THERE EXIST ORIENTED PLANAR GRAPHS WITH ORIENTED CHROMATIC NUMBER AT LEAST SIXTEEN

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Abstract. We prove that there exist oriented planar graphs with oriented chromatic number at least 16. Using a result of Raspaud and Sopena [*Good and semi-strong colorings of oriented planar graphs, Inf. Processing Letters 51, 171–174, 1994*], this gives that the oriented chromatic number of the family of oriented planar graphs lies between 16 and 80.

Keywords. combinatorial problems, oriented graph colouring, graph homomorphism.

1 Introduction

We consider oriented graphs, that is digraphs having no loops and no opposite arcs. If G is an oriented graph, we denote by $V(G)$ its set of vertices and by $E(G)$ its set of arcs. If xy is an arc in $E(G)$, we say that y is a *successor* of x and that x is a *predecessor* of y.

An *oriented k-colouring* of an oriented graph G is a mapping $c : V(G) \longrightarrow \{1, 2, \dots, k\}$ such that (i) if $xy \in E(G)$ then $c(x) \neq c(y)$ and (ii) if $xy, zt \in E(G)$ then $c(x) = c(t) \Longrightarrow c(y) \neq c(z)$. With every oriented k-colouring c of G one can associate a digraph H_c, called the *colour-graph* of c, with vertex set $V(H_c) = \{c(x),\ x \in V(G)\}$ and arc set $E(H_c) = \{c(x)c(y),\ xy \in E(G)\}$. Thanks to conditions (i) and (ii), H_c is an oriented graph. The oriented k-colouring c can then be viewed as a homomorphism (that is an arc-preserving vertex mapping) from G to H_c. Similarly, every homomorphism of G to an oriented graph H on k vertices can be viewed as an oriented k-colouring of G, using the vertices of H as colours. Oriented colourings have been introduced in [5] and studied by several authors (see [7] for a general overview). In particular, various problems related to planar graphs are discussed in [1], [2] and [4].

The *oriented chromatic number* $\vec{\chi}(G)$ of an oriented graph G is defined as the smallest k such that G admits an oriented k-colouring or, equivalently, as the smallest order of an oriented graph H such that G admits a homomorphism to H. The oriented chromatic number $\vec{\chi}(\mathcal{F})$ of a family $\mathcal{F}$ of oriented graphs is then defined as the largest oriented chromatic number among the oriented chromatic numbers of its members ($\vec{\chi}(\mathcal{F})$ can thus be infinite). Let us denote by $\mathcal{P}$ the family of oriented planar graphs. It has been proved in [6] that $\vec{\chi}(\mathcal{P}) \leq 80$.

From the definition of oriented k-colourings, we get that if xyz is a directed 2-path in G ($xy, yz \in E(G)$) then $c(x) \neq c(y) \neq c(z) \neq c(x)$ for every oriented k-colouring c of G. In other words, any two vertices that are linked in G by a directed path of length 1 or 2 must be assigned distinct colours. Let G_1 be the oriented graph having one vertex and no arc and G_i, $i \geq 2$, be the oriented graph obtained by taking two disjoint copies of G_{i-1} and a new vertex x_i and adding all the arcs from the vertices of the first copy towards x_i and all the arcs from x_i towards the vertices of the second copy (see Figure 1). By construction, and considering the observation above, we get $\vec{\chi}(G_i) = 2^i - 1$ for every $i \geq 1$. Note in particular that the graph G_3 is outerplanar and that the graph G_4 is thus planar, so that $\vec{\chi}(\mathcal{P}) \geq 15$. In section 2, we shall prove the following improvement of this lower bound:

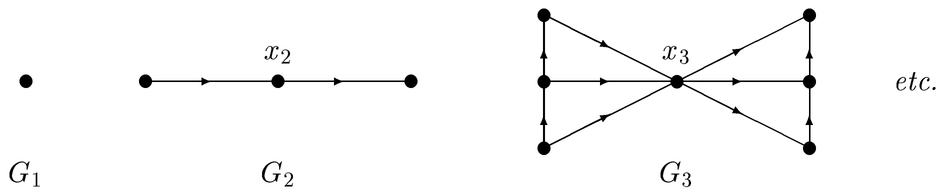

Figure 1: A sequence of oriented graphs G_i, $i \geq 1$, with $\vec{\chi}(G_i) = 2^i - 1$

Theorem 1 $\vec{\chi}(\mathcal{P}) \geq 16$.

