TWO PROBLEMS ON PLANAR GRAPHS

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Abstract. About ten years ago, I was attending an ALTEC conference on Algorithms in Budapest. At this time, I was puzzled by the problem of determining the oriented chromatic number of oriented planar graphs. Jarik Nešetřil was attending the same conference and I approached him for the first time, thinking that he could be attracted to my problem. Beyond my expectations, Jarik was extremely enthusiastic about it, and we spent most of the remaining free time of the conference discussing planar graphs. This was the starting point of a very fruitful collaboration, together with my colleague André Raspaud, regularly stimulated by Jarik's numerous questions which allowed us to consider many new interesting problems related to oriented colourings. Ten years after, the original problem is still open and I am particularly pleased to include it in this volume in honour of Jarik's 55th birthday. I have also included a more recent problem concerning the size of k-dominating sets in planar graphs.

Keywords. Oriented graph colouring, graph homomorphism, k-dominating set, planar graphs.

1 The oriented chromatic number of oriented planar graphs

Our first problem deals with oriented graphs, that is digraphs having no loops and no opposite arcs. If G is an oriented graph, we denote by $V(G)$ its set of vertices and by $E(G)$ its set of arcs. An *oriented k-colouring* of an oriented graph G is a mapping $c : V(G) \longrightarrow \{1, 2, \dots, k\}$ such that

(i) if $xy \in E(G)$ then $c(x) \neq c(y)$, and

(ii) if $xy, zt \in E(G)$ then $c(x) = c(t) \Longrightarrow c(y) \neq c(z)$.

With every oriented k-colouring c of G one can associate a digraph H_c with vertex set $V(H_c) = \{c(x),\ x \in V(G)\}$ and arc set $E(H_c) = \{c(x)c(y),\ xy \in E(G)\}$. Thanks to conditions ($i$) and ($ii$), H_c is an oriented graph. The oriented k-colouring c can be viewed as a homomorphism (that is an arc-preserving vertex mapping) from G to H_c. Similarly, every homomorphism of G to an oriented graph H on k vertices can be viewed as an oriented k-colouring of G, using the vertices of H as colours. Oriented colourings have been introduced in [7] and studied by several authors (see [9] for a general overview). In particular, various problems related to planar graphs are discussed in [1], [2] and [6].

The *oriented chromatic number* $\vec{\chi}(G)$ of an oriented graph G is defined as the smallest k such that G admits an oriented k-colouring or, equivalently, as the smallest order of an oriented graph H such that G admits a homomorphism to H. The oriented chromatic number $\vec{\chi}(\mathcal{F})$ of a family $\mathcal{F}$ of oriented graphs is then defined as the largest oriented chromatic number among the oriented chromatic numbers of its members ($\vec{\chi}(\mathcal{F})$ can thus be infinite). Our first problem is the following:

Problem 1 *Determine the oriented chromatic number of the family of oriented planar graphs.*

Let us denote by $\mathcal{O}$ the family of oriented outerplanar graphs and by $\mathcal{P}$ the family of oriented planar graphs. It has been proved in [8] that $\vec{\chi}(\mathcal{O}) = 7$ and in [7] that $\vec{\chi}(\mathcal{P}) \leq 80$.

From the definition of oriented k-colourings, it is not difficult to observe that if xyz is a directed 2-path in G ($xy, yz \in E(G)$) then $c(x) \neq c(y) \neq c(z) \neq c(x)$ for every k-colouring c of G. In other words, any two vertices that are linked in G by a directed path of length 1 or 2 must be assigned distinct colours. Let G_1 be the oriented graph having one vertex and no arc and G_i, $i \geq 2$, be the

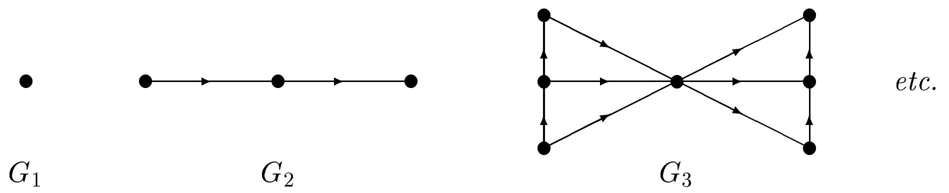

Figure 1: A sequence of oriented graphs G_i, $i \geq 1$, with $\vec{\chi}(G_i) = 2^i - 1$

oriented graph obtained by taking two disjoint copies of G_{i-1}, a new vertex x_i, and adding all the arcs from the vertices of the first copy towards x_i and all the arcs from x_i towards the vertices of the second copy (see Figure 1). By construction, and considering the observation above, we get $\vec{\chi}(G_i) = 2^i - 1$ for every $i \geq 1$. Note in particular that the graph G_3 is outerplanar, so that $\vec{\chi}(\mathcal{O}) \geq 7$, and that the graph G_4 is planar, so that $\vec{\chi}(\mathcal{P}) \geq 15$. We have proved in [10] that $\vec{\chi}(\mathcal{P}) > 15$. Recently, T.H. Marshall [5] improved this result by showing that $\vec{\chi}(\mathcal{P}) > 16$. Concerning the family of oriented planar graphs, we thus have:

Theorem 2 $17 \leq \vec{\chi}(\mathcal{P}) \leq 80$.

