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THE CHROMATIC NUMBER OF ORIENTED GRAPHS[1]

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Abstract. We introduce in this paper the notion of the chromatic number of an oriented graph G (that is of an antisymmetric directed graph) defined as the minimum order of an oriented graph H such that G admits a homomorphism to H. We study the chromatic number of oriented k-trees and of oriented graphs with bounded degree. We show that there exist oriented k-trees with chromatic number at least $2^{k+1}-1$ and that every oriented k-tree has chromatic number at most $(k+1)\times 2^k$. For 2-trees and 3-trees we decrease these upper bounds respectively to 7 and 16 and show that these new bounds are tight. As a particular case, we obtain that oriented outerplanar graphs have chromatic number at most 7 and that this bound is tight too. We then show that every oriented graph with maximum degree k has chromatic number at most $(2k-1)\times 2^{2k-2}$. For oriented graphs with maximum degree 2 we decrease this bound to 5 and show that this new bound is tight. For oriented graphs with maximum degree 3 we decrease this bound to 16 and conjecture that there exists no such connected graph with chromatic number greater than 7.

Keywords. Graph coloring, Graph homomorphism, Oriented coloring.

1 Introduction

In this paper, we introduce and study the notion of the chromatic number of antisymmetric directed graphs, called *oriented graphs* in the following. These graphs are in fact *orientations* of undirected graphs, obtained by arbitrarily giving one of the two possible orientations to the edges of an undirected graph. For every graph G, the sets of vertices and edges (or arcs) of G will be respectively denoted by $V(G)$ and $E(G)$. The number of vertices of a graph G will be called the *order* of G.

Let G and H be two oriented graphs. A *homomorphism* of G to H is a mapping μ from $V(G)$ to $V(H)$ such that for every arc (x,y) in $E(G)$, $(\mu x, \mu y)$ is an arc in $E(H)$. If there exists a homomorphism of G to H, we will write $G \to H$ and say that G is *H-colorable*. We will often call such a graph H a *color-graph* and its vertices will be called *colors*. The mapping μ will be referred to as a *H-coloring*, or simply a *coloring*, of G. Note that if a graph G is H-colorable, then every subgraph G' of G is also H-colorable since the restriction of a homomorphism of G to H to the vertices of G' is a homomorphism of G' to H.

One useful hint the reader should have in mind when considering oriented colorings is that two vertices that are linked by a directed path of length 1 or 2 must be assigned distinct colors in every oriented coloring. To see that, observe that from the definition of homomorphisms of oriented graphs we get that every coloring μ of an oriented graph G must satisfy (i) $xy \in E(G) \Longrightarrow \mu x \neq \mu y$ and (ii) $xy \in E(G)$, $zt \in E(G)$, $\mu x = \mu t \Longrightarrow \mu y \neq \mu z$ (note that this second condition comes from the fact that we only consider oriented color-graphs). For instance, it is not difficult to check that in every coloring of the directed cycle C_5 on 5 vertices all the vertices must be assigned distinct colors, since every two vertices are linked by a directed path of length 1 or 2.

Graph homomorphisms have been extensively studied in the last past years. Many papers have been devoted to the study of the complexity of H-coloring. This problem can be stated as follows :

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given a fixed graph H, what is the complexity of deciding whether a graph G is H-colorable or not? This problem has been solved in the undirected case by Hell and Nešetřil [14]. They proved that this problem is polynomially decidable when the graph H is bipartite and NP-complete otherwise. In the directed case, only partial answers have been given [3, 4, 12, 15, 16, 17, 18, 24, 25] and the question is still open.

For an oriented graph H, we will denote by $[H]$ the set of oriented graphs that are H-colorable: $[H] = \{G : G \to H\}$. We define the *chromatic number* of an oriented graph G, denoted by $\chi(G)$, as the order of the smallest oriented graph H such that $G \in [H]$. Note that this notion is a natural generalisation of the usual notion of chromatic number in the undirected case since an undirected graph U has chromatic number n if and only if it admits a homomorphism to the complete graph K_n and there exists no homomorphism of U to K_{n-1}.

These sets $[H]$ together with the inclusion operator have a lattice structure and have been studied in [9, 22, 31, 32] (there they were called *color-classes*). The structure of this lattice is better known in the undirected case than in the directed one. The main result in [31] shows that the hierarchy of undirected color-classes is *dense*, that is given any two color-classes $[G_1]$ and $[G_2]$ with $[G_1] \subsetneq [G_2]$, there exists a graph G_3 such that $[G_1] \subsetneq [G_3] \subsetneq [G_2]$ (except in the particular case where $G_1 = K_1$ and $G_2 = K_2$). This is no longer true in the directed case and there exist infinite sequences of "immediate predecessors" within the directed hierarchy [22, 25].

In this paper we study the chromatic number of oriented partial k-trees and of oriented k-bounded graphs (that is graphs with maximum degree k). In section 2 we give some basic definitions. Section 3 is devoted to the study of oriented partial k-trees and section 4 to the study of oriented k-bounded graphs.