Let us say that an oriented graph U is *$\mathcal{F}$-universal* for some family $\mathcal{F}$ of oriented graphs if every oriented graph $F \in \mathcal{F}$ admits a homomorphism to U. If the family $\mathcal{F}$ has oriented chromatic number at most k, a $\mathcal{F}$-universal oriented graph on k vertices does not necessarily exist. (Consider for instance the family $\mathcal{G}_k$ of all oriented graphs on k vertices; the oriented chromatic number of $\mathcal{G}_k$ is clearly at most k but there exists no $\mathcal{G}_k$-universal oriented graph on k vertices since such a graph would contain all tournaments on k vertices as subgraphs). However, if we consider the family $\mathcal{P}$ of oriented planar graphs such a universal graph exists:

Proposition 2 *For every $k > 0$, if $\vec{\chi}(\mathcal{P}) \leq k$ then there exists a $\mathcal{P}$-universal graph on k vertices.*

Proof. Suppose to the contrary that for every oriented graph U on k vertices there exists some oriented planar graph P_U which does not map homomorphically to U. Denote by ℓ the number of oriented graphs on k vertices and let $\mathcal{G}_k = \{U_1, U_2, \ldots, U_\ell\}$. Consider the oriented graph P obtained by taking one copy of each graph P_{U_1}, P_{U_2}, $\ldots$, P_{U_ℓ} and gluing them by identifying one vertex of each of them into a unique vertex. The graph P thus obtained is clearly planar and therefore there exists some oriented graph U_i in $\mathcal{G}_k$ such that $P \longrightarrow U_i$. Considering the restriction to P_{U_i} of such a homomorphism, we get a homomorphism from P_{U_i} to U_i which contradicts our assumption. □

2 Minimal $\mathcal{P}$-universal graphs

In this section we prove Theorem 1 using a series of Lemmas. For every oriented graph G and every vertex $x \in V(G)$ we denote by $\Gamma^+_G(x)$ the subgraph of G induced by the successors of x and by $\Gamma^-_G(x)$ the subgraph of G induced by the predecessors of x. For any two oriented graphs G_1 and G_2, we denote by $G_1 \cap G_2$ the graph given by $V(G_1 \cap G_2) = V(G_1) \cap V(G_2)$ and $E(G_1 \cap G_2) = E(G_1) \cap E(G_2)$.

Let U be a minimal $\mathcal{P}$-universal oriented graph (by minimal, we means here that no proper subgraph of U is $\mathcal{P}$-universal).

Lemma 3 *There exists an oriented planar graph P^* such that for every homomorphism $\varphi : P^* \longrightarrow U$ and every arc $uv \in E(U)$ there exists an arc $xy \in E(P^*)$ such that $\varphi(x) = u$ and $\varphi(y) = v$.*

Proof. Let uv be any arc in $E(U)$. Since U is minimal, there exists an oriented planar graph P_{uv} such that every homomorphism from P_{uv} to U has to use the arc uv (otherwise, the arc uv could be deleted from U). We now construct the oriented graph P^* as follows: take one copy of P_{uv} for every $uv \in E(U)$ and glue them together by choosing one vertex in each copy and identifying them into a unique vertex. The graph P^* thus obtained is clearly planar and every homomorphism from P^* to U induces a homomorphism from P_{uv} to U for every $uv \in E(U)$. Therefore, every such homomorphism has to use all the arcs of U and we are done. □

