# Acircuitic directed star arboricity of planar graphs with large girth 

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#### Abstract

A directed star forest is a forest all of whose components are stars with arcs emanating from the center to the leaves. The acircuitic directed star arboricity of an oriented graph $G$ is the minimum number of edge-disjoint directed star forests whose union covers all edges of $G$ and such that the union of two such forests is acircuitic. We show that graphs with maximum average degree less than $\frac{7}{3}$ (resp. $\frac{133}{41}$ ) have acircuitic directed star arboricity at most $\Delta$ (resp. $\Delta+1$ ); this implies that planar graphs of girth at least 14 (resp. 6) have acircuitic directed star arboricity at most $\Delta$ (resp. $\Delta+1$ ).


Keywords: Oriented graph, acyclic edge-coloring, star arboricity, planar graph.

## 1 Introduction

We consider finite simple loopless undirected or oriented graphs, that is graphs with no cycle of length one or two. For an undirected graph (resp. an oriented graph) $G$, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set (resp. by $A(G)$ its arc set).

An edge-coloring of an undirected graph $G$ is proper if no pair of incident edges receive the same color. A proper edge-coloring of $G$ is acyclic if the graph induced by any two color classes is a forest. The minimum number of colors needed to acyclically edge-color $G$ is the acyclic chromatic index of $G$ and is denoted by $\chi_{a}^{\prime}(G)$. The notion of acyclic coloring was introduced by Grünbaum in [7].

The best known upper bound on acyclic edge-coloring is due to Alon et al. [3] where they proved that every graph has an acyclic (16 $\Delta$ )-edge-coloring. More recently, Muthu et al. [10] obtained better results for graphs with large girth: in particular, for every undirected graph $G$ with girth $g \geq 9, \chi_{a}^{\prime}(G) \leq 6 \Delta(G)$.

Acyclic chromatic index can be related to the notion of star arboricity. The notion of star arboricity was introduced by Akiyama and Kano in [1]. It is defined as the minimum number of edge-disjoint star forests needed to cover $E(G)$. They proved that complete graphs on $n$ vertices can be decomposed into $\left\lceil\frac{n}{2}\right\rceil+1$ star forests. This notion was studied by many authors for some graph families such as $d$-regular graphs, complete regular multipartite graphs or planar graphs (see [2, 4, 5]).

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Considering planar graphs, Algor and Alon showed in [2] that every planar graph can be decomposed into 6 star forests. This bound was decreased to 5 by Hakimi et al. in [9] and this bound is tight.

In [2], Algor and Alon introduced the notion of directed star arboricity of a digraph $G$, defined as the minimum number of edge-disjoint directed star forests needed to cover $A(G)$ (a directed star is a star with arcs emanating from the center to the leaves). In [8], Guiduli showed that every digraph with indegree and outdegree both less than $D$ has directed star arboricity at most $D+20 \log D+84$.
We introduced in [11] the new notion of acircuitic directed star arboricity of an oriented graph $G$, denoted by $\operatorname{adst}(G)$, defined as the minimum number of edge-disjoint directed star forests needed to cover $A(G)$ in such a way that the union of any two such forests contains no circuit. The study of this notion arises from the study of oriented arc-colorings since every oriented graph with acircuitic directed star arboricity at most $k$ admits an oriented ( $k \cdot 2^{k-1}$ )-arc-coloring (see [11] for more details).

It is easy to see that each color class of an acyclic edge-coloring of $G$ is a matching (therefore a star forest), and the graph induced by any two such matchings is a forest. We thus get that for every orientation $\vec{G}$ of an undirected graph $G$ with $\chi_{a}^{\prime}(G) \leq k, \operatorname{adst}(\vec{G}) \leq k$.

In this paper, we consider the acyclic chromatic index and the acircuitic directed star arboricity of oriented graphs with bounded maximum average degree.
Definition 1 Let $G$ be a graph. The maximum average degree of $G$, denoted by $\operatorname{mad}(G)$, is given by

$$
\operatorname{mad}(G)=\max _{H \subseteq G}\left\{\frac{2|A(H)|}{|V(H)|}\right\}
$$

We prove the following:
Theorem 1 Let $G$ be an undirected graph (which is not a cycle) with $\operatorname{mad}(G) \leq \frac{7}{3}$. Then $\chi_{a}^{\prime}(G)=$ $\Delta(G)$.

Therefore, we get:
Corollary 1 Let $G$ be an oriented graph (which is not a cycle) with $\operatorname{mad}(G) \leq \frac{7}{3}$. Then $\operatorname{adst}(G) \leq$ $\Delta(G)$.

For graphs with larger maximum average degree, we prove the following:
Theorem 2 Let $G$ be an oriented graph with $\operatorname{mad}(G)<\frac{133}{41}$. Then adst $(G) \leq \Delta(G)+1$.
In case of planar graphs, the maximum average degree parameter is linked to the girth (the shortest cycle of the graph) as follows [6]:
Observation 1 Let $G$ be a planar graph with girth $g$, then $\operatorname{mad}(G)<\frac{2 g}{g-2}$.
Therefore, we get:

## Corollary 2

(1) Let $G$ be an oriented planar graph (which is not a cycle) with girth $g \geq 14$. Then $\operatorname{adst}(G) \leq \Delta(G)$.
(2) Let $G$ be an oriented planar graph with girth $g \geq 6$. Then $\operatorname{adst}(G) \leq \Delta(G)+1$.

Moreover, we can easily see that for every undirected graph $G$, there exist an orientation $\vec{G}$ with acircuitic directed star arboricity at least $\Delta(G)$ (consider $\vec{G}$ containing a star of degree $\Delta(G)$ with arcs emanating from the leaves to the center). Therefore, for oriented planar graphs with girth at least 14 , the bound is tight, and for oriented planar graphs $G$ with girth at least 6 , we get $\Delta(G) \leq \operatorname{adst}(G) \leq \Delta(G)+1$.

This paper is organised as follows: we introduce the main definitions and notation in the next section and prove Theorem 1 and 2 in Sections 3 and 4.

## 2 Definitions and notation

Let $G$ be an oriented graph. We denote by $\overrightarrow{u v}$ the arc from $u$ to $v$ or simply $u v$ whenever its orientation is not relevant (therefore $u v=\overrightarrow{u v}$ or $u v=\overrightarrow{v u}$ ).

For a vertex $v$ of $G$, we denote by $d^{-}(v)$ the indegree of $v$, by $d^{+}(v)$ its outdegree and by $d(v)$ its degree, that is $d(v)=d^{+}(v)+d^{-}(v)$. A vertex of degree $k$ (resp. at most $k$, at least $k$ ) is called a
 vertex. A neighbor of degree $k$ (resp. at most $k$, at least $k$ ) of a vertex $u$ is called a $k$-neighbor (resp. $\leq_{k \text {-neighbor, }} \geq_{k \text {-neighbor) of } u \text {. We denote by } N^{-}(v) \text { the set of incoming neighbors of } v\left(N^{-}(v)=, ~=~=~\right.}^{\text {n }}$. $\{u \in V \mid \overrightarrow{u v} \in A(G)\})$, by $N^{+}(v)$ the set of outgoing neighbors of $v\left(N^{+}(v)=\{w \in V \mid \overrightarrow{v w} \in A(G)\}\right)$ and by $N(v)$ the set of neighbors of $v$, that is $N(v)=N^{-}(v) \cup N^{+}(v)$. A source vertex is a vertex $v$ with $d^{-}(v)=0$. The maximum degree and minimum degree of a graph $G$ are respectively denoted by $\Delta(G)$ and $\delta(G)$.