2 Preliminaries

Let $\mathcal{F}$ be any (finite or infinite) family of oriented graphs. We define the chromatic number $\chi(\mathcal{F})$ of $\mathcal{F}$ as the maximum chromatic number of a graph in $\mathcal{F}$ (note that this number may be unbounded when $\mathcal{F}$ is infinite). We will say that such a family is *colorable*, or more precisely *C-colorable*, if there exists an oriented graph C such that $\mathcal{F} \subseteq [C]$. A family $\mathcal{F}$ has a finite chromatic number if and only if it is C-colorable for some C: the chromatic number of $\mathcal{F}$ is obviously bounded by the order of C and conversely, if $\mathcal{F}$ has a finite chromatic number, say k, it is C-colorable for C being an oriented graph containing all oriented graphs of order at most k as subgraphs (such a graph has at most $k \times 2^{\frac{k-1}{2}}$ vertices when k is odd and $\frac{3}{2\sqrt{2}}k \times 2^{\frac{k-1}{2}}$ vertices when k is even [23]).

The reader should note here that if a family $\mathcal{F}$ has a finite chromatic number k, the smallest graph C such that $\mathcal{F}$ is C-colorable may be of order strictly greater than k. For instance, the family $\mathcal{F}_k$ of all graphs with order at most k obviously has chromatic number k; however, there is no graph C of order k such that every tournament on k vertices is C-colorable (such a graph C should contain all those tournaments as subgraphs). A family $\mathcal{F}$ will be said to be *optimally colorable* if there exists a graph C of order $\chi(\mathcal{F})$ such that $\mathcal{F}$ is C-colorable.

The following proposition provides a sufficient condition for a colorable family of graphs to be optimally colorable. We will say that a family $\mathcal{F}$ is *complete* if for every two graphs F_1 and F_2 in $\mathcal{F}$, there exists a graph F^* in $\mathcal{F}$ containing F_1 and F_2 as subgraphs. Then we have:

Proposition 1 *Every complete family $\mathcal{F}$ of graphs which is colorable is optimally colorable.*

Proof. Since $\mathcal{F}$ is colorable, $\chi(\mathcal{F})$ is finite and every graph F in $\mathcal{F}$ is colorable by a color-graph C_F having at most $\chi(\mathcal{F})$ colors. If $\mathcal{F}$ is not optimally colorable, then there exists a finite number of graphs in $\mathcal{F}$, say $F_1, F_2, \ldots, F_n$, such that for every color-graph C having at most $\chi(\mathcal{F})$ colors at least one of them is not C-colorable (it is sufficient here to consider a finite number of graphs since there exists a finite number of color-graphs having at most $\chi(\mathcal{F})$ colors). Since $\mathcal{F}$ is complete there exists a graph

F^* in $\mathcal{F}$ which contains all the F_i's as subgraphs. The graph F^* is C_{F^*}-colorable for some color-graph C_{F^*} having at most $\chi(\mathcal{F})$ vertices and all the F_i's are thus C_{F^*}-colorable, a contradiction. □

For instance, the family of planar graphs is complete (let F^* be the disjoint union of F_1 and F_2) as well as the family of connected planar graphs (let x_i be any vertex of F_i; the graph F^* is then obtained by taking a disjoint copy of each graph F_i and identifying the x_i's to a single vertex). In [27], we have shown with A. Raspaud that the family of oriented planar graphs is colorable and has chromatic number at most 80. This bound can be significantly improved under a high girth assumption [26]:

Theorem 2 (Nešetřil et al.) *Every oriented planar graph with girth at least 6 (resp. 7,11,16) has chromatic number at most 32 (resp. 12,7,5).*

The chromatic number of an undirected graph is clearly a lower bound for the chromatic number of any of its orientations. The gap between the chromatic number of an undirected graph and the chromatic number of its orientations can be arbitrarily large, as illustrated by the following result:

Proposition 3 *The family of oriented bipartite graphs is not colorable.*

Proof. Let $K_{n,n}$ be the orientation of the complete bipartite graph on $2n$ vertices defined as follows: let $V(K_{n,n}) = \{1, 2, \ldots, n\} \cup \{1', 2', \ldots, n'\}$ and let $ij' \in E(K_{n,n})$ if $i = j$ and $j'i \in E(K_{n,n})$ otherwise. The chromatic number of $K_{n,n}$ is $2n$ since every two vertices are joined by a directed path of length at most 2 and thus must be assigned distinct colors. Therefore, the family of oriented bipartite graphs has unbounded chromatic number. □

Let G be an oriented (resp. undirected) graph; a *k-clique subgraph* of G is a sequence $X = (x_1, x_2, \ldots, x_k)$ of pairwise distinct vertices of G such that the subgraph induced by these vertices is a tournament (resp. a complete graph). In this paper we will extensively use the following notion:

Definition 4 An *orientation vector* of size n is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ in $\{0,1\}^n$; let G be an oriented graph and $X = (x_1, x_2, \ldots, x_n)$ be a sequence of pairwise distinct vertices of G. A vertex y of G is said to be an *α-successor of X* if for every i, $1 \leq i \leq n$, we have $\alpha_i = 1 \Longrightarrow x_i y \in E(G)$ and $\alpha_i = 0 \Longrightarrow y x_i \in E(G)$.

3 The chromatic number of oriented partial k-trees

3.1 The general case

The notion of *undirected k-tree* can be defined as follows [5]: the complete graph K_k with k vertices is a k-tree; if G is a k-tree then the graph G' obtained from G by adding a new vertex v linked to every vertex of a k-clique subgraph of G is a k-tree and there are no further k-trees. A subgraph of a k-tree is called a *partial k-tree.* Partial k-trees have been extensively studied in the last past years, since they often lead to polynomial algorithms for problems which are known to be NP-complete in the general case (see [30]). The notion of a 1-tree obviously corresponds to the usual notion of a tree. The class of outerplanar graphs is strictly contained in the class of partial 2-trees. This notion of k-tree seems to play an essential role in the study of the complexity of the H-coloring problem [10, 16, 24].

