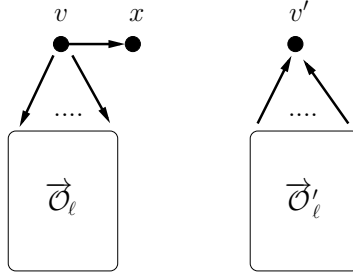
Note that Edwards also proved in [11] that there is no interpolation theorem with respect to complete directed colourings of oriented graphs.

Similarly to the undirected case [7, Chapter 12], the following result indicates under which conditions the equality  $\chi_o(\epsilon(\vec{G})) = \chi_o(\vec{G})$  holds for an elementary homomorphism  $\epsilon$  of an oriented graph  $\vec{G}$ :

**Theorem 22** *Let  $\vec{G}$  be an oriented graph and  $u$  and  $v$  be any two vertices in  $V(\vec{G})$  with  $d_o(\vec{G}, u, v) > 2$ . We then have  $\chi_o(\epsilon_{u=v}(\vec{G})) = \chi_o(\vec{G})$  if and only if there exists an oriented  $\chi_o(\vec{G})$ -colouring  $\gamma$  of  $\vec{G}$  such that  $\gamma(u) = \gamma(v)$ .*

**Proof.** Let  $k = \chi_o(\vec{G})$  and suppose first that there exists an oriented  $k$ -colouring  $\gamma$  of  $\vec{G}$  such that  $\gamma(u) = \gamma(v)$ . The mapping  $\gamma' : V(\epsilon_{u=v}(\vec{G})) \rightarrow \{1, 2, \dots, k\}$  given by  $\gamma'(w) = \gamma(w)$  for every  $w$  in  $V(\epsilon(\vec{G}))$  is clearly an oriented colouring of  $\epsilon_{u=v}(\vec{G})$  and, therefore,  $\chi_o(\epsilon_{u=v}(\vec{G})) \leq \chi_o(\vec{G})$ . Since  $\chi_o(\vec{G}) \leq \chi_o(\epsilon_{u=v}(\vec{G}))$  by Theorem 20, we get  $\chi_o(\epsilon_{u=v}(\vec{G})) = \chi_o(\vec{G})$ .

Conversely, suppose that  $\chi_o(\epsilon_{u=v}(\vec{G})) = \chi_o(\vec{G}) = k$  and let  $\gamma'$  be an oriented  $k$ -colouring of  $\epsilon_{u=v}(\vec{G})$ . The mapping  $\gamma : V(\vec{G}) \rightarrow \{1, 2, \dots, k\}$  given by  $\gamma(v) = \gamma(u) = \gamma'(u)$  and  $\gamma(w) = \gamma'(w)$  for every  $w$  in  $V(\vec{G}) \setminus \{u, v\}$  is again clearly an oriented colouring of  $\vec{G}$  and this colouring satisfies the required condition.  $\square$



The oriented graph  $\vec{G}_k$

Figure 6: Construction for Theorem 24

If  $\vec{G}$  is an oriented graph,  $u$  and  $v$  two vertices of  $\vec{G}$  at directed distance at least 3, we may either identify  $u$  and  $v$ , add the arc  $\vec{uv}$  or add the arc  $\vec{vu}$  and still get an oriented graph. The following result shows that at least one of these graphs has the same oriented chromatic number as  $\vec{G}$ :

**Theorem 23** *Let  $\vec{G}$  be an oriented graph and  $u$  and  $v$  be any two vertices in  $V(\vec{G})$  with  $d_o(\vec{G}, u, v) > 2$ . We then have*

$$\chi_o(\vec{G}) = \min \{ \chi_o(\epsilon_{u=v}(\vec{G})), \chi_o(\vec{G} + \vec{uv}), \chi_o(\vec{G} + \vec{vu}) \}.$$

**Proof.** If  $\chi_o(\vec{G}) = \chi_o(\epsilon_{u=v}(\vec{G}))$ , then the statement is true since  $\chi_o(\vec{G}) \leq \chi_o(\vec{G} + \vec{uv})$  and  $\chi_o(\vec{G}) \leq \chi_o(\vec{G} + \vec{vu})$  for every oriented graph  $\vec{G}$  and each pair of vertices  $u, v$  with  $d_o(\vec{G}, u, v) > 2$ .