For an undirected (resp. oriented) graph $G$ and a vertex $v$ of $V(G)$, we denote by $G \backslash v$ the graph obtained from $G$ by removing $v$ together with the set of its incident edges (resp. arcs); similarly, for an edge (resp. arc) $a$ of $E(G)$ (resp. $A(G)$ ), $G \backslash a$ denotes the graph obtained from $G$ by removing $a$. These two notions are extended to sets in a standard way: for a set of vertices $V^{\prime} \subseteq V(G), G \backslash V^{\prime}$ denotes the graph obtained from $G$ by successively removing all vertices of $V^{\prime}$ and their incident edges (resp. arcs), and for a set of edges (resp. arcs) $W \subseteq E(G)$ (resp. $W \subseteq A(G)$ ), $G \backslash W$ denotes the graph obtained from $G$ by removing all edges (resp. arcs) of $W$.

The notions of arboricity discussed in the previous section may be defined in terms of arc-coloring or partitions of the set of the arcs. More precisely, a $k$-directed-star-coloring (or simply $k$-dst-coloring) of an oriented graph $G$ is a partition of $A(G)$ into $k$ directed star forests $\left\{F_{1}, F_{2}, \ldots, F_{k}\right\}$. Equivalently, a $k$-dst-coloring of $G$ is a $k$-coloring $f$ of $A(G)$ such that $(i) \overrightarrow{u v}, \overrightarrow{v w} \in A(G) \Rightarrow f(\overrightarrow{u v}) \neq f(\overrightarrow{v w})$, and (ii) $\overrightarrow{u v}, \vec{w} \in A(G) \Rightarrow f(\overrightarrow{u v}) \neq f(\overrightarrow{w v})$. The directed star arboricity of $G$, denoted by dst $(G)$, is the smallest $k$ for which $G$ admits a $k$-dst-coloring.

An oriented graph $G$ is acircuitic if it does not contain any circuit (that is a cycle with all arcs having the same direction). A $k$-acircuitic-directed-star-coloring (or simply $k$-adst-coloring) of a graph $G$ is a partition of $A(G)$ into $k$ directed star forests $\left\{F_{1}, F_{2}, \ldots, F_{k}\right\}$ such that for all $i, j \in[1, k], F_{i} \cup F_{j}$ is acircuitic. Equivalently, a $k$-adst-coloring of $G$ is a $k$-dst-coloring of $G$ such that no circuit in $G$ is bichromatic. The acircuitic directed star arboricity of $G$, denoted by $\operatorname{adst}(G)$, is the smallest $k$ for which $G$ admits a $k$-adst-coloring.

For an undirected graph $G$, a given edge-coloring $f$ of $G$ and a vertex $v \in V(G)$, we denote by $C_{f}(v)$ the set of incident colors of $v$ (i.e. $C_{f}(v)=\{f(u v) \mid u \in N(v)\}$ ). For an oriented graph $G$, a given arc-coloring $f$ of $G$ and a vertex $v \in V(G)$, we denote by $C_{f}^{-}(v)$ the set of incoming colors of $v$ (i.e. $\left.C_{f}^{-}(v)=\left\{f(\overrightarrow{u v}) \mid u \in N^{-}(v)\right\}\right)$ and by $C_{f}^{+}(v)$ the set of outgoing colors of $v$ (i.e. $C_{f}^{+}(v)=\{f(\overrightarrow{v u}) \mid$
$\left.\left.u \in N^{+}(v)\right\}\right)$ (therefore $C_{f}(v)=C_{f}^{+}(v) \cup C_{f}^{+}(v)$ ).

## 3 Proof of Theorem 1

We use the method of reducible configurations and discharging procedures to prove Theorem 1.
In the rest of this section, let $H$ be a minimal (with respect to inclusion as a subgraph) counter-example to Theorem 1 and $\Delta=\Delta(H)$. We shall show that $H$ does not contain a set of reducible configurations.

The drawing conventions for a forbidden configuration are the following: a vertex whose neighbors are totally specified will be black (i.e. vertex of fixed degree), whereas a vertex whose neighbors are partially specified will be white. Moreover, an edge will represent an arc with any of its two possible orientations.

In all the proofs which follow, we shall proceed similarly. We suppose that $H$ contains some configurations and, for each of them, we consider a reduction $H^{\prime}$ of $H$ obtained from $H$ by removing vertices and/or arcs. Since $H^{\prime}$ is a subgraph of $H, \operatorname{mad}\left(H^{\prime}\right) \leq \operatorname{mad}(H)$. Therefore, due to the minimality of $H$, there exists an acyclic $\Delta\left(H^{\prime}\right)$-edge-coloring $f$ of $H^{\prime}$; moreover, $\Delta\left(H^{\prime}\right) \leq \Delta$ and thus $f$ is an acyclic $\Delta$ -edge-coloring of $H^{\prime}$. The coloring $f$ is therefore a partial acyclic $\Delta$-edge-coloring of $H$, that is an acyclic edge-coloring on some subset $S$ of $A(H)$, and we show how to extend it to an acyclic $\Delta$-edge-coloring of $H$.

The existence of such a coloring $f$ of $H$ showed that $H$ does not contain each considered configuration.


Fig. 1: Forbidden configurations for Theorem 1

Lemma 1 The graph $H$ does not contain a 1-vertex.
Proof: Suppose that $H$ contains a dangling edge $u v$ and let $H^{\prime}=H \backslash u v$. We can easily extend $f$ since $u v$ is adjacent to at most $\Delta-1$ colors.

Since $H$ is not a circuit and does not contains 1 -vertices, we get $\Delta \geq 3$ (therefore, we have at least three colors).

Lemma 2 The graph $H$ does not contain the configuration (a) of Figure 1.
Proof: Let $H^{\prime}=H \backslash v$. We can set $f(u v)=c_{1}$ for some $c_{1} \notin\left\{f\left(u_{1} u\right), f\left(w w_{1}\right)\right\}$ since $\mid\left\{f\left(u_{1} u\right)\right.$, $\left.f\left(w w_{1}\right)\right\} \mid \leq 2$. We can finally set $f(v w)=c_{2}$ for some $c_{2} \notin\left\{f\left(u_{1} u\right), f\left(w w_{1}\right)\right\}$ since we have at least one available color ( $f$ is acyclic since $C_{f}(v) \cap C_{f}(w)=\emptyset$ ).


Fig. 2: Extented acyclic edge-coloring
Lemma 3 The graph $H$ does not contain the configuration (b) of Figure 1.
Proof: Let $H^{\prime}=H \backslash\left\{x, v_{2}, w_{2}\right\}$. Let us consider the two graphs of Figure 2. The bold colors are the colors of the partial acyclic $\Delta$-edge-coloring $f$, and the notation $\mathbf{a} / \mathbf{b}$ means that we consider two cases on the same configuration.

We consider four distinct cases depending on the coloring of $u u_{1}, v v_{1}$ and $w w_{1}:(1,1,1),(1,1,2)$, $(1,2,2)$ and $(1,2,3)$. Figure 2 shows the four possible cases and gives a possible extension of the required acyclic edge-coloring.

We now prove that every graph with no 1-vertex that does not contain the configurations of Figure 1 has $\operatorname{mad} \geq \frac{7}{3}$.