The two following results about k-trees will be useful in the sequel:

Proposition 5 *The family of partial k-trees is complete.*

Proof. Since partial k-trees are subgraphs of k-trees, it suffices to consider k-trees. Let T_1 and T_2 be k-trees and X_1 be any k-clique subgraph of T_1. Consider the graph T_3 obtained from T_1 by applying, starting from X_1, the same sequence of vertex additions that leads from K_k to T_2. The graph T_3 thus obtained is a partial k-tree containing both T_1 and T_2 as subgraphs. □

Proposition 6 *Let T_1 be a partial k-tree, $X = (x_1, x_2, \ldots, x_k)$ a k-clique subgraph of T_1 and T_2 a partial k'-tree ($k' < k$). The graph T_3 obtained from T_1 and T_2 by adding edges from every vertex of T_2 to vertices x_1, x_2, $\ldots$, $x_{k-k'}$ is also a partial k-tree.*

Proof. Let U_1 be a k-tree containing T_1 as a subgraph and U_2 a k'-tree containing T_2 as a subgraph. We construct a k-tree U_3 as follows: starting with U_1, we add k' new vertices, say $y_1, \ldots, y'_k$, in such a way that for every j, $1 \leq j \leq k'$, y_j is linked to vertices $x_1, x_2, \ldots, x_{k-k'}, y_1, y_2, \ldots, y_{j-1}$. The k-tree thus obtained contains $(x_1, \ldots, x_{k-k'}, y_1, \ldots, y'_k)$ as a k-clique subgraph. Starting from $(y_1, \ldots, y'_k)$ we then apply the same sequence of vertex additions that leads from $K_{k'}$ to U_2 except that every such new vertex is additionally linked to $(x_1, \ldots, x_{k-k'})$. It is then not difficult to check that the k-tree U_3 thus obtained contains T_3 as a subgraph, which completes the proof. □

By construction, every k-tree G distinct from the complete graph K_k has a vertex v of degree k whose neighbourhood is a k-clique subgraph in G, and whose deletion leads to another k-tree, denoted by $G - v$. It is then folklore to establish by induction that every such undirected k-tree has chromatic number $k + 1$ (or k when it is the complete graph K_k): the result is obvious for K_k and K_{k+1} and if $G - v$ has chromatic number $k + 1$ then, considering any $(k + 1)$-coloring of $G - v$, there always remains a free color to be assigned to v, leading to a $(k+1)$-coloring of G. In order to apply the same technique in the oriented case, we must be able to find in the color-graph an adequate color c to be assigned to v, whatever the orientations of the arcs linking v to $G - v$ are. The following property P_k will ensure that such a color can always be found.

Definition 7 We will say that a color-graph C satisfies property P_k for some $k > 0$ if for every oriented n-clique subgraph $(c_1, c_2, \ldots, c_n)$ in C with $1 \leq n \leq k$, and every orientation vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of size n, there exists a color c in $V(C)$ which is an α-successor of $(c_1, c_2, \ldots, c_n)$.

Note that every color-graph satisfying property P_k also satisfies property $P_{k'}$ for every $k' < k$. By convention, we will assume that every graph with at least one vertex satisfies property P_0 and that the null-graph (with no vertices) does not. Then, we have:

Theorem 8 *If C is a color-graph satisfying property P_k then every oriented partial k-tree is C-colorable.*

Proof. Note first that it suffices to consider the case of k-trees, since partial k-trees are subgraphs of k-trees. We proceed by induction on the number of vertices n of a k-tree G. If $n = k$, then G is a tournament with vertex set $\{x_1, x_2, \ldots, x_n\}$. Giving arbitrarily any color c to x_1, property P_k ensures that we can achieve a C-coloring of G: if $x_1, \ldots, x_i$ are already colored, the color of x_{i+1} is obtained by using the orientation vector $(\alpha_1, \ldots, \alpha_i)$ defined by $\alpha_j = 0$ iff $x_{i+1}x_j \in E(G)$. Suppose now that every oriented k-tree having at most n vertices ($n \geq k$) is C-colorable and let H be any oriented k-tree with $n + 1$ vertices. Then there exists a vertex v in H with degree k. The induction hypothesis ensures that $H - v$ is C-colorable and property P_k says that one can find in C an adequate color c to be assigned to v (by choosing an orientation vector according to the direction of the arcs linking v to its neighbours in G), leading to a C-coloring of H. □

We are now going to propose a combinatorial construction leading for every k to color-graphs satisfying property P_k and having $(k + 1) \times 2^k$ vertices (or colors). Let k be any given positive integer; we define the color-graph Z_k as follows: let $V(Z_k)$ be the subset of $\{1, 2, \ldots, k + 1\} \times \{0, 1\}^{k+1}$ consisting of elements $(a, x_1, \ldots, x_{k+1})$ such that $x_a = 0$, and let $E(Z_k)$ be such that $((a, x_1, \ldots, x_{k+1}), (b, y_1, \ldots, y_{k+1})) \in E(Z_k)$ iff ($a > b$ and $x_b \neq y_a$) or ($a < b$ and $x_b = y_a$). Then, the graph Z_k thus defined has exactly $(k + 1) \times 2^k$ vertices and we have:

Proposition 9 *For every $k > 0$, the color-graph Z_k satisfies property P_k.*

Proof. Let $c = (a, x_1, \ldots, x_{k+1})$ be any vertex of Z_k. For every i, $1 \leq i \leq k+1$, we will denote by $c(i)$ the component x_i of c and by $Id(c)$ (the *identity* of c) the component a. Let us first note that only vertices with distinct identities are linked in Z_k. Hence, the oriented n-clique subgraphs ($1 \leq n \leq k$) in Z_k are exactly the sets of vertices with pairwise distinct identities. Let $(c_1, c_2, \ldots, _c n)$ be such a set of vertices and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be any orientation vector. Let d denote any identity which does not appear in $(c_1, c_2, \ldots, c_n)$. Then, the vertex c defined by:

(*i*) $Id(c) = d$,

(*ii*) $\forall\, i$, $1 \leq i \leq n$, if ($d < Id(c_i)$ and $\alpha_i = 1$) or ($d > Id(ci)$ and $\alpha_i = 0$)
then $c(Id(c_i)) = c_i(d)$
else $c(Id(c_i)) = 1 - c_i(d)$,

is an α-successor of $(c_1, c_2, \ldots, c_n)$ and the result follows. □

We are now able to prove the following:

Theorem 10 *For every k, the family $\mathcal{T}_k$ of oriented partial k-trees is optimally colorable and has chromatic number at most $(k+1) \times 2^k$.*

Proof. From Theorem 8 and Proposition 9, the family $\mathcal{T}_k$ is colorable and has chromatic number at most $(k+1) \times 2^k$. By Proposition 5 it is also complete and by Proposition 1 it is optimally colorable. □

Let H be an oriented graph; for every vertex $x \in V(H)$, we will denote by $\Gamma^+(H, x)$ (resp. $\Gamma^-(H, x)$) the oriented subgraph induced by the successors (resp. predecessors) of x in H. Then we have:

Theorem 11 *For every $k > 0$, if a color-graph C satisfies property P_k then for every color c in $V(C)$, both $\Gamma^+(C, c)$ and $\Gamma^-(C, c)$ satisfy property P_{k-1}.*

Proof. Let c be any color in $V(C)$, $(c_1, c_2, \ldots, c_n)$, $n \leq k-1$, be any n-clique subgraph of $\Gamma^+(C, c)$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be any orientation vector of size n. Since C satisfies P_k, the $(n+1)$-clique $(c, c_1, c_2, \ldots, c_n)$ has a $(1, \alpha_1, \alpha_2, \ldots, \alpha_n)$-successor in C which is also an α-successor of $(c_1, c_2, \ldots, c_n)$ in $\Gamma^+(C, c)$. A similar argument shows that $\Gamma^-(C, c)$ also satisfies P_{k-1}. □

This theorem leads to the following lower bound for the number of vertices of a graph satisfying property P_k:

Corollary 12 *Every color-graph C satisfying property P_k must have at least $2^{k+1} - 1$ colors.*

Proof. We proceed by induction on k. The smallest graph satisfying P_1 is the directed cycle C_3 on three vertices. Let now C be a graph satisfying P_k. By Theorem 11 and using the induction hypothesis, every vertex in C must have at least $2^k - 1$ successors and $2^k - 1$ predecessors. The number of vertices of C is therefore at least $2 \times (2^k - 1) + 1 = 2^{k+1} - 1$ and the result follows. □

This lower bound is also a lower bound for the chromatic number of oriented partial k-trees since we have:

Theorem 13 *For every $k > 0$, there exists an oriented k-tree T_k with $\chi(T_k) \geq 2^{k+1} - 1$.*

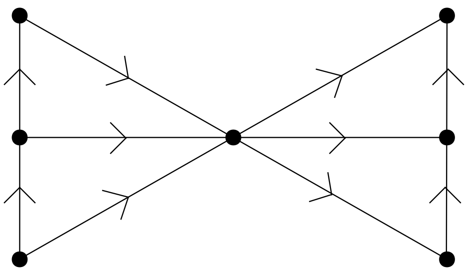

Figure 1: An oriented graph with chromatic number 7

Proof. We proceed by induction on k. For $k = 1$, it suffices to consider the directed path on three vertices (whose vertices must be assigned distinct colors). Suppose now that the result holds up to $(k-1)$-trees and let T_{k-1} be an oriented $(k-1)$-tree with chromatic number at least $2^{k+1} - 1$. Let T_k be the oriented k-tree obtained by taking two copies of T_{k-1} and a new vertex x and adding arcs from x towards every vertex of the first copy and from every vertex of the second copy towards x (according to Proposition 6, this graph is indeed a k-tree). Then every vertex in the first copy must be assigned a color distinct from those assigned to the vertices of the second copy. Moreover, vertex x must be assigned a color distinct from those assigned to all other vertices. Hence, the number of colors we need for coloring T_k is at least $2 \times (2^k - 1) + 1 = 2^{k+1} - 1$ and the result follows. $\square$

This lower bound is tight for oriented trees since every oriented tree is C_3-colorable, for C_3 being the directed cycle on three vertices. In the following subsection we will show that this lower bound is also tight for oriented 2-trees. However we will show that this is no longer true for oriented 3-trees and that the exact value of the chromatic number in this case is 16.

3.2 The case of oriented 2-trees

The smallest color-graph satisfying property P_1 is the directed cycle C_3 on three vertices. By Corollary 12, we know that every color-graph satisfying P_2 must have at least 7 vertices. The unique color-graph on 7 vertices which satisfies P_2 is the tournament QR_7 [11], build from the non-zero quadratic residues of 7 as follows:

- $V(QR_7) = \{0, 1, \ldots, 6\}$,
- $\forall\, i, j \in V(QR_7),\ ij \in E(QR_7) \Longleftrightarrow j - i \equiv 1, 2 \text{ or } 4 \pmod 7$.