Lemma 4 *For every vertex $u \in V(U)$, $|V(\Gamma^+_U(x))| \geq 7$ and $|V(\Gamma^-_U(x))| \geq 7$.*

Proof. Consider the oriented graph P obtained from P^* by adding, for every vertex $x \in V(P^*)$, two copies of the oriented graph G_3 (given in Figure 1) together with arcs from x towards every vertex of the first copy and arcs from every vertex of the second copy towards x. Since G_3 is outerplanar, the oriented graph P thus obtained is planar. Every homomorphism from P to U has to use every arc, and thus every vertex, of U since P^* is a subgraph of P. Considering the fact that $\vec{\chi}(G_3) = 7$ we get the desired result. □

Lemma 4 shows in particular that if U is an oriented graph on 15 vertices then U is a tournament and every vertex $u \in V(U)$ has exactly 7 predecessors and 7 successors.

Lemma 5 *If U is a tournament on 15 vertices then for every arc $uv \in E(U)$, for every $A \in \{\Gamma_U^+(u), \Gamma_U^-(u)\}$, $B \in \{\Gamma_U^+(v), \Gamma_U^-(v)\}$, $|V(A \cap B)| \geq 3$. Moreover, if $|V(A \cap B)| = 3$ then the subgraph $A \cap B$ is the directed cycle on three vertices.*

Proof. Let P_4 be the directed path on 4 vertices. Every oriented colouring of P_4 has to use at least 3 colours and, if c is an oriented 3-colouring of P_4, the colour-graph of c has to be the directed cycle on 3 vertices.

Let now P be the oriented graph obtained from P^* by adding, for every arc $xy \in E(P^*)$, four copies of P_4, say P_4^1, P_4^2, P_4^3 and P_4^4 with arcs from x towards every vertex of P_4^i if $i \in \{1,2\}$, arcs from every vertex of P_4^i towards x if $i \in \{3,4\}$, arcs from y towards every vertex of P_4^i if $i \in \{1,3\}$, and arcs from every vertex of P_4^i towards y if $i \in \{2,4\}$. The graph P thus obtained is clearly planar and, since every homomorphism from P to U has to use all the arcs of U, we get the desired result. □

Lemma 5 shows in particular that if U is a tournament on 15 vertices then for any two vertices u and v in $V(U)$, one of the four subgraphs $\Gamma_U^+(u) \cap \Gamma_U^+(v)$, $\Gamma_U^+(u) \cap \Gamma_U^-(v)$, $\Gamma_U^-(u) \cap \Gamma_U^+(v)$ and $\Gamma_U^-(u) \cap \Gamma_U^-(v)$ has exactly four vertices while the three others are the directed cycle on three vertices.

Lemma 6 *If U is a tournament on 15 vertices then every three vertices $u, v, w \in V(U)$ have a common successor.*

Proof. We have two cases to consider depending on whether u, v, w induce a transitive tournament or a directed cycle.

Case 1. *u, v, w induce a transitive tournament.*

Without loss of generality, we may assume that $uv, uw, vw \in E(U)$. If $\Gamma_U^+(u) \cap \Gamma_U^+(v)$ is the directed cycle on 3 vertices (Lemma 5) we are done since $w \in V(\Gamma_U^+(u) \cap \Gamma_U^+(v))$ and the successor of w in this cycle is also a successor of u and v.

Suppose now that $\Gamma_U^+(u) \cap \Gamma_U^+(v)$ has 4 vertices. If w has a successor in $\Gamma_U^+(u) \cap \Gamma_U^+(v)$ then this successor is again also a successor of u and v. If w has no successor in $\Gamma_U^+(u) \cap \Gamma_U^+(v)$ let us count the predecessors of w in U: w has 3 predecessors in $\Gamma_U^+(u) \cap \Gamma_U^+(v)$, 3 predecessors in $\Gamma_U^-(u) \cap \Gamma_U^-(w)$, which, including u and v, gives 8 predecessors. This contradicts Lemma 4.