We assign to each vertex $v$ an initial charge $w(v)=d(v)$. Now, we use the following discharging rules:
Rule 1. Each 3-vertex gives $\frac{1}{3}$ to each of its 3-neighbors having a 2-neighbor;
Rule 2. Each 3-vertex gives $\frac{1}{6}$ to each of its other 2-neighbors.
We denote by $w^{*}(v)$ the new charge of the vertex $v$ after the discharging procedure.
Let $v$ be a $k$-vertex; we shall prove that the new charge $w^{*}(v)$ of each vertex $v$ is at least $\frac{7}{3}$. We consider the following cases (recall that $H$ has no 1 -vertices):

- if $k=2$, then $w(v)=2$ and $w^{*}(v) \geq 2+\min \left\{\frac{1}{3} ; 2 \cdot \frac{1}{6}\right\}=\frac{7}{3}$, since by Lemma 2 a 2 -vertex has at least one $\geq 3$-neighbor;
- if $k=3$, then $w(v)=3$ and $w^{*}(v) \geq 3-\max \left\{2 \cdot \frac{1}{3} ; \frac{1}{3}+2 \cdot \frac{1}{6} ; 3 \cdot \frac{1}{6}\right\}=\frac{7}{3}$;
- if $k \geq 4$, then $w(v)=k$ and $w^{*}(v) \geq k-k \cdot \frac{1}{3}>\frac{7}{3}$.

Observe that

$$
\sum_{v \in V(H)} w(v)=\sum_{v \in V(H)} w^{*}(v)=\sum_{v \in V(H)} d(v)
$$

and

$$
\operatorname{mad}(H) \geq \frac{2|A(H)|}{|V(H)|}=\frac{\sum_{v \in V(H)} d(v)}{|V(H)|}=\frac{\sum_{v \in V(H)} w^{*}(v)}{|V(H)|} \geq \frac{\frac{7}{3}|V(H)|}{|V(H)|}=\frac{7}{3}
$$

The contradiction with the hypothesis $\operatorname{mad}(\mathrm{H})<\frac{7}{3}$ completes the proof of Theorem 1.
Therefore, every graph (which is not a cycle) with mad $\leq \frac{7}{3}$ admits an acyclic $\Delta$-edge-coloring and thus a $\Delta$-adst-coloring.

## 4 Proof of Theorem 2

To prove this theorem, we use the same technique as before.
Let $H$ be a minimal (with respect to inclusion as a subgraph) counter-example to Theorem 2 and $\Delta=$ $\Delta(H)$. We introduce here two new notions. For an uncolored arc $\overrightarrow{u v}$ of $H$, a color $c$ is called an available color for $\overrightarrow{u v}$ if it satisfies the constraints of a $d s t$-coloring for $\overrightarrow{u v}$; we denote by $F^{f}(\overrightarrow{u v})=C_{f}^{-}(u) \cup C_{f}(v)$ the set of forbidden colors of $\overrightarrow{u v}$; we can easily see that any color $c \notin F^{f}(\overrightarrow{u v})$ is an available color for $\overrightarrow{u v}$. A color $c$ is called a feasible color for $\overrightarrow{u v}$ if assigning color $c$ to $\overrightarrow{u v}$ results in a partial adst-coloring. Note that a feasible color is necessarily an available color whereas the contrary is false. We can also note that for a given partial adst-coloring $f$ of $H$ and an uncolored arc $\overrightarrow{u v}$, if $C_{f}^{-}(u) \cap C_{f}^{+}(v)=\emptyset$, then any available color for $\overrightarrow{u v}$ is also feasible for $\overrightarrow{u v}$ (indeed, any circuit which contains $\overrightarrow{u v}$ will not be bichromatic).

In almost all cases, the completion process of $f$ will be the following: for an uncolored arc $\overrightarrow{u v}$, we shall initially determine its available colors, and then ensure, by recoloring some arcs if necessary, that any such color is also feasible for $\overrightarrow{u v}$.

We shall show in the following lemmas that $H$ contains none of the configurations depicted in Figure 3.
In [11], we proved that subcubic graphs (graphs of maximum degree at most three) admit a 4 -adstcoloring. Hence, we may assume that $\Delta \geq 4$ in our arguments.
Lemma 4 The graph $H$ does not contain any of the configurations (a) and (b) of Figure 3.
Proof: Let $H^{\prime}=H \backslash u u_{1}$. We can set $f\left(u u_{1}\right)=c$ for some $c \notin F^{f}\left(u u_{1}\right)$ since each available color for $u u_{1}$ is also feasible (no circuit in $H$ contains $u u_{1}$ ) and $\left|F^{f}\left(u u_{1}\right)\right|=\left|C_{f}\left(u_{1}\right)\right| \leq \Delta-1$.

Lemma 5 The graph H does not contain the configuration (c) of Figure 3.
Proof: Let $H^{\prime}=H \backslash \overrightarrow{u_{1} \vec{u}}$. We can set $f\left(\overrightarrow{u_{1} \vec{u}}\right)=c$ for some $c \notin F^{f}\left(\overrightarrow{u_{1}} \vec{u}\right)$ since each available color for $\overrightarrow{u_{1} u}$ is also feasible (no circuit in $H$ contains $\overrightarrow{u_{1} \vec{u}}$ ) and $\left|F^{f}\left(\overrightarrow{u_{1} u}\right)\right|=\left|C_{f}^{-}\left(u_{1}\right) \cup C_{f}(u)\right| \leq \Delta-1+1=\Delta$.

Note that by the two previous lemmas, if $u$ is a 2-vertex in $H$, then $u$ has one incoming and one outgoing $\operatorname{arc}\left(\delta^{-}(u)=\delta^{+}(u)=1\right)$.
Lemma 6 The graph H does not contain the configuration (d) of Figure 3.
Proof: Let $H^{\prime}=H \backslash \overrightarrow{v u}$. We can set $f(\overrightarrow{v u})=c$ for some $c \notin F^{f}(v u)$ since each available color for $\overrightarrow{v u}$ is also feasible (no circuit in $H$ contains $\overrightarrow{v u}$ ) and $\left|F^{f}(\overrightarrow{v u})\right|=\left|C_{f}^{-}(v) \cup C_{f}(u)\right| \leq 1+\Delta-1=\Delta$.

Lemma 7 The graph $H$ does not contain the configuration (e) of Figure 3.
Proof: Let $H^{\prime}=H \backslash \overrightarrow{v w}$. Suppose first that $f(\overrightarrow{u v}) \notin C_{f}^{+}(w)$; therefore $C_{f}^{-}(v) \cap C_{f}^{+}(w)=\emptyset$ and thus any available color for $\overrightarrow{v w}$ is also feasible: we can set $f(\overrightarrow{v w})=c$ for some $c \notin F^{f}(v w)$ since $\left|F^{f}(\overrightarrow{v w})\right|=\left|C_{f}^{-}(v) \cup C_{f}(w)\right| \leq 1+\Delta-1=\Delta$.
$k \leq \Delta$

(b)

(c)

$$
k \leq \Delta-1, \quad k^{\prime} \leq \Delta-1
$$



(f)
(e)

(h)
(k)


(i)

(1)

(j)

(m)

Fig. 3: Forbidden configurations for Theorem 2
$k \leq \Delta-2$

(n)

$$
k \leq \Delta-2
$$


(o)

$$
k \leq \Delta-2
$$


(p)

Fig. 3: Forbidden configurations for Theorem 2 (cont'd)
Suppose now that $f(\overrightarrow{u v}) \in C^{+}(w)$. W.l.o.g., let $f(\overrightarrow{u v})=f\left(\overrightarrow{w w_{1}}\right)=\ldots=f\left(\overrightarrow{w w_{n}}\right)$ for some $n$, $1 \leq n \leq k$. Since $\left|F^{f}\left(\overrightarrow{w w_{i}}\right)\right| \leq \Delta-1$ for $1 \leq i \leq n$, we can recolor $f\left(\overrightarrow{w w_{i}}\right)=c_{i}$ for some $c_{i} \notin F^{f}\left(\overrightarrow{w w_{i}}\right) \cup f(\overrightarrow{u v})$, that leads us to the previous case.