This tournament is a rotational tournament and it is vertex-transitive (for every two vertices x and y, there exists an automorphism that maps x to y). Then, we have:

Theorem 14 *The chromatic number of the family of oriented 2-trees is 7. Moreover, this family is optimally colorable by QR_7.*

Proof. It is easy to check that QR_7 satisfies property P_2 [2]. Hence, every oriented 2-tree is QR_7-colorable and thus has chromatic number at most 7. By Theorem 13, we know that there exists oriented 2-trees with chromatic number 7 and the result follows. $\square$

Figure 1 depicts such a 2-tree (obtained by the construction given in the proof of Theorem 13). One can check that all vertices must be assigned distinct colors.

The family of outerplanar graphs is strictly included in the family of 2-trees. Since the graph in Figure 1 is outerplanar, we also have:

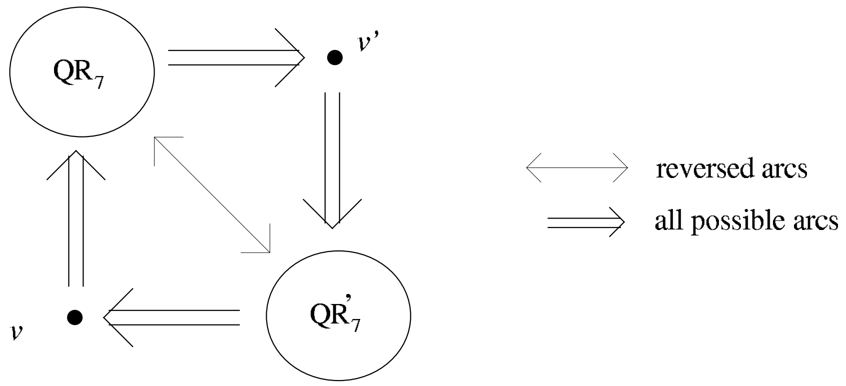

Figure 2: Tromp's construction

Corollary 15 *The chromatic number of the family of oriented outerplanar graphs is 7. Moreover, this family is optimally colorable by QR_7.*

This result improves a result given in [27] where it is shown that every oriented outerplanar graph has chromatic number at most 12.

Property P_2 says that for every orientation vector α of size 2, every arc in QR_7 has an α-successor. A subdivision of a graph G is obtained by repeatedly subdividing some edges with extra vertices of degree 2 (that is an edge xy is replaced by two edges xz and zy, where z is a new vertex). If G is QR_7- colorable, then so is every subdivision G' of G and we have:

Theorem 16 *Let $\mathcal{F}$ be any family of oriented graphs which is QR_7-colorable. Then the family $\mathcal{F}'$ of all oriented subdivisions of the graphs in $\mathcal{F}$ is also QR_7-colorable.*

3.3 The case of oriented 3-trees

The smallest color-graph satisfying P_3 has 16 vertices and is obtained by using a construction proposed by Tromp [29]. Let T_{16} be the oriented graph defined by:

(*i*) $V(T_{16}) = \{0, 1, \dots, 6\} \cup \{0', 1', \dots, 6'\} \cup \{v, v'\}$,

(*ii*) $\forall\, i, j \in \{0, 1, \dots, 6\}$, $ij \in E(QR_7) \Longrightarrow ij \in E(T_{16})$ and $i'j' \in E(T_16)$
(we have two disjoint copies of the color-graph QR_7),

(*iii*) $\forall\, i \in \{0, 1, \dots, 6\}$, $vi, v'i', i'v$ and $iv' \in E(T_{16})$,

(*iv*) $\forall\, i, j \in \{0, 1, \dots, 6\}$, $ij \in E(QR_7) \Longrightarrow j'i \in E(T_{16})$ and $ji' \in E(T_{16})$
(the arcs between the two copies of QR_7 are "reversed").

This construction is illustrated by Figure 2. The following propositions are not difficult to check [29]:

Proposition 17 *For every $i \in \{0, 1, \dots, 6, v\}$, the graph T_{16} is such that:*

1. *$\Gamma^+(T_{16}, i)$ and $\Gamma^-(T_{16}, i)$ are both isomorphic to QR_7,*
2. *i and i' are not joined by an arc,*
3. *$\Gamma^+(T_{16}, i) = \Gamma^-(T_{16}, i')$ and $\Gamma^+(T_{16}, i') = \Gamma^-(T_{16}, i)$.*

Proposition 18 *The graph T_{16} satisfies property P_3.*

There exists no color-graph with 15 vertices which satisfies property P_3. By Theorem 11, such a graph would have to be a tournament such that every vertex x has 7 successors ($\Gamma^+(x)$ must be isomorphic to QR_7), every two distinct vertices x and y have 3 successors in common ($\Gamma^+(x) \cap \Gamma^+(y)$ must be isomorphic to C_3) and every three distinct vertices x, y and z have 1 successor in common ($\Gamma^+(x) \cap \Gamma^+(y) \cap \Gamma^{(}y)$ must contain at least one vertex). Such a tournament cannot exist by the following result, due to Brown and Reid [7] (simply consider the case $n = 1$):

Theorem 19 (Brown and Reid) *For every positive integer n, there exists no tournament T with order $8n + 7$ satisfying:*

