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GOOD AND SEMI-STRONG COLORINGS OF ORIENTED PLANAR GRAPHS

André RASPAUD and Éric SOPENA
LaBRI, Université Bordeaux I, 351 cours de la Libération, 33405 Talence Cedex, France.

Abstract. A $k-$*coloring* of an oriented graph $G = (V, A)$ is an assignment c of one of the colors $1, 2, \ldots, k$ to each vertex of the graph such that, for every arc (x, y) of G, $c(x) \neq c(y)$. The $k-$*coloring* is *good* if for every arc (x, y) of G there is no arc $(z, t) \in A$ such that $c(x) = c(t)$ and $c(y) = c(z)$. A $k-$*coloring* is said to be *semi*$-$*strong* if for every vertex x of G, $c(z) \neq c(t)$ for any pair $\{z, t\}$ of vertices of $N^-(x)$.

We show that every oriented planar graph has a good coloring using at most 5×2^4 colors and that every oriented planar graph $G = (V, A)$ with $d^-(x) \leq 3$ for every $x \in V$ has a good and semi-strong coloring using at most $4 \times 5 \times 2^4$ colors.

Keywords. Combinatorial problems, Graph coloring.

1 Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. We assume that G is finite and simple (without loops and multiple edges). A $k-$*coloring* of the vertices is an assignment of one of the colors $1, 2, \ldots, k$ to each vertex so that no two adjacent vertices receive the same color. A $k-$*coloring* is said to be *acyclic* if the subgraph induced by the vertices with any two colors has no cycle. If $G = (V, A)$ is an oriented graph, a $k-$*coloring* c of G is a $k-$*coloring* of the underlying unoriented graph and a $k-$*coloring* is *good* if for every arc (x, y) of G there is no arc $(z, t) \in A$ such that $c(x) = c(t)$ and $c(y) = c(z)$. Let $x \in V$, we denote by $N^+(x)$ (resp. $N^-(x)$) the set of vertices y (resp. z) such that $(x, y) \in A$ (resp. $(z, x) \in A$) and $d^+(x) = |N^+(x)|$, $d^-(x) = |N^-(x)|$. A $k-$*coloring* c of an oriented graph G is said to be *semi*$-$*strong* if for every vertex x of G, $c(z) \neq c(t)$ for any pair $\{z, t\}$ of vertices of $N^-(x)$.
In this paper we give a first answer to a question coming from a work of B. Courcelle [3] concerning the monadic second order logic of graphs. In [3] B. Courcelle showed that every planar oriented graph with $d^-(x) \leq 3$ for every $x \in V$ has a good and semi-strong coloring which uses at most $4^3 \times 3^{63}$ colors. The question is: can we improve this upper bound? We have the following result:

Theorem 1 *Every oriented planar graph has a good coloring using at most* 5×2^4 *colors.*

Theorem 2 *Every planar graph can be oriented in such a way that the oriented graph thus obtained has a good and semi-strong coloring using at most* $4 \times 5 \times 2^4$ *colors.*

To prove those theorems we use the following result [2]:

Theorem 3 (Borodin - 79) *Every planar graph has an acyclic* $5-$*coloring.*

This property was conjectured in [4] and several authors have tried to solve it [1, 7, 9].

As a matter of fact, this coloring problem is a particular case of the general H-coloring problem considered by several authors (see e.g. [6, 8]) in the case of directed or undirected graphs : let

$G = (V, A)$ and $H = (W, B)$ be two directed (resp. undirected) graphs ; we say that G can be H-colored if there exists a mapping μ from V to W such that $(x, y) \in A \implies (\mu x, \mu y) \in B$ (resp. $\{x, y\} \in A \implies \{\mu x, \mu y\} \in B$). In the undirected case the usual notion of k-coloring hence corresponds to that of K_k-coloring. In this paper, we study the H-coloring problem in the directed case when the orientation of the considered graphs is antisymmetric (in that case, the coloring of any two neighbour vertices induces the direction of the corresponding edge). Hence, our problem can be stated as follows : can we find a minimal (with respect to the number of vertices) oriented graph M such that any oriented planar graph G is M-colorable ?