Note that by the previous lemma, if $\overrightarrow{u v w}$ is a directed path in $H$, then one of $v$ and $w$ must have at least two incoming arcs $\left(\delta^{-}(v) \cdot \delta^{-}(w) \geq 2\right)$.
Lemma 8 The graph $H$ does not contain the configuration $(f)$ of Figure 3.
Proof: Let $H^{\prime}=H \backslash u v$. Suppose first that $f\left(u u_{1}\right) \neq f\left(v v_{1}\right)$; we can set $f(u v)=c$ for some $c \notin F^{f}(u v)$ since each available color for $\overrightarrow{u v}$ is also feasible (we have $C_{f}(u) \cap C_{f}(v)=\emptyset$ ) and $\left|F^{f}(u v)\right|=2<\Delta+1$.
Suppose now that $f\left(u u_{1}\right)=f\left(v v_{1}\right)$. Let $c \notin F^{f}(u v) \cup C_{f}\left(v_{1}\right)=C_{f}\left(v_{1}\right)$; we have $\left|C_{f}\left(v_{1}\right)\right| \leq \Delta$, and thus such a color exists and is an available color for $u v$; moreover, since $c \notin C_{f}\left(v_{1}\right), c$ is also a feasible color for $u v$ and therefore we can set $f(u v)=c$.

Lemma 9 The graph H does not contain the configuration $(g)$ of Figure 3.
Proof: Let $H^{\prime}=H \backslash u v$. Let $f\left(u u_{1}\right)=a, f\left(u u_{2}\right)=b$ and $f\left(v v_{1}\right)=c$. By Lemmas 4, 5 and 7, there exists $i \in\{1,2\}$ such that $\overrightarrow{u_{i} \vec{u}} \in A(H)$; w.l.o.g., let $\overrightarrow{u_{1} \vec{u}} \in(H)$; that implies $a \neq b$. We have two cases to consider:

1. $|\{a, b, c\}|=3$.

We have $C_{f}(u) \cap C_{f}(v)=\emptyset$ and therefore any available color for $\overrightarrow{u v}$ is feasible for $\overrightarrow{u v}$ (no bichromatic circuit in $H$ contains $\overrightarrow{u v}$ ). We can thus set $f(u v)=d$ for some $d \notin F^{f}(u v)$ since $\left|F^{f}(u v)\right| \leq 3<\Delta$.
2. $|\{a, b, c\}|=2$.
W.l.o.g. we assume that $a=c$. Suppose that $\overrightarrow{u v v_{1}}$ is a directed path; thus, by Lemma 7, we have $\left|d^{-}\left(v_{1}\right)\right| \geq 2$. Let $d \notin S=F^{f}(\overrightarrow{u v}) \cup C_{f}^{+}\left(v_{1}\right)$; we have $|S| \leq 2+\Delta-2=\Delta$, and thus such a color exists and is an available color for $u v$; moreover, since $d \notin C_{f}^{+}\left(v_{1}\right), d$ is also a feasible color for $u v$ and therefore we can set $f(u v)=d$.

Suppose now that $\overrightarrow{v_{1} v u}$ is directed path; we have $d^{-}(u)=2$ by Lemma 6. We can recolor $f\left(\overrightarrow{v_{1}} \vec{v}\right)=$ $d$ for some $d \notin T=F^{f}\left(\overrightarrow{v_{1}} \vec{v}\right) \cup\{c\}$ since $|T| \leq \Delta-1+1=\Delta$; this implies $C_{f}^{-}(u) \cap C_{f}^{+}(v)=\emptyset$ and therefore any available color for $\overrightarrow{v u}$ is feasible for $\overrightarrow{v u}$ (no bichromatic circuit in $H$ contains $\overrightarrow{v u}$ ): so, we set $f(\overrightarrow{v u})=e$ for some $e \notin F^{f}(\overrightarrow{v u})$ since $\left|F^{f}(\overrightarrow{v u})\right| \leq 3<\Delta$.

Lemma 10 The graph H does not contain the configuration (h) of Figure 3.
Proof: Let $H^{\prime}=H \backslash \overrightarrow{u v}$. By Lemma 4, we have $\overrightarrow{u_{2} u} \in A(H)$, and by Lemma 7, $d^{-}(v) \geq 2$ (w.l.o.g. assume $\overrightarrow{v_{1} v} \in A(H)$.

Suppose that either $\overrightarrow{v_{2} v} \in A(H)$ or $f\left(\overrightarrow{u_{2} u}\right) \neq f\left(\overrightarrow{v v_{2}}\right)$; therefore, $C_{f}^{-}(u) \cap C_{f}^{+}(v)=\emptyset$ and thus any available color for $\overrightarrow{u v}$ is also feasible; we can thus set $f(\overrightarrow{u v})=c$ for some $c \notin F^{f}(\overrightarrow{u v})$ since $\left|F^{f}(u v)\right| \leq 3<\Delta$.

Suppose now that $f\left(\overrightarrow{u_{2} u}\right)=f\left(\overrightarrow{v v_{2}}\right)$. Consider the set of colors $S=F^{f}(\overrightarrow{u v}) \cup\left(C_{f}^{-}\left(u_{2}\right) \cap C_{f}^{+}\left(v_{2}\right)\right)$. Clearly, any color $c \notin S$ is feasible for $\overrightarrow{u v}$. If $|S| \leq \Delta$, we can set $f(\overrightarrow{u v})=c$ for some $c \notin S$. If $|S|=\Delta+1$, it implies $C_{f}^{-}\left(u_{2}\right)=C_{f}^{+}\left(v_{2}\right), d^{-}\left(u_{2}\right)=d^{+}\left(v_{2}\right)=\Delta-1$ and $f\left(\overrightarrow{\left.v_{1} \vec{v}\right)} \notin C_{f}^{+}\left(v_{2}\right)\right.$. In this case, we erase $f\left(\overrightarrow{v v_{2}}\right)$ and pick one outgoing arc from $v_{2}$, denoted by $\overrightarrow{v_{2} v_{2}^{\prime}}$. Denote $f\left(\overrightarrow{v_{2} v_{2}}\right)=d$; since $\left|F^{f}\left(v_{2} v_{2}^{\prime}\right)\right|=\left|C_{f}^{+}\left(v_{2}\right)\right| \leq \Delta-1$, we can recolor $f\left(\overrightarrow{v_{2} v_{2}^{\prime}}\right)=e$ for some $e \notin F^{f}\left(v_{2} v_{2}^{\prime}\right) \cup\{d\}$. Thus, the color $d$ is feasible for $\overrightarrow{v v_{2}}$; indeed, $\left.f\left(\overrightarrow{v_{1}} \vec{v}\right) \notin\left\{f\left(v_{2} w\right) \mid w \in N^{+}\left(v_{2}\right) \backslash v_{2}^{\prime}\right)\right\}$ and $d \notin C_{f}^{+}\left(v_{2}^{\prime}\right)$. Thus, we set $f\left(\overrightarrow{v v_{2}}\right)=d$, which leads us to the previous case $\left(f\left(\overrightarrow{u_{2} u}\right) \neq f\left(\overrightarrow{v v_{2}}\right)\right)$.