(i) *every vertex has $4n + 3$ successors,*

(ii) *every two distinct vertices have $2n + 1$ common successors and*

[*(iii)* *every three distinct vertices have n common successors.*

In fact, for the particular case we are concerned with (that is $n = 1$), the third condition can be weakened as:

(iii') every three distinct vertices inducing a transitive tournament have a common successor.

and we can prove (see [1]):

Proposition 20 *There exists no tournament T on 15 vertices such that for every vertex x in T, $\Gamma^+(T, x)$ and $\Gamma^-(T, x)$ are both isomorphic to QR_7.*

We are now able to prove the main result of this subsection:

Theorem 21 *The chromatic number of the family of oriented 3-trees is 16. Moreover, this family is optimally colorable by T_{16}.*

Proof. By Theorem 8 and Proposition 18, every oriented 3-tree is T_{16}-colorable and then has chromatic number at most 16. In order to get the result we now construct a 3-tree having 3255 vertices and whose chromatic number is exactly 16.

Step 1: we first build an oriented partial 2-tree T_2 in such a way that the only color-graph C of order 7 such that T_2 is C-colorable is QR_7 (see Figure 3). Let P_3 be the directed path on three vertices and G_0 be the partial 2-tree depicted by Figure 1. We now join every vertex x of G_0 to two distinct copies of P_3 by adding all possible arcs from x towards the first copy and from the second copy towards x (see Figure 3a). Since P_3 is a tree, the graph G_1 thus obtained is a partial 2-tree. To every arc linking such a vertex x to one of its two corresponding copies of P_3 we now add four α-successors, one for each possible orientation vector of size 2 (see Figure 3b). The graph G_2 thus obtained is still a 2-tree and if C is a color-graph such that G_2 is C-colorable then for every arc c_1c_2 in C and every orientation vector α of size 2, there exists an α-successor of c_1c_2 in C. The only color-graph of order 7 satisfying this property is QR_7 and we therefore set $T_2 = G_2$ (the number of vertices of T_2 is 217).

Step 2: Consider now the graph H_1 obtained from two distinct copies of the graph G_0 and a vertex v by adding all possible arcs from v towards the first copy and from the second copy towards v. This graph is a partial 3-tree whose 15 vertices must be assigned distinct colors. We then join every vertex x of H_1 to two distinct copies of T_2 by adding all possible arcs from x towards the first copy and from the second copy towards x, and get a new partial 3-tree H_2 having 3255 vertices. Suppose that C is a color-graph having 15 vertices such that H_2 is C-colorable; every color c in C is such that both $\Gamma^+(C, c)$ and $\Gamma^-(C, c)$ are isomorphic to QR_7. We know by Theorem 19 that such a graph cannot exist. Hence, H_2 has chromatic number 16 and the result follows.

□

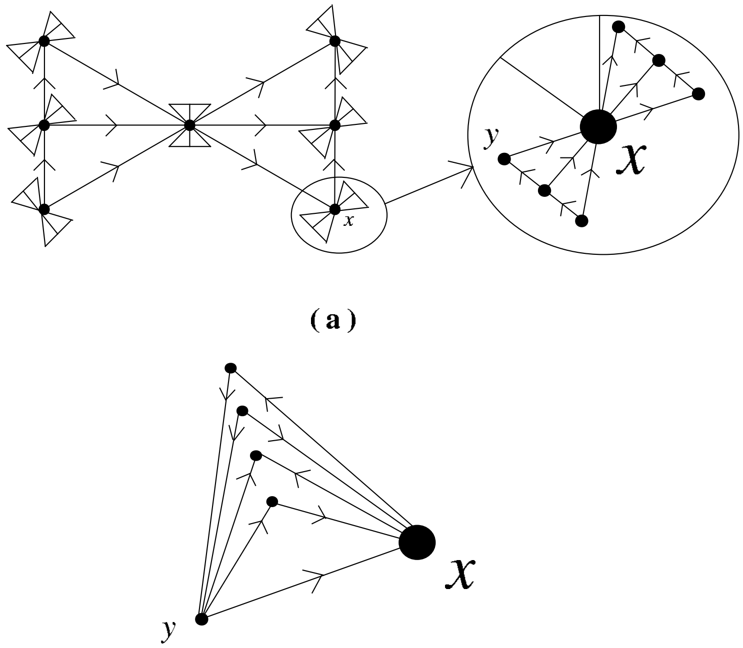

Figure 3: Construction of an oriented 3-tree with chromatic number 16

4 The chromatic number of oriented k-bounded graphs

We first prove the following general upper bound:

Theorem 22 *The family of oriented k-bounded graphs is Z_{2k-2}-colorable and thus has chromatic number at most* $(2k-1) \times 2^{2k-2}$.

Proof. We proceed by induction on the number of vertices n of an oriented k-bounded graph. If $n = 1$ there is nothing to prove. Let now G be an oriented k-bounded graph of order $n+1$. Let x be any vertex of G and let $y_1, y_2, \ldots, y_j$, $j \leq k$, denote the neighbours of x in G. By the induction hypothesis, $G - x$ is Z_{2k-2}-colorable. Since every y_i has degree at most $k-1$ in $G - x$, we can color these vertices in such a way that every two of them get distinct identities since at least k distinct identities may be chosen for each of them (recall here that the identity is the first component of a color in Z_{2k-2}). Finally, property P_{2k-2} ensures that we can find a color for vertex x. □

For small values of k, this result can easily be improved:

Theorem 23 *The family of oriented 2-bounded graphs has chromatic number 5 and is optimally colorable by the tournament T depicted in Figure 4.*

Proof. Oriented 2-bounded graphs are in fact oriented paths and oriented cycles. Every oriented path has chromatic number at most 3 since it is C_3-colorable. Every oriented cycle having at most 4 vertices is contained in T as a subgraph and is then T-colorable. Let us now show that every oriented cycle having at least 5 vertices is T-colorable.