Other families of oriented graphs have been studied in [11] where some optimal solutions are proposed.

2 Proofs of the theorems

2.1 Theorem 1

We will prove in fact a stronger result:

Theorem 4 *If $G = (V, A)$ is an oriented graph whose underlying unoriented graph has an acyclic k-coloring then it has a good coloring using at most $k.2^{k-1}$ colors.*

It is clear that this result with theorem 3 implies the theorem 1. We first establish the easy following lemma:

Lemma 5 *Let F be a forest, c be a 2-coloring of F using colors i and j $(i < j)$, and $\widehat{F}$ be any oriented forest obtained by giving an orientation to F. Then there exists a good 4-coloring f of $\widehat{F}$ using colors $\{i, j\} \times \{0, 1\}$ and such that for any vertex x in $\widehat{F}$, $f^1(x) = c(x)$ (where $f^1(x)$ denotes the first component of $f(x)$).*

Note that by oriented forest we mean any oriented graph whose underlying undirected graph is a forest.

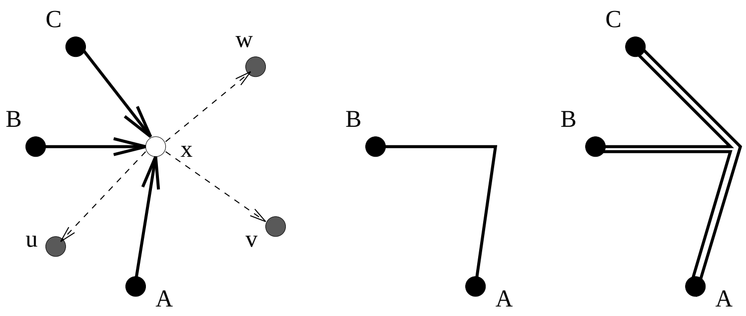
Proof. The coloring f can easily be obtained from c by using the following rule : if (x, y) is an arc in $\widehat{F}$ then let $f(x) = (c(x), \alpha)$, $f(y) = (c(y), \beta)$ (where $\alpha, \beta \in \{0, 1\}$) with $\alpha = \beta$ if and only if $c(x) < c(y)$. $\square$

Proof of the theorem. Let $G = (V, A)$ be an oriented graph and $V_1, \ldots, V_k$ be the k color classes of V, by hypothesis $F_{i,j} = G[V_i \cup V_j]$ for $i, j \in [1, k]$ and $i \neq j$ (the subgraph induced by $V_i \cup V_j$) is a forest. There are $\frac{k(k-1)}{2}$ such forests, since $F_{i,j}$ and $F_{j,i}$ denote the same forest. Hence by the previous lemma $F_{i,j}$ has a good 4-coloring, denoted by $f_{i,j} = f_{j,i}$, which satisfies : for any $x \in V_i$ (resp. V_j), $f^1_{i,j}(x) = i$ (resp. j). For each $x \in V$ there is $i \in [1, k]$ such that $x \in V_i$ and x belongs to $(k-1)$ forests $F_{i,j}$, $j \in [1, k]$, $i \neq j$. We define now the following coloring c of G :
Let $x \in V$ and i such that $x \in V_i$ then

$$c(x) = (i, f^2_{i,i_1}(x), \ldots, f^2_{i,i_{k-1}}(x)),$$

where $\{i_1, \ldots, i_{k-1}\}$ is the set $\{1, \ldots, k\} \setminus \{i\}$, with $i_1 < i_2 < \cdots < i_{k-1}$ and where $f^2_{i,j}(x)$ denotes the second component of $f_{i,j}(x)$. This is a coloring of G. We must prove that it is a good coloring. If it is not a good coloring of G there are four vertices x, y, z, t of G such that $(x, y) \in A$ and $(z, t) \in A$ with $c(x) = c(t)$ and $c(y) = c(z)$. By the definition of c it is clear that x, t (resp. y, z) belong to the same V_i (resp. V_j). Hence

$$c(x) = c(t) = (i, f^2_{i,i_1}(x), \ldots, f^2_{i,i_{k-1}}(x))$$



(a) (b) (c)