Lemma 11 The graph $H$ does not contain the configuration (i) of Figure 3.
Proof: Let $H^{\prime}=H \backslash u$. By Lemma 10, $u$ has two incoming arcs and one outgoing arc (w.l.o.g. assume $\overrightarrow{v u}, \overrightarrow{w u}, \overrightarrow{u x} \in A(H)$ ). We first set $f(\overrightarrow{v u})=c$ for some $c \notin S=F^{f}(\overrightarrow{v u}) \cup\left\{f\left(x x_{1}\right), f\left(x x_{2}\right)\right\}$ since $|S| \leq 2+2 \leq \Delta$. Let $T=F^{f}(\overrightarrow{w u}) \cup\left\{f\left(x x_{1}\right), f\left(x x_{2}\right)\right\}$. We consider two cases:

1. If $|T| \leq \Delta$, we can set $f(\overrightarrow{w u})=d$ for some $d \notin T$. Therefore $C_{f}^{-}(u) \cap C_{f}^{+}(x)=\emptyset$, and thus any available color for $\overrightarrow{u x}$ is feasible and we can set $f(\overrightarrow{u x})=e$ for some $e \notin F^{f}(\overrightarrow{u x})$ since $\left|F^{f}(\overrightarrow{u x})\right| \leq 4 \leq \Delta$.
2. If $|T|=\Delta+1$, this implies $\Delta=4$. Assume w.l.o.g. that $C_{f}^{-}(v)=C_{f}^{-}(w)=\{1,2\}$ and $C_{f}^{+}(x)=\{3,4\}$. We can recolor in this case $f(\overrightarrow{v u})=3$, and set $f(\overrightarrow{w u})=4$; finally, the color 5 is clearly feasible for $\overrightarrow{u x}$ and we set $f(\overrightarrow{u x})=5$.

Lemma 12 The graph $H$ does not contain the configuration ( $j$ ) of Figure 3.

Proof: By Lemmas 4 and 10 , we have $d^{+}(u) \leq 1$. We consider three cases:

1. Suppose first that $u$ is a sink and let $H^{\prime}=H \backslash w u$. Since $C_{f}^{-}(w) \cap C_{f}^{+}(u)=\emptyset$, any available color for $\overrightarrow{w u}$ is feasible and thus we can set $f(\overrightarrow{w u})=c$ for some $c \notin F^{f}(w u)$ since $\left|F^{f}(w u)\right| \leq 4 \leq \Delta$.
2. Suppose now that one of the arcs $\overrightarrow{u v}, \overrightarrow{u w}$ belongs to $A(H)$ (w.l.o.g. assume $\overrightarrow{u v} \in A(H)$ ) and let $H^{\prime}=H \backslash w u$. By Lemma 10, we have $\overrightarrow{w_{1} w}, \overrightarrow{w_{2} w} \in A(H)$. Let $H^{\prime}=H \backslash \overrightarrow{w u}$.
If $C_{f}^{-}(w) \cap C_{f}^{+}(u)=\emptyset$, any available color for $\overrightarrow{w u}$ is also feasible, and therefore, we can set $f(\overrightarrow{w u})=c$ for some $c \notin F^{f}(f(\overrightarrow{w u}))$ since $\left|F^{f}(\overrightarrow{w u})\right| \leq 4 \leq \Delta$.
Otherwise, $f(\overrightarrow{u v}) \in C_{f}^{-}(w)$ (w.l.o.g. assume $f(\overrightarrow{u v})=f\left(\overrightarrow{\left.w_{1} \vec{w}\right)}\right.$ ). Let $S=F^{f}(\overrightarrow{w u}) \cup C_{f}^{+}(v)$. Any color $c \notin S$ is feasible for $\overrightarrow{w u}$ since $c \notin C_{f}^{+}(v)$ and thus, if $|S| \leq \Delta$, we can set $f(\overrightarrow{w u})=c$. Otherwise, we have $|S|=\Delta+1$, which implies $\Delta=4$ and $d^{+}(v)=2$; therefore, we may thus assume w.l.o.g. that $f\left(\overrightarrow{w_{1}} \vec{w}\right)=f(\overrightarrow{u v})=1, f\left(\overrightarrow{w_{2}} \vec{w}\right)=2, f\left(\overrightarrow{u_{1}} \vec{u}\right)=3, f\left(\overrightarrow{v v_{1}}\right)=4$ and $f(\overrightarrow{v x})=5$. Then, we erase the colors $f(\overrightarrow{u v})$ and $f\left(\overrightarrow{v v_{1}}\right)$ and set $f(\overrightarrow{u v})=4$; now, any color $c \notin F^{f}\left(\overrightarrow{v v_{1}}\right)$ is feasible since $C_{f}^{-}(v) \cap C_{f}^{+}\left(v_{1}\right)=\emptyset$, and we can set $f\left(\overrightarrow{v v_{1}}\right)=c$ since $\left|F^{f}\left(\overrightarrow{v v_{1}}\right)\right| \leq \Delta$. This leads us to the previous case $\left(f(\overrightarrow{u v}) \notin\left\{f\left(w_{1} w\right), f\left(w_{2} w\right)\right\}\right)$.
3. Suppose finally that $\overrightarrow{u u_{1}} \in A(H)$; by Lemmas 4 and 10 , we have $\overrightarrow{v_{1} v}, \overrightarrow{x v}, \overrightarrow{w_{1} w}, \overrightarrow{w_{2} w} \in A(H)$. Since $k \leq \Delta-2$, we have $d(x) \leq \Delta-1$. Let $H^{\prime}=H \backslash v$. We first set $f\left(\overrightarrow{v_{1}} \vec{v}\right)=c$ for some $c \notin S=F^{f}\left(\overrightarrow{v_{1}} \vec{v}\right) \cup f\left(\overrightarrow{u u_{1}}\right)$ since $|S| \leq \Delta-1+1=\Delta$ and $f(\overrightarrow{x v})=d$ for some $d \notin T=F^{f}(x v) \cup f\left(\overrightarrow{u u_{1}}\right)=C_{f}^{-}(x) \cup\left\{f\left(\overrightarrow{v_{1} v}\right), f\left(\overrightarrow{u u_{1}}\right)\right\}$ since $|T| \leq \Delta-2+2=\Delta$. Therefore, any available color for $\overrightarrow{v u}$ is also feasible since $C_{f}^{-}(v) \cap C_{f}^{+}(u)=\emptyset$; we can thus set $f(\overrightarrow{v u})=e$ for some $e \notin F^{f}(\overrightarrow{v u})$.

Lemma 13 The graph $H$ does not contain the configuration ( $k$ ) of Figure 3.
Proof: Let $H^{\prime}=H \backslash v$. By Lemmas 4 and 5, we have $\overrightarrow{v_{1} v} \in A(H)$ and by Lemma 6 we have $d^{+}(u)>0$ (w.l.o.g. assume $\overrightarrow{u u_{1}} \in A(H)$ ). We shall consider three cases depending on the cardinality of the color $\operatorname{set} C_{f}^{+}(u)$.

1. $\left|C_{f}^{+}(u)\right|=1$.