Let $C = (x_1, x_2, \ldots, x_n)$ be any such oriented cycle. For every vertex x_i, we define the level of x_i, denoted $l(x_i)$, as the number of forward arcs minus the number of backward arcs along the path $x_1 x_2 \ldots x_i$, taken modulo 3 (we have $l(x_1) = 0$). We are going to color the path $x_1 x_2 \ldots x_{n-2}$ with

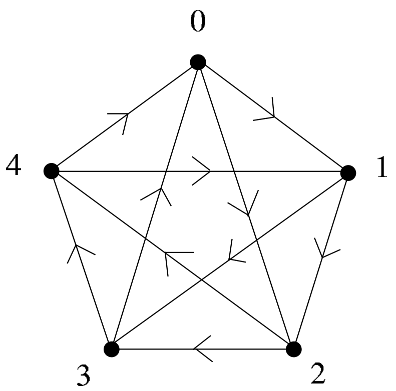

Figure 4: The tournament T

one of the three C_3-subgraphs of T induced by vertices 013, 014 or 024. Then, according to the level of x_{n-2}, and to the type of arcs in the path $x_{n-2}x_{n-1}x_nx_1$, we complete the coloring of C as shown in Figure 5 (other cases are obtained by considering the cycle C in the opposite direction). The C_3-subgraph we use is given in each case. Note that the color of any of the x_i's, $2 \leq i \leq n-2$ is uniquely determined by the color of x_1 (all vertices with same level are assigned the same color). $\square$

In order to prove that every oriented 3-bounded graph is T_{16}-colorable, we need the following lemma:

Lemma 24 *For every arc uv in T_{16} and every orientation vector α of size 2, there exist three distinct α-successors of uv.*

Proof. It is not difficult to check that the graph T_{16} is arc-transitive (for every two arcs xx' and yy' there exists an automorphism of T_{16} that maps x to y and x' to y'). Hence, it suffices to consider one arc, say 01. Note also that if x (resp. x') is a (α_1, α_2)-successor of uv, then x' (resp. x) is a $(1-\alpha_1, 1-\alpha_2)$-successor of uv. The (0,0)-successors of 01 are v, 6 and 2', and the (0,1)-successors of 01 are 3, 5 and 4', which completes the proof. $\square$

Theorem 25 *The family of oriented 3-bounded graphs is T_{16}-colorable and thus has chromatic number at most 16.*

Proof. Let G be an oriented 3-bounded graph. We proceed by induction on the order n of G. If $n = 1$, there is nothing to prove. Suppose the result is true for every oriented 3-bounded graph with order at most n. Let now G be an oriented 3-bounded graph with order $n+1$. If G has a vertex v with degree 1, then every T_{16}-coloring of $G - v$ can be extended to a T_{16}-coloring of G. If G has no vertex with degree 1 but a vertex v with degree 2, then we first color the graph G^*, obtained from $G - v$ by adding an arc linking the two neighbours of v if such an arc does not exist. Then every T_{16}-coloring of G^* can be extended to a T_{16}-coloring of G since T_{16} satisfies property P_3 (see Section 3). Finally, if G is 3-regular, let v be any vertex of degree 3. Since every neighbour of v has degree 2 in $G - v$, Lemma 24 ensures that we can construct a T_{16}-coloring of $G - v$ such that the colors of the three neighbours of v induce a 3-clique subgraph of T_{16} (since we have three possible choices for each of them, we can ensure that colors i and i' are not simultaneously chosen, for every $i \in \{0, \ldots, 6, v\}$). This coloring can then be extended to a T_{16}-coloring of G since T_{16} satisfies property P_3. $\square$

Theorem 25 improves a previous result given in [27]. We have recently proved with Laurence Vignal that every oriented 3-bounded graph has chromatic number at most 11 [28]. However, we do