Every improvement of these bounds seems, at least up to now, to be particularly challenging!

Let us say that an oriented graph U is *$\mathcal{F}$-universal* for some family $\mathcal{F}$ of oriented graphs if every graph $F \in \mathcal{F}$ admits a homomorphism to U. If the family $\mathcal{F}$ has oriented chromatic number at most k, a $\mathcal{F}$-universal oriented graph on k vertices does not necessarily exist. (Consider for instance the family $\mathcal{G}_k$ of all oriented graphs on k vertices; the oriented chromatic number of $\mathcal{G}_k$ is clearly at most k but there exists no $\mathcal{G}_k$-universal oriented graph on k vertices since such a graph would contain all tournaments on k vertices as subgraphs). However, if we consider the family $\mathcal{P}$ of oriented planar graphs (or, similarly, the family $\mathcal{O}$ of oriented outerplanar graphs) such a universal graph exists. To see that, suppose to the contrary that for every oriented graph U on k vertices there exists some oriented planar graph P_U which does not map homomorphically to U. Denote by ℓ the number of oriented graphs on k vertices and let $\mathcal{G}_k = \{U_1, U_2, \ldots, U_\ell\}$. Consider the oriented graph P obtained by taking one copy of each graph P_{U_1}, P_{U_2}, ..., P_{U_ℓ} and gluing them by identifying one vertex of each of them into a unique vertex. The graph P thus obtained is clearly planar and therefore there exists some oriented graph $U = U_i$ in $\mathcal{G}_k$ such that $P \longrightarrow U$. Considering the restriction to P_{U_i} of such a homomorphism, we get a homomorphism from P_{U_i} to U_i which contradicts our assumption.

All the above-mentionned results concerning the oriented chromatic number of oriented planar graphs have been obtained by considering such universal graphs. One way to improve Theorem 2 is thus either to exhibit some $\mathcal{P}$-universal oriented graph having less than 80 vertices, or to prove that no oriented graph on 17 (or more) vertices can be $\mathcal{P}$-universal.

2 k-dominating sets in planar graphs

We now consider undirected graphs. Let k be a non-negative integer; we say that a set $S \subseteq V(G)$ *k-dominates* a graph G if every vertex x in G is at distance at most k from the set S (the distance $d_G(x,S)$ from x to S is given by $d_G(x,S) = min\{d_G(x,s),\ s \in S\}$). Using this terminology, a dominating set of G is a 1-dominating set of G. We are interested in the minimum size of such k-dominating sets. For that, we define the parameter

$$\gamma_k(G) \ = \ min\{|S|,\ S\ k\text{-dominates } G\}$$

for every graph G. Let us denote by $D(G)$ the *diameter* of G, that is the maximum length of a shortest path in G. We have the following conjecture [3]:

Conjecture 3 *There exists some constant c_0 such that for every planar graph G and every $k \geq D(G)/2$, $\gamma_k(G) \leq c_0$.*

Our second problem is therefore the following

Problem 4 *Prove (or disprove) Conjecture 3.*

Observe that if a set S k-dominates a graph G then it also k'-dominates G for every $k' \geq k$. Therefore, considering the case $k = \lceil \frac{D(G)}{2} \rceil$ in Conjecture 3 is enough.

Consider the tree $T_{n,p}$, $n \geq 2$, $p \geq 2$, obtained from the star $K_{1,n}$ by replacing each edge by a path of length p. We clearly have $D(T_{n,p}) = 2p$. Since every leaf in $T_{n,p}$ is at distance p from the central vertex, every $(p-1)$-dominating set of $T_{n,p}$ must contain one vertex per branch. Therefore, $\gamma_{p-1}(T_{n,p}) \geq n$. This shows that the condition $k \geq D(G)/2$ in Conjecture 3 is necessary in order to get a k-dominating set of bounded size.

Recall that a *center* in a tree T is a vertex whose maximal distance to other vertices is minimum (every tree has at most two centers). If c is a center of T, we clearly have $d_T(c,x) \leq D(T)/2$ for every vertex $x \in V(T)$ and thus $\gamma_{\lceil \frac{D(T)}{2} \rceil} = 1$ for every tree T.

In case of outerplanar graphs, it is proved in [3] that $\gamma_{\lceil \frac{D(G)}{2} \rceil}(G) \leq 2$ for every outerplanar graph G (considering the cycle on four vertices shows that this bound is tight).

For planar graphs, MacGillivray and Seyffarth [4] proved that $\gamma_{\lceil \frac{D(G)}{2} \rceil}(G) \leq 3$ if $D(G) = 2$ (and this bound is tight) and that $\gamma_{\lceil \frac{D(G)}{2} \rceil}(G) \leq 10$ if $D(G) = 3$. In [3], we have proved that $\gamma_{\lceil \frac{D(G)}{2} \rceil}(G) \leq 23$ if $D(G) = 4$ and that $\gamma_{\lceil \frac{D(G)}{2} \rceil}(G) \leq 41$ if $D(G) = 7$. The only general result we have concerning k-dominating sets of bounded size for planar graphs is the following:

Theorem 5 ([3]) *For every planar graph G, for every $\varepsilon > 0$, for every $k > (\frac{5}{7} + \varepsilon)D$, $\gamma_k(G) \leq \frac{3}{\varepsilon} + 6$.*

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