We set $f\left(\overrightarrow{v_{1} v}\right)=c$ for some $c \notin S=F^{f}\left(\overrightarrow{v_{1}}\right) \cup C_{f}^{+}(u)$ since $|S| \leq \Delta-1+1=\Delta$. Then, we have $C_{f}^{-}(v) \cap C_{f}^{+}(u)=\emptyset$ and thus any available color for $\overrightarrow{v u}$ is also feasible. We then set $f(\overrightarrow{v u})=d$ for some $d \notin F^{f}(\overrightarrow{v u})$ since $\left|F^{f}(\overrightarrow{v u})\right|=2<\Delta$.
2. $\left|C_{f}^{+}(u)\right|=2$.
W.1.o.g., assume $f\left(\overrightarrow{u u_{1}}\right)=1$ and $f\left(\overrightarrow{u u_{2}}\right)=2$. Let $S=F^{f}\left(\overrightarrow{v_{1}}\right) \cup C_{f}^{+}(u)=C_{f}^{-}\left(v_{1}\right) \cup C_{f}(v) \cup$ $C_{f}^{+}(u)=C_{f}^{-}\left(v_{1}\right) \cup C_{f}^{+}(u)$. Clearly, if $|S| \leq \Delta$, we can proceed as in the previous case.
If $|S|=\Delta+1, C_{f}^{-}\left(v_{1}\right)=\{3,4, \ldots, \Delta+1\}$. Suppose first that $\overrightarrow{u_{3}} \in A(H)$ and w.l.o.g. assume $f\left(\overrightarrow{u u_{3}}\right)=2$. We can recolor $f\left(\overrightarrow{u u_{1}}\right)=c$ for some $c \notin F^{f}\left(\overrightarrow{u u_{1}}\right) \cup\{1\}$ since $\left|F^{f}\left(\overrightarrow{u u_{1}}\right)\right| \leq \Delta-1$. This implies $|S| \leq \Delta$ and therefore, we can also proceed as in the previous case.
Suppose now that $\overrightarrow{u_{3} \vec{u}} \in A(H)$ and assume w.l.o.g. $f\left(\overrightarrow{u_{3} \vec{u}}\right)=3$. Let $T=\{4,5, \ldots, \Delta+1\}$. If there exists some $i \in\{1,2\}$ such that $T \nsubseteq C_{f}^{+}\left(u_{i}\right)$, we have $\left|T \backslash C_{f}^{+}\left(u_{i}\right)\right| \neq \emptyset$ and therefore, we can pick a color $c \in T \backslash C_{f}^{+}\left(u_{i}\right)$ and set $f\left(\overrightarrow{v_{1} v}\right)=i$ and $f(\overrightarrow{v u})=c$ : the color $c$ is feasible for $\overrightarrow{v u}$ since $f\left(\overrightarrow{u u_{3-i}}\right) \neq f\left(\overrightarrow{v_{1} v}\right)$ and $\left.c \notin C_{f}^{+}\left(u_{i}\right)\right)$.

Finally, we can suppose that for all $i \in\{1,2\}, T \subseteq C_{f}^{+}\left(u_{i}\right)$. We first erase $f\left(\overrightarrow{v v_{1}}\right)$ and consider four subcases:

- $C_{f}^{+}\left(u_{1}\right)=T$ and $2 \notin C_{f}^{-}\left(u_{1}\right)$.

We can recolor $f\left(\overrightarrow{u u_{1}}\right)=2$ (the color 2 is feasible for $\overrightarrow{u u_{1}}$ since $3 \notin C_{f}^{+}\left(u_{1}\right)$ ) and set $f\left(\overrightarrow{v_{1} \vec{v}}\right)=1$. We thus have $C_{f}^{-}(v) \cap C_{f}^{+}(u)=\emptyset$ which implies that any available color for $\overrightarrow{v u}$ is also feasible; we can set $f(\overrightarrow{v u})=c$ for some $c \notin F^{f}(\overrightarrow{v u})$ since $\left|F^{f}(\overrightarrow{v u})\right|=3<\Delta$.

- $C_{f}^{+}\left(u_{1}\right)=T \cup\{2\}$.

Let $\overrightarrow{u_{1} w}$ denote the outgoing arc of $u_{1}$ with $f\left(\overrightarrow{u_{1}} \vec{w}\right)=2$. We first erase $f\left(\overrightarrow{u u_{1}}\right)$ and we recolor $f\left(\overrightarrow{u_{1} w}\right)=c$ for some $c \notin U=F^{f}\left(\overrightarrow{u_{1} \vec{w}}\right) \cup\{2\}$ (we have $|U| \leq \Delta-1+1=\Delta$ ); then, 2 is a feasible color since $\left.f\left(\overrightarrow{u_{3}} \vec{u}\right) \notin\left\{f\left(u_{1} x\right) \mid x \in N^{+}\left(u_{1}\right) \backslash w\right)\right\}$ and $2 \notin C_{f}^{+}(w)$ and we set $f\left(\overrightarrow{u u_{1}}\right)=2$. We can then set $f\left(\overrightarrow{v_{1}} \vec{v}\right)=1$, which implies $C_{f}^{-}(v) \cap C_{f}^{+}(u)=\emptyset$. Therefore any available color for $\overrightarrow{v u}$ is also feasible and we can set $f(\overrightarrow{v u})=c$ for some $c \notin F^{f}(\overrightarrow{v u})$ since $\left|F^{f}(\overrightarrow{v u})\right|=3<\Delta$.

- $C_{f}^{+}\left(u_{1}\right)=T \cup\{3\}$.

Let $\overrightarrow{u_{1} w}$ denote the outgoing arc of $u_{1}$ with $f\left(\overrightarrow{u_{1} w}\right)=3$. We can set $f\left(\overrightarrow{u_{1} w}\right)=c$ for some $c \notin U=F^{f}\left(\overrightarrow{u_{1} w}\right) \cup\{3\}$ (we have $|U| \leq \Delta-1+1$ ). At this point, we have $C_{f}^{+}\left(u_{1}\right)=T$ and $2 \notin C_{f}^{-}\left(u_{1}\right)$, or $C_{f}^{+}\left(u_{1}\right)=T \cup\{2\}$ : these two subcases were already considered before.

- $C_{f}^{+}\left(u_{1}\right)=T$ and $C_{f}^{-}\left(u_{1}\right)=\{2\}$.

By considering the three previous subcases on the vertex $u_{2}$, we deduce that we have $C_{f}^{+}\left(v_{2}\right)=$ $T$ and $C_{f}^{-}\left(v_{2}\right)=\{1\}$. In this case, we erase $f\left(\overrightarrow{u u_{1}}\right)$ and $f\left(\overrightarrow{u_{1}}\right)$ and recolor $f\left(\overrightarrow{u_{3} \vec{u}}\right)=c$ for some $c \notin V=F^{f}\left(\overrightarrow{u_{3}} \vec{u}\right) \cup\{3\}$ (we have $|V| \leq \Delta-1+1$ ). We thus have $C_{f}^{-}(u) \cap C_{f}^{+}\left(u_{i}\right)=\emptyset$ for $i=1,2$; therefore the color 3 is feasible for $\overrightarrow{u_{1}}$ and $\overrightarrow{u_{2}}$ since $3 \notin F^{f}\left(\overrightarrow{u u_{i}}\right)$ and we can set $f\left(\overrightarrow{u u_{1}}\right)=3, f\left(\overrightarrow{u u_{2}}\right)=3$. Finally, we set $f\left(\overrightarrow{v_{1} v}\right)=1$, which implies $C_{f}^{-}(v) \cap C_{f}^{+}(u)=\emptyset$ : we can set $f(\overrightarrow{v u})=d$ for some $d \notin F^{f}(\overrightarrow{v u})$ since $\left|F^{f}(\overrightarrow{v u})\right| \leq 3<\Delta$.
3. $\left|C_{f}^{+}(u)\right|=3$.
W.l.o.g. assume $f\left(\overrightarrow{u u_{1}}\right)=1, f\left(\overrightarrow{u u_{2}}\right)=2, f\left(\overrightarrow{u_{3}}\right)=3$. Let $S=F^{f}\left(\overrightarrow{v_{1}}\right) \cup C_{f}^{+}(u)$. If $|S| \leq \Delta$, we can set $f\left(\overrightarrow{v_{1}} \vec{v}\right)=c$ for some $c \notin S$ and $f(\overrightarrow{v u})=d$ for some $d \notin F^{f}(\overrightarrow{v u})$ since any available color for $\overrightarrow{v u}$ is also feasible (we have $C_{f}^{-}(v) \cap C_{f}^{+}(u)=\emptyset$ ) and $F^{f}(\overrightarrow{v u})=4 \leq \Delta$. If $|S|=$ $\Delta+1$, this implies $\{4,5, \ldots, \Delta+1\} \subseteq C_{f}^{-}\left(v_{1}\right)$ and therefore at least two colors of $C_{f}^{+}(u)$ do not belong to $C_{f}^{-}\left(v_{1}\right)$. Suppose w.l.o.g. that $1 \notin C_{f}^{-}\left(v_{1}\right)$. We then set $f\left(\overrightarrow{u u_{1}}\right)=c$ for some $c \notin T=F^{f}\left(\overrightarrow{u u_{1}}\right) \cup\{1\}$ since $|T| \leq \Delta-1+1=\Delta$. We can then set $f\left(\overrightarrow{v_{1}} \vec{v}\right)=1$, which implies $C_{f}^{-}(v) \cap C_{f}^{+}(u)=\emptyset$. We finally set $f(\overrightarrow{v u})=d$ for some $d \notin F^{f}(\overrightarrow{v u})$ since $\left|F^{f}(\overrightarrow{v u})\right| \leq 4 \leq \Delta$.