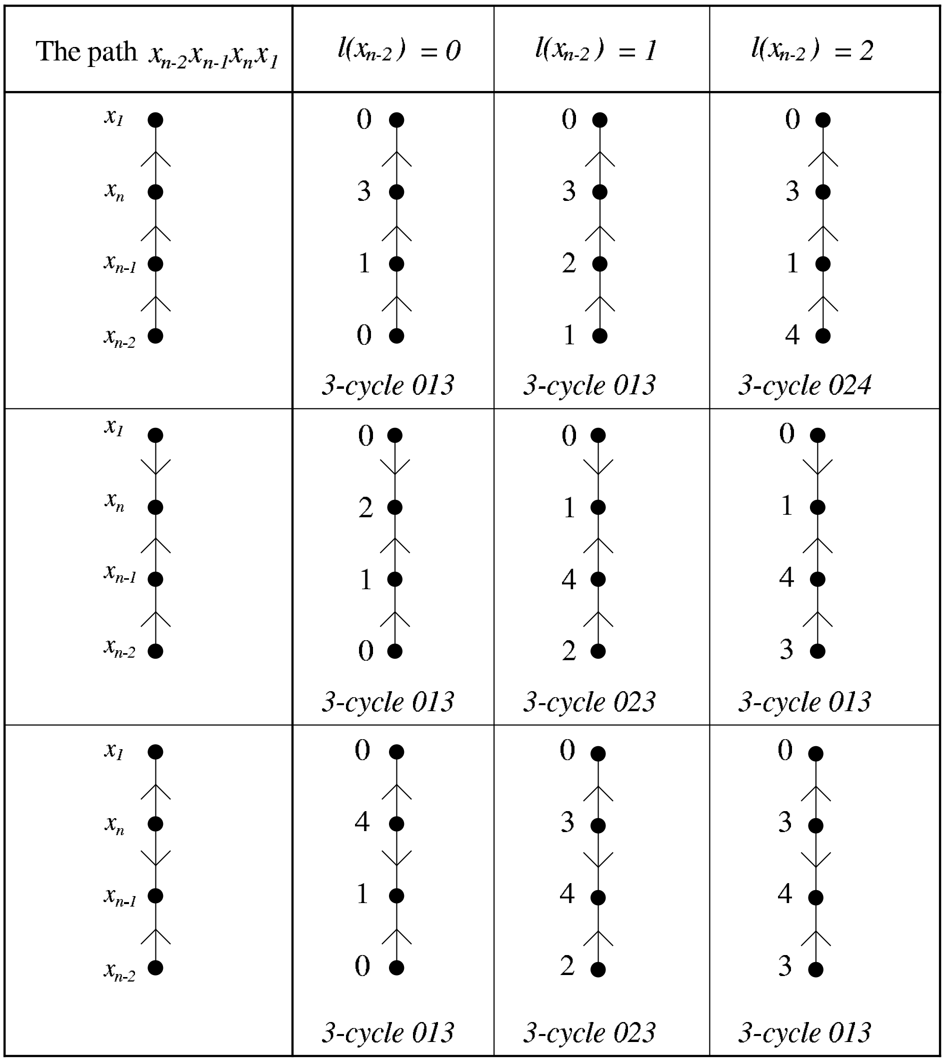

Figure 5: T-coloring of oriented cycles

not know any such connected graph with chromatic number greater than 7. Therefore, we conjecture:

Conjecture 26 *The family of oriented 3-bounded connected graphs has chromatic number 7.*

The family of oriented k-bounded graphs is obviously complete, and thus optimally colorable, since the disjoint union of two k-bounded graphs is still k-bounded. However, we do not know up to now whether the family of oriented connected k-bounded graphs is optimally colorable (the fact that this family is not complete is easily seen by considering the case of k-regular graphs). Hence, it would be possible that there exists no color-graph C on 7 vertices such that every oriented connected 3-bounded graph is C-colorable. In this case, the chromatic number of (not necessarily connected) 3-bounded graphs would be strictly greater than 7.

5 Conclusions

In this paper, we have introduced a new generalisation of the notion of chromatic number for oriented graphs. We have given upper bounds for the chromatic number of some families of graphs and a natural question is now to find optimal bounds for these values. In particular, if we denote by $f(k)$ the chromatic number of oriented k-bounded graphs, we do not know whether the function f is polynomial or exponential (recall that this function is linear in the undirected case). In the case of k-trees, the construction of optimal color-graphs satisfying property P_k is still an open problem for $k > 3$. In [2] we have constructed such color-graphs, for $k = 4$ and 5, having respectively 40 and 96 colors. For $k > 5$, the best known color-graphs are the graphs Z_k we have used in Section 3.

In [13], Häggkvist and Hell introduced the notion of A-mote graphs, that is graphs G such that there is no homomorphism $A \to G$. They proved that whenever the graph A is connected, the family $\mathcal{F}$ of all A-mote k-bounded graphs is universal (there exists a A-mote universal graph U, that is a color-graph in our terminology, such that every graph G in $\mathcal{F}$ admits a homomorphism to U). If A is the directed two-cycle, then A-mote k-bounded graphs are exactly oriented k-bounded graphs. Hence, the fact that the family of oriented k-bounded graphs is colorable was implicitly contained in their paper. They were essentially interested in minimising the maximum degree of universal graphs on the contrary to our approach in which we focus on the minimum order of universal graphs.

Finally, it should be noted that most of the graphs we have used as optimal color-graphs (excepted the tournament T in Theorem 23) play a central role in the theory of bounded timestamp systems [1, 2, 8, 19, 21, 33]. Timestamp systems have proven to be an useful tool in the design of many distributed algorithms [19]. An oriented graph G is a timestamp system of order k if every set $\{x_1, \ldots, x_n\}$, $n < k$, of vertices inducing a transitive tournament in G has a common successor. This property is obviously weaker than the property P_k we have considered: every graph satisfying P_k is a timestamp system of order $k+1$. However, most of the best known solutions for the timestamping problem satisfy property P_k. This is namely the case for the graphs Z_k we used in section 3, which have been introduced by Zielonka [33] as a generalisation of a construction proposed by Lamport [21]. The graph Z_4 (with 80 vertices) has also been used in [27] as a color-graph for the family of oriented planar graphs. Hence, any progress in the determination of the chromatic number of oriented partial k-trees would lead to immediate applications in the theory of timestamp systems.

Bibliographic note. Since the first version of this paper, Kostochka et al. proved in [20] that every oriented k-bounded graph has chromatic number at most $2k^2 2^k$ and that for every k there exist oriented k-bounded graphs with chromatic number at least $2^{\frac{k}{2}}$. The function f we discussed above is thus exponential. In [6], Borodin et al. improved Theorem 2 by showing that every oriented planar graph with girth 5 (resp. 6, 8, 11) has chromatic number at most 19 (resp. 11, 7, 5).

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