Case 2. *u, v, w induce a directed cycle.*

Assume to the contrary that u, v and w have no common successor in U. Considering the structure of the graph U (see Figure 2), we will show that we necessary get a contradiction to Lemma 4.

Let us denote by U_{uv} (resp. U_{vw}, U_{wu}) the set $V(\Gamma_U^+(u) \cap \Gamma_U^+(v))$ (resp. $V(\Gamma_U^+(v) \cap \Gamma_U^+(w))$, $V(\Gamma_U^+(w) \cap \Gamma_U^+(u))$). These three sets are clearly disjoint. By Lemma 5, we know that each of these sets contains at least 3 vertices and, by our assumption, that w is a successor of U_{uv}, u is a successor of U_{vw} and v is a successor of U_{wu}.

Let us now count the successors of u: v is a successor of u, as well as the vertices of U_{uv} and U_{wu}. Therefore, U_{uv} and U_{wu} necessarily contain exactly 3 vertices each, otherwise Lemma 4 would be contradicted. Considering in a similar way the successors of v, or w, we get that the third set U_{vw} must also contain exactly 3 vertices. By Lemma 5 we thus get that each of these three sets induces

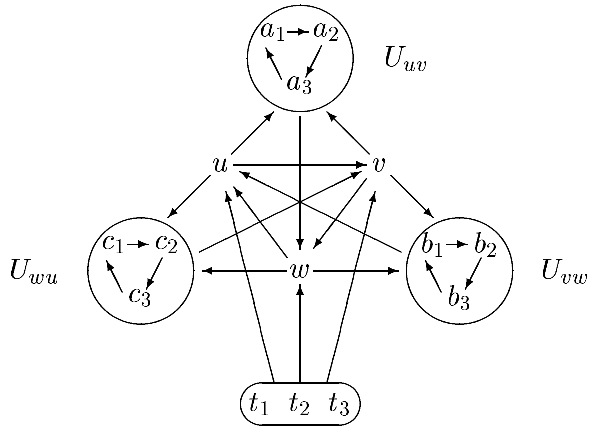

Figure 2: The structure of the graph U

a directed cycle on 3 vertices. Let then U_{uv} be the directed cycle $a_1a_2a_3$, U_{vw} be the directed cycle $b_1b_2b_3$ and U_{wu} be the directed cycle $c_1c_2c_3$.

Let us denote by t_1, t_2 and t_3 the three remaining vertices in U. Since u, v and w have already 7 successors, the vertices t_1, t_2 and t_3 are necessarily predecessors of u, v and w (otherwise Lemma 4 would be contradicted).

Let us finally consider the common successors of u and any a_i, $1 \leq i \leq 3$. We know by Lemma 5 that there are at least 3 such common successors. One of them is $a_{i+1 \ (mod\ 3)}$ and the at least two others necessarily belong to U_{wu}. Similarly, considering the common successors of u and any c_i, $1 \leq i \leq 3$, we get that each c_i has at least two successors in U_{uv}. The number of arcs between U_{uv} and U_{wu} is therefore at least 12, a contradiction since each of these sets has only 3 vertices. $\square$

From Lemmas 4, 5 and 6, we get that if U is a tournament on 15 vertices then every vertex has 7 successors, every two vertices have at least 3 common successors and every three vertices have at least one common successor. The proof of Theorem 1 follows from the fact that such a tournament cannot exist, as shown by the following result of Brown and Reid (considering the case $n = 1$):

Theorem 7 ([3]) *There exists no tournament on $8n + 7$ vertices such that every vertex has at least $4n + 3$ successors, every two vertices have at least $2n + 1$ common successors and every three vertices have at least n common successors.*

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