Lemma 14 The graph $H$ does not contain the configuration (l) of Figure 3.
Proof: Let $H^{\prime}=H \backslash v$. By Lemma 13, we have $\overrightarrow{u v} \in A(H)$ and $d\left(v_{1}\right) \geq 5$ (therefore, $\Delta \geq 5$ ), and by Lemma 7 we have $d^{-}\left(v_{1}\right) \geq 2$. Let $S=F^{f}(\overrightarrow{u v}) \cup C_{f}^{+}\left(v_{1}\right)$. If $|S| \leq \Delta$, we can set $f(\overrightarrow{u v})=c$ for
some $c \notin S$, which implies $C_{f}^{-}(v) \cap C_{f}^{+}\left(v_{1}\right)=\emptyset$ : any available color for $\overrightarrow{v v_{1}}$ is also feasible and thus we can set and $f\left(\overrightarrow{v v_{1}}\right)=d$ for some $d \notin F^{f}\left(\overrightarrow{v v_{1}}\right)$ since $\left|F^{f}\left(\overrightarrow{v v_{1}}\right)\right| \leq \Delta$. If $|S|=\Delta+1$, it means that $C_{f}^{-}(u)=3$ (w.l.o.g. assume $f\left(\overrightarrow{u_{1} u}\right)=1, f\left(\overrightarrow{u_{2} u}\right)=2, f(\overrightarrow{w u})=3$ ) and $\left|C_{f}^{+}\left(v_{1}\right)\right|=\Delta-2$ (w.l.o.g. assume $C_{f}^{+}\left(v_{1}\right)=\{4,5, \ldots, \Delta+1\}$ ). In this case, we erase $f(\overrightarrow{w u})$, which implies $|S| \leq 2+\Delta-2=\Delta$ and we set $f(\overrightarrow{u v})=3$. Then, we set $f(\overrightarrow{w u})=d$ for some $d \notin F^{f}(\overrightarrow{w u})$ since $\left|F^{f}(\overrightarrow{w u})\right| \leq 5 \leq \Delta$. Finally, since $3 \notin C_{f}^{+}\left(v_{1}\right)$, any available color for $\overrightarrow{v v_{1}}$ is also feasible: we can set $f\left(\overrightarrow{v v_{1}}\right)=\bar{d}$ for some $d \notin F^{f}\left(\overrightarrow{v v_{1}}\right)$ since $\left|F^{f}\left(\overrightarrow{v v_{1}}\right)\right| \leq \Delta$.

Note that on the four last configurations (m),(n),(o) and (p) of Figure 3, by Lemmas 4 and 5, we have a directed 2-path linking $u$ and $v_{i}^{\prime}$, for all $i \in[1, k]$.
Lemma 15 The graph H does not contain the configuration (m) of Figure 3.
Proof: Let $H^{\prime}=H \backslash v_{1}$. We consider two cases:

1. Suppose first that there exists $i \in[1, k]$ such that $\overrightarrow{u v_{i} v_{i}^{\prime}}$ is a directed path in $H$ (w.l.o.g. assume $\left.\overrightarrow{u v_{1} v_{1}^{\prime}}\right)$. We can set $f\left(\overrightarrow{v_{1} v_{1}^{\prime}}\right)=c$ for some $c \notin F^{f}\left(\overrightarrow{v_{1} v_{1}^{\prime}}\right)$ since $\left|F^{f}\left(\overrightarrow{v_{1} v_{1}^{\prime}}\right)\right| \leq \Delta-1$. Clearly, any feasible color for $\overrightarrow{u v_{1}}$ does not belong to $f\left(\overrightarrow{v_{1} v_{1}^{\prime}}\right) \cup C_{f}^{-}(u) \cup\left\{f\left(\overrightarrow{v_{j}^{\prime} v_{j}}\right) \mid f\left(\overrightarrow{v_{j} u}\right)=c\right\}$; therefore at most $\Delta$ colors are not feasible and it remains at least one feasible color to set $f\left(\overrightarrow{u v_{1}}\right)$.
2. Suppose now that $\overrightarrow{v_{i}^{\prime} v_{i} u}$ is a directed path in $H$ for all $i \in[1, k]$. By Lemma 6, we have $\overrightarrow{u u_{1}} \in$ $A(H)$. We can set $f\left(\overrightarrow{v_{1}^{\prime} v_{1}}\right)=c$ for some $c \notin S=F^{f}\left(\overrightarrow{v_{1}^{\prime} v_{1}}\right) \cup\left\{f\left(\overrightarrow{u u_{1}}\right)\right\}$ since $|S| \leq \Delta-1+1=\Delta$. Thus, we have $C_{f}^{-}\left(v_{1}\right) \cap C_{f}^{+}(u)=\emptyset$ which implies that any available color for $\overrightarrow{v_{1} u}$ is also feasible; we then set $f\left(\overrightarrow{v_{1} u}\right)=d$ for some $d \notin F^{f}\left(\overrightarrow{v_{1} u}\right)$ since $\left|F^{f}\left(\overrightarrow{v_{1}} \vec{v}\right)\right| \leq \Delta$.

Lemma 16 The graph H does not contain the configuration (n) of Figure 3.
Proof: Suppose first that $\overrightarrow{u v_{i}} \in A(H)$ for all $i \in[1, k]$. By Lemma 7, we have $d^{-}\left(v_{i}^{\prime}\right) \geq 2$ for all $i \in[1, k]$. Let $H^{\prime}=H \backslash v_{1}$. We can set $f\left(\overrightarrow{u v_{1}}\right)=c$ for some $c \notin S=F^{f}\left(\overrightarrow{u v_{1}}\right) \cup C_{f}^{+}\left(v_{1}^{\prime}\right)$ since $|S| \leq 2+\Delta-2=\Delta$. Then, we have $C_{f}^{-}\left(v_{1}\right) \cap C_{f}^{+}\left(v_{1}^{\prime}\right)=\emptyset$ and therefore any available color for $\overrightarrow{v_{1} v_{1}^{\prime}}$ is also feasible: we can set $f\left(\overrightarrow{v_{1} v_{1}^{\prime}}\right)=d$ for some $d \notin F^{f}\left(\overrightarrow{v_{1} v_{1}^{\prime}}\right)$ since $\left|F^{f}\left(\overrightarrow{v_{1} v_{1}^{\prime}}\right)\right| \leq \Delta$.

Suppose now that there exists $i \in[1, k]$ such that $\overrightarrow{v_{i} u} \in A(H)$ (w.l.o.g. assume $\overrightarrow{v_{2} u} \in A(H)$ ). We shall consider two cases depending on the orientations of $u u_{1}$ and $u u_{2}$ :

- $\overrightarrow{u u_{1}}, \overrightarrow{u u_{2}} \in A(H)$.