Otherwise,  $\chi_o(\epsilon_{u=v}(\vec{G})) = \chi_o(\vec{G}) + 1$  by Theorem 20. Moreover, by Theorem 22, every oriented  $\chi_o(\vec{G})$ -colouring of  $\vec{G}$  assigns distinct colours to  $u$  and  $v$ . Every such colouring  $\gamma$  is thus an oriented colouring of  $\vec{G} + \vec{vu}$  or  $\vec{G} + \vec{uv}$ , depending on whether there exists an arc  $\vec{wx}$  with  $\gamma(u) = \gamma(x)$  and  $\gamma(v) = \gamma(w)$ , or  $\gamma(u) = \gamma(w)$  and  $\gamma(v) = \gamma(x)$ , respectively (if none of these situations occurs,  $\gamma$  is an oriented colouring of both  $\vec{G} + \vec{vu}$  and  $\vec{G} + \vec{uv}$ ).  $\square$

Let us now consider elementary homomorphisms and (oriented) achromatic numbers. In [15], Harary and Hedetniemi proved that for every undirected graph  $G$  and every elementary homomorphism  $\epsilon$  of  $G$ ,  $\psi(G) - 2 \leq \psi(\epsilon(G)) \leq \psi(G)$ . The rightmost inequality also holds for the oriented achromatic number, as observed in Proposition 3 for oriented graphs and in Proposition 6 for undirected graphs. On the other hand, the leftmost inequality no longer holds for the oriented achromatic number of oriented graphs, and the difference between  $\psi_o(\vec{G})$  and  $\psi_o(\epsilon(\vec{G}))$  can be made arbitrarily large:

**Theorem 24** *For every integer  $k, k \geq 1$ , there exists an oriented graph  $\vec{G}_k$  and an elementary homomorphism  $\epsilon_k$  of  $\vec{G}_k$  such that  $\psi_o(\vec{G}_k) \geq \psi_o(\epsilon_k(\vec{G}_k)) + k$ .*

**Proof.** Let  $\vec{G}_k$  be the oriented graph depicted in Figure 6 where  $\vec{O}_\ell$  and  $\vec{O}'_\ell$  are two copies of an oriented o-clique  $\vec{O}_\ell$  such that  $2\psi_o(\vec{O}_\ell) \geq \psi_o(2\vec{O}_\ell) + k + 1$ , where  $2\vec{O}_\ell$  denotes the disjoint union of two copies of  $\vec{O}_\ell$  (the existence of such a clique was already discussed in the proof of Theorem 14).

Let  $\epsilon_k = \epsilon_{x=v'}$ . Since the two vertices  $v$  and  $v'$  (identified with  $x$ ) must be assigned distinct colours in every complete oriented colouring of  $\vec{G}_k$ , we have  $\psi_o(\epsilon_k(\vec{G}_k)) \leq q + 2$ , where  $q = \psi_o(2\vec{O}_\ell)$ .

Using again similar arguments as in the proof of Theorem 14, we get that  $\psi_o(\vec{G}_k) \geq 2p + 1$ , where  $p = \psi_o(\vec{O}_\ell)$ .