Figure 1

and

$$c(y) = c(z) = (j, f^2_{j,j_1}(y), \ldots, f^2_{j,j_{k-1}}(y)).$$

Moreover (x, y) and (z, t) belong to the same forest $F_{i,j}$ for these two integers in $[1, k]$. We deduce that $f^2_{i,j}(x) = f^2_{i,j}(t)$ and $f^2_{j,i}(y) = f^2_{j,i}(z)$, which is impossible because $f_{i,j} = f_{j,i}$ is a good coloring. It is easy to see that the coloring c uses at most $k.2^{k-1}$ colors. This completes the proof. □

2.2 Theorem 2

The arboricity $a(G)$ of a graph G is the minimal number of forests needed to cover G. By a theorem of Nash-Williams [10] it is easy to prove that for a planar graph G $a(G) \leq 3$ [3]. We can then deduce the following easy lemma [3]:

Lemma 6 *If $G = (V, E)$ is a planar graph then it can be oriented in such a way that $d^-(x) \leq 3$ for every $x \in V$.*

We prove now that if $G = (V, A)$ is an oriented planar graph such that $d^-(x) \leq 3$ for every $x \in V$ then it has a good and semi strong coloring using at most $4 \times 5 \times 2^4$ colors.
We use a technique developed in [3]. By theorem 1 G has a good 80-coloring c. Let V_i be a set of vertices of a color class for some $i \in [1, 80]$. Let G_i be the graph with vertex set V_i and edge set E_i defined as follows: there is an edge linking the vertices x and y of G_i if and only if $\{x, y\} \subset N^-(z)$ for some $z \in V$. It is not difficult to see that G_i is planar : consider a vertex x in G which has a predecessor A in V_i. Note that in this case we have $x \notin V_i$ and $N^+(x) \cap V_i = \emptyset$. Since $d^-(x) \leq 3$, x can induce edges in E_i between at most three vertices, say A, B and C, provided these vertices are i-colored by c (see Figure 1.a). We can then obtain a planar drawing of G_i by using an initial planar drawing of G as follows : (i) if only one predecessor of x belongs to V_i, x cannot induce any edge in E_i, (ii) if two predecessors of x, say A, B, belong to V_i the corresponding edge can be drawn as shown in Figure 1.b and (iii) if all the three predecessors of x belong to V_i then the three corresponding edges can be drawn as shown in Figure 1.c. Moreover, it is easy to see that no other edge in G_i will cross the edges thus constructed.

Hence G_i has a 4-coloring c_i. We define the following coloring C of G : for every $x \in V$, $C(x) = (c_i(x), c(x))$ where i is such that $x \in V_i$. This coloring is good because of c and semi-strong because of the c_i's. C uses at most $4 \times 5 \times 2^4$ colors. This completes the proof. □

3 Remarks

A planar graph G is outerplanar if it can be embedded in the plane so that all its vertices lie on the same face [5]. In [4] it is proved that every outerplanar graph has an acyclic 3-coloring. Then by using the theorem 4 we have the following result:

Corollary 7 *If G is an oriented outerplanar graph then it has a good coloring using at most* 3×2^2 *colors.*

In the same way, if a graph G is a graph with maximal degree 3 then G has an acyclic 4-coloring [4]. So we have:

Corollary 8 *If G is a graph with maximal degree 3 then every oriented graph obtained from G has a good coloring using at most* 4×2^3 *colors.*

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