Let $H^{\prime}=H \backslash v_{1}$. We can set $f\left(\overrightarrow{v_{1} v_{1}^{\prime}}\right)=c$ for some $c \notin F^{f}\left(\overrightarrow{v_{1} v_{1}^{\prime}}\right)$ since $\left|F^{f}\left(\overrightarrow{v_{1} v_{1}^{\prime}}\right)\right| \leq \Delta-1$. Clearly, any feasible color for $\overrightarrow{u v_{1}}$ does not belong to $f\left(\overrightarrow{v_{1} v_{1}^{\prime}}\right) \cup C_{f}^{-}(u) \cup\left\{f\left(\overrightarrow{v_{j}^{\prime} v_{j}}\right) \mid f\left(\overrightarrow{v_{j} \vec{u}}\right)=c\right\}$; therefore at most $\Delta$ colors are not feasible for $\overrightarrow{u v_{1}}$ and it remains at least one feasible color to set $f\left(\overrightarrow{u v_{1}}\right)$.

- $\overrightarrow{u_{1} u} \in A(H)$.

Let $H^{\prime}=H \backslash v_{2}$. We can set $f\left(\overrightarrow{v_{2}^{\prime} v_{2}}\right)=c$ for some $c \notin S=F^{f}\left(\overrightarrow{v_{2}^{\prime} v_{2}}\right) \cup\left\{f\left(u u_{2}\right)\right\}$ since $|S| \leq \Delta-1+1=\Delta$. Clearly, any feasible color for $\overrightarrow{v_{2} u}$ does not belong to $f\left(\overrightarrow{v_{2}^{\prime} v_{2}}\right) \cup C_{f}^{-}(u) \cup$ $\left\{f\left(\overrightarrow{v_{j} v_{j}^{\prime}}\right) \mid f\left(\overrightarrow{u v_{j}}\right)=c\right\}$; therefore at most $\Delta$ colors are not feasible for $\overrightarrow{v_{2} \vec{u}}$ and it remains at least 1 feasible color to set $f\left(\overrightarrow{v_{2} u}\right)$.

Lemma 17 The graph H does not contain the configuration (o) of Figure 3.
Proof: Let $H^{\prime}=H \backslash v_{1}$. By Lemma 16, we have $\overrightarrow{v_{i} \vec{u}} \in A(H)$ for all $i \in[1, k]$. We can set $f\left(\overrightarrow{v_{1} v_{1}^{\prime}}\right)=c$ for some $c \notin S=F^{f}\left(\overrightarrow{v_{1} v_{1}^{\prime}}\right) \cup\left\{f\left(u u_{2}\right)\right\}$ since $|S| \leq \Delta-1+1=\Delta$. Therefore, $C_{f}^{-}\left(v_{1}\right) \cap C_{f}^{+}(u)=\emptyset$ and any available color for $\overrightarrow{v_{1} u}$ is also feasible; we thus set $f\left(\overrightarrow{v_{1} u}\right)=d$ for some $d \notin F^{f}\left(\overrightarrow{v_{1} u}\right)$.

Lemma 18 The graph $H$ does not contain the configuration (p) of Figure 3.
Proof: Let $H^{\prime}=H \backslash u w$. By Lemmas 16 and 17, we have $\overrightarrow{v_{i} u} \in A(H)$ for all $i \in[1, k-2]$, and $\overrightarrow{u u_{1}}, \overrightarrow{u w} \in A(H)$. Clearly, any feasible color for $\overrightarrow{u u_{2}}$ does not belong to $F^{f}\left(\overrightarrow{u u_{2}}\right) \cup\left\{f\left(\overrightarrow{v_{j}^{\prime} v_{j}}\right) \mid f\left(\overrightarrow{v_{j}} \vec{u}\right) \in\right.$ $\left.\left\{f\left(w w_{1}\right), f\left(w w_{2}\right)\right\}\right\}$; therefore at most $\Delta$ colors are not feasible and thus it remains at least 1 feasible color to set $f(\overrightarrow{u w})$.
We now prove that every graph that does not contain the configurations of Figure 3 has mad $\geq \frac{133}{41}$.
We call a strong $k$-vertex a $k$-vertex which is adjacent to at most $k-32$-vertices, and a weak $k$-vertex a $k[k-2]$-vertex. We also call strong 3-vertex a 3-vertex which is adjacent to at most one 3 -vertex, and weak 3-vertex a 3-vertex adjacent to two 3-vertices.

Note that by Lemma 16, if $H$ contains a weak $k$-vertex $u$, the $k-22$-vertices $v_{i}$ adjacent to $u$, $1 \leq i \leq k-2$, are directed towards $u$ (i.e. $\overrightarrow{v_{i} u} \in A(H) \forall i \in[1, k-2]$ ). Therefore, since $H$ contains no source vertex (by Lemmas 4 and 5), a 2 -vertex cannot be adjacent to two weak $k$-vertices.

We assign to each vertex $v$ an initial charge $w(v)=d(v)$. Now, we use the following discharging rules:
Rule 1. Each strong 3-vertex gives $\frac{2}{41}$ to each of its 3-neighbors;
Rule 2. Each 4-vertex gives $\frac{27}{41}$ to each of its 2-neighbors and $\frac{6}{41}$ to each of its 3-neighbors;
Rule 3. Each $\geq_{5}$-vertex gives $\frac{6}{41}$ to each of its 3-neighbors;
Rule 4. Each weak $\geq 5$-vertex gives $\frac{24}{41}$ to each of its 2-neighbors;
Rule 5. Each strong $\geq 5$-vertex gives $\frac{27}{41}$ to each of its 2-neighbors.
We denote by $w^{*}(v)$ the new charge of the vertex $v$ after the discharging procedure.
Let $H$ be a minimal counter-example to Theorem 2 and $v$ be a $k$-vertex; we shall prove that the new charge $w^{*}(v)$ of each vertex $v$ is at least $\frac{133}{41}$. We consider the following cases (recall that $H$ has no 1 -vertices):

- if $k=2$, then $w(v)=2$ and $w^{*}(v) \geq 2+\frac{24}{41}+\frac{27}{41}=\frac{133}{41}$, since by Lemma 16 a 2 -vertex is not adjacent to two weak $\geq 5$-vertices;
- if $k=3$ and $v$ is a strong 3-vertex, then $w(v)=3$ and $w^{*}(v) \geq 3+2 \cdot \frac{6}{41}-\frac{2}{41}=\frac{133}{41}$;
- if $k=3$ and $v$ is a weak 3-vertex, then $w(v)=3$ and $w^{*}(v)=3+2 \cdot \frac{2}{41}+\frac{6}{41}=\frac{133}{41}$, since by Lemma 12 a weak 3-vertex is adjacent to two strong 3 -vertices;
- if $k=4$ and $v$ is adjacent to a 2-vertex, then $w(v)=4$ and $w^{*}(v) \geq 4-\frac{27}{41}=\frac{137}{41}$, since by Lemma 13, a 4 -vertex is linked to a 2 -vertex by an outgoing arc and therefore, by Lemmas 16 and 17, a 4-vertex can have only one 2-neighbor; moreover, by Lemma 14, a 4-vertex is not adjacent to a 3-vertex if it is already adjacent to 2 -vertices;
- if $k=4$ and $v$ is not adjacent to a 2-vertex, then $w(v)=4$ and $w^{*}(v) \geq 4-4 \cdot \frac{6}{41}=\frac{140}{41}$;
- if $k \geq 5$ and $v$ is a strong k-vertex, then $w(v)=k$ and $w^{*}(v) \geq k-3 \cdot \frac{6}{41}-(k-3) \cdot \frac{27}{41} \geq \frac{133}{41}$;
- if $k \geq 5$ and $v$ is a weak k-vertex, then $w(v)=k$ and $w^{*}(v) \geq k-(k-2) \cdot \frac{24}{41} \geq \frac{133}{41}$, since by Lemma 18 the two other neighbors of a weak $k$-vertex are $\geq 4$-vertices.

As in the proof of Theorem 1, we get that $\operatorname{mad}(H) \geq \frac{133}{41}$. The contradiction concludes the proof of Theorem 2.

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