Finally, since  $2p - q \geq k + 1$ , we get  $\psi_o(\vec{G}_k) - \psi_o(\epsilon_k(\vec{G}_k)) \geq 2p + 1 - (q + 2) \geq k$ .  $\square$

Despite the fact that the difference between  $\psi_o(\vec{G})$  and  $\psi_o(\epsilon(\vec{G}))$  can be made arbitrarily large, we still have the following result, similarly to the undirected case:

**Theorem 25** *If  $\vec{G}$  is an oriented graph which is not an o-clique, then there exists an elementary homomorphism  $\epsilon$  of  $\vec{G}$  such that  $\psi_o(\epsilon(\vec{G})) = \psi_o(\vec{G})$ .*

**Proof.** By Proposition 3, we know that  $\psi_o(\epsilon(\vec{G})) \leq \psi_o(\vec{G})$  for every elementary homomorphism  $\epsilon$  of  $\vec{G}$ .

Let now  $\gamma$  be any optimal complete oriented colouring of  $\vec{G}$  using  $\psi_o(\vec{G})$  colours. Since  $\vec{G}$  is not an o-clique,  $\psi_o(\vec{G}) < |V(\vec{G})|$  and, therefore, there exist two vertices  $u$  and  $v$  in  $V(\vec{G})$  with  $\gamma(u) = \gamma(v)$ . The colouring of  $\epsilon_{u=v}(\vec{G})$  induced by  $\gamma$  is obviously a complete oriented colouring so that  $\psi_o(\epsilon_{u=v}(\vec{G})) \geq \psi_o(\vec{G})$  and the result follows.  $\square$

**Corollary 26** *If  $G$  is an undirected graph such that no orientation of  $G$  is an o-clique, then there exists an elementary homomorphism  $\epsilon$  of  $G$  such that  $\psi_o(\epsilon(G)) = \psi_o(G)$ .*

**Proof.** By Proposition 6 we know that  $\psi_o(\epsilon(G)) \leq \psi_o(G)$ . Let  $\vec{G}$  be any orientation of  $G$  such that  $\psi_o(\vec{G}) = \psi_o(G)$ . Since  $\vec{G}$  is not an o-clique, we get by Theorem 25 that there exists an elementary homomorphism  $\epsilon$  of  $\vec{G}$  such that  $\psi_o(\epsilon(\vec{G})) = \psi_o(\vec{G})$ . Note that  $\epsilon$  is also an elementary homomorphism of  $G$  and that  $\epsilon(\vec{G})$  is an orientation of  $\epsilon(G)$ . Hence,  $\psi_o(\epsilon(G)) \geq \psi_o(\epsilon(\vec{G})) = \psi_o(\vec{G}) = \psi_o(G)$  and the result follows.  $\square$

## 6 Discussion

In this paper, we introduced and studied the new notions of complete colourings and the oriented achromatic number of oriented graphs. We provided some first results on this topic and we would like to mention possible directions for future work.

Many results concerning the achromatic number of undirected graphs make use of the fact that, for every integer  $k$ , the only clique of order  $k$  is the complete graph  $K_k$ . Since the size (number of edges) of  $K_k$  is  $\binom{k}{2}$ , the achromatic number of a graph of order  $n \leq \binom{k}{2}$  is necessarily less than  $k$ . The situation is less clear for the oriented achromatic number of oriented graphs since we do not know the exact size  $f(k)$  of an optimal o-clique of order  $k$ . (Theorem 1 just tells us that  $f(k) \approx k \log k$  when  $k$  tends to infinity.) The study of this function  $f$  is therefore particularly challenging.

Another direction of research, related to the previous one, would be to study the oriented achromatic number of some “simple” classes of oriented graphs, such as paths, cycles, trees or graphs with bounded degree. Several results for the achromatic number of these families are given in [4, 5, 12, 17, 21, 22].

Finally, it would be interesting to study in more details the computational complexity of the problem of determining the oriented achromatic number of an oriented graph for various classes of oriented graphs.

**Acknowledgements.** The author thanks the anonymous referees for their helpful comments on the first version of this paper.

## References

- [1] J. Bensmail, R. Duvignau and S. Kirgizov. The complexity of deciding whether a graph admits an orientation with fixed weak diameter. Preprint (2013), available at <http://hal.archives-ouvertes.fr/hal-00824250>.
- [2] V.N. Bhave. On the pseudoachromatic number of a graph. *Fundamenta Mathematicae* **102** (1979), 159–164.
- [3] O.V. Borodin, D. Von-Der-Flaass, A.V. Kostochka, A. Raspaud and E. Sopena. On deeply critical oriented graphs. *J. Combin. Theory Ser. B* **81** (2001), 150–155.
- [4] N. Cairnie and K. Edwards. Some results on the achromatic number. *J. Graph Theory* **26** (1997), 129–136.
- [5] N. Cairnie and K. Edwards. The achromatic number of bounded degree trees. *Discrete Math* **188** (1998), 87–97.
- [6] V. Chvatal and C. Thomassen. Distances in orientations of graphs. *J. Combin. Theory Ser. B* **24** (1978), 61–75.
- [7] G. Chartrand and P. Zhang. *Chromatic Graph Theory*. CRC Press, Chapman & Hall (2009).
- [8] B. Courcelle. The monadic second order logic of graphs VI: On several representations of graphs by relational structures. *Discrete Appl. Math.* **54** (1994), 117–149.
- [9] K.J. Edwards. The harmonious chromatic number and the achromatic number. In *Surveys in Combinatorics, 1997 (Invited papers for the 16th British Combinatorial Conference)*, ed. R.A. Bailey, *LMS Lecture Notes* **241**, Cambridge University Press, Cambridge (1997), 13–47.
- [10] K.J. Edwards. *Harmonious Colourings and Achromatic Number*. Online bibliography available at <http://www.computing.dundee.ac.uk/staff/kedwards/>.
- [11] K.J. Edwards. Harmonious chromatic number of directed graphs. *Discrete Appl. Math.* **161** (2013), 369–376.
- [12] M. Farber, G. Hahn, P. Hell and D. Miller. Concerning the achromatic number of graphs. *J. Combin. Theory Ser. B* **40** (1986), 21–39.
- [13] Z. Füredi, P. Horak, C.M. Pareek and X. Zhu. Minimal oriented graphs of diameter 2. *Graphs Combin.* **14** (1998), 345–350.
- [14] D.P. Geller and H.V. Kronk. Further results on the achromatic number. *Fundamenta Mathematicae* **85** (1974), 285–290.
- [15] F. Harary and S.T. Hedetniemi. Achromatic number of a graph. *J. Comb. Theory* **8** (1970), 154–161.
- [16] F. Harary, S.T. Hedetniemi and G. Prins. An interpolation theorem for graphical homomorphisms. *Portugaliae Mathematica* **26** (1967), 453–462.
- [17] P. Hell and D. Miller. Graph with given achromatic number. *Discrete Math.* **16** (1976), 195–207.
- [18] F. Hugues and G. MacGillivray. The achromatic number of graphs: a survey and some new results. *Bull. Inst. Combin. Appl.* **19** (1997), 27–56.

- [19] W.F. Klostermeyer and G. MacGillivray. Analogues of cliques for oriented coloring. *Discuss. Math. Graph Theory* **24** (2004), 373–387.
- [20] A.V. Kostochka, T. Luczak, G. Simonyi and E. Sopena. On the minimum number of edges giving maximum oriented chromatic number. In proc. Dimacs/Dimatia conference “Contemporary Trends in Discrete Mathematics”, Štířin, Czech Rep., May 1997, *Dimacs Series in Discrete Math. and Theoret. Comput. Sci.* **49** (1997), 179–182.
- [21] J. Lee and Y. Shin. The achromatic number of the union of cycles. *Discrete Appl. Math.* **143** (2004), 330–335.
- [22] G. MacGillivray and A. Rodriguez. The achromatic number of the union of paths. *Discrete Math.* **231** (2001), 331–335.
- [23] E. Sopena. The chromatic number of oriented graphs. *J. Graph Theory* **25** (1997), 191–205.
- [24] E. Sopena. Oriented graph colouring. *Discrete Math.* **229(1-3)** (2001), 359–369.