

# Singleton Theorem Using Models

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# Introduction

## Singleton Theorem [Statman'82]

For every lambda term  $M$ , there exists a finite standard model  $\mathcal{D}$  and a variable assignment  $\nu$  such that  $M$  is uniquely determined in  $\mathcal{D}$  and  $\nu$ .

**Motivation:** Standard models are strong enough to identify single terms (up to  $\beta, \eta$ -reductions).

**Method:** Construction of  $\mathcal{D}$  for  $M$  by induction on the Böhm tree of  $M$ .

# Simply typed $\lambda$ terms

## Types $\tau$

$$\tau ::= 0 \mid \tau \rightarrow \tau$$

## Terms

- **Variables:**  $x^\alpha, y^\alpha, \dots$
- **$\lambda$ -abstraction:**  $\lambda x^\alpha. M^\beta$
- **Application:**  $MN : \beta$ ; if  $M : \alpha \rightarrow \beta$  and  $N : \alpha$

## Remarks

- We can have more than one basic type.
- Constants can be added without any problems.

# Standard Models

## Standard Finite Model $\mathcal{D} = (D_\alpha)_{\alpha \in \tau}$

- $D_0$ : a finite set of elements of the basic type.
- $D_{\alpha \rightarrow \beta}$ : the set of functions from  $D_\alpha$  to  $D_\beta$ .

## Variable assignment

A **variable assignment** is a function  $v$  associating to a variable of type  $\alpha$  an element of  $D_\alpha$ .

Notation:  $v[d/x^\alpha]$ .

# Interpretation

## Interpretation

Interpretation of a term  $M$  of type  $\alpha$  in a model  $\mathcal{D}$  and variable assignment  $v$   
 $\llbracket M \rrbracket_{\mathcal{D}}^v \in D_{\alpha}$ :

- $\llbracket x^{\alpha} \rrbracket_{\mathcal{D}}^v = v(x^{\alpha})$
- $\llbracket MN \rrbracket_{\mathcal{D}}^v = \llbracket M \rrbracket_{\mathcal{D}}^v \llbracket N \rrbracket_{\mathcal{D}}^v$
- $\llbracket \lambda x^{\alpha}. M \rrbracket_{\mathcal{D}}^v$  is a function mapping an element  $d \in D_{\alpha}$  to  $\llbracket M \rrbracket_{\mathcal{D}}^{v[d/x^{\alpha}]}$ .
- **$\beta$ -reduction**  $(\lambda x. M)N \rightarrow_{\beta} M[N/x]$ .
- **$\eta$ -reduction**  $\lambda x. Mx \rightarrow_{\eta} M$ , provided  $x$  is not free in  $M$ .

## $\eta$ -long form

Using  $\lambda$  to make the functions explicit:

$$\lambda x^{\alpha}. z^{\alpha \rightarrow \beta} x \quad \text{instead of} \quad z^{\alpha \rightarrow \beta}$$

# Böhm Trees

Observe that a term in a  $\beta$ -normal, and  $\eta$ -long form is of a shape:

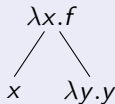
$$\lambda \vec{x}. z M_1 \dots M_k,$$

where  $z$  is a variable,  $z M_1 \dots M_k : 0$ , and the sequence  $\lambda \vec{x}$  may be empty.

## Böhm Trees

If  $M = \lambda \vec{x}. z M_1 \dots M_k$ , then the root of  $BT(M)$  is labeled  $\lambda \vec{x}. z$  and has  $BT(M_1), \dots, BT(M_k)$  as its children.

Example:  $\lambda x. (f \ x \ (\lambda y. y))$



## Remark

$BT(M)$  is a particular way of representing terms in a normal form as a tree.

# Statement of the Theorem

## Uniquely determined

$M$  is said to be *uniquely determined* in a model  $\mathcal{D}$  with a variable assignment  $\nu$  if for all lambda terms  $N$ ,  $\llbracket N \rrbracket_{\mathcal{D}}^{\nu} = \llbracket M \rrbracket_{\mathcal{D}}^{\nu}$  iff  $N =_{\beta\eta} M$ .

## Singleton Theorem [Statman'82]

For every lambda term  $M$ , there exists a standard finite model  $\mathcal{D}$  and a variable assignment  $\nu$  such that  $M$  is uniquely determined in  $\mathcal{D}$  and  $\nu$ .

# Basic Idea

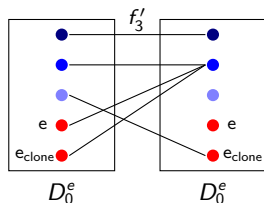
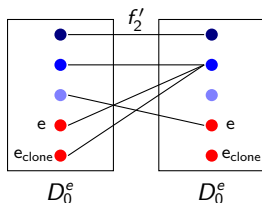
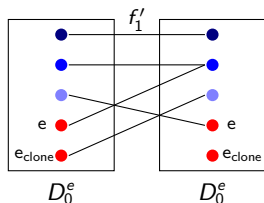
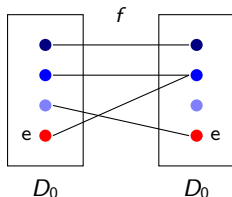
- We consider a lambda term  $M$  in  $\eta$ -long normal form.
- We assume that we have a model  $\mathcal{D}$  and an interpretation in which all subterms of  $M$  are uniquely determined.
- We add “an element” to  $\mathcal{D}$ , and alter the interpretation to make  $M$  uniquely determined too.

# The Extended Model

## Model $\mathcal{D}^e$

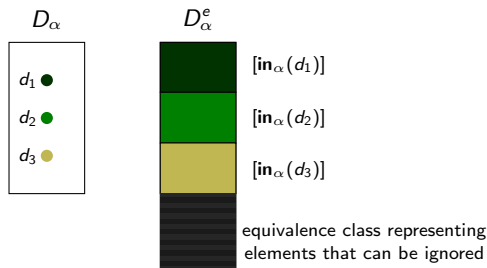
Given a model  $\mathcal{D} = (D_\alpha)_{\alpha \in \tau}$  and an element  $e \in D_0$  the **extended model**  $\mathcal{D}^e = (D_\alpha^e)_{\alpha \in \tau}$  is determined by:

$$D_0^e = D_0 \uplus \{e_{clone}\}$$



# Visualizing a set $D_\alpha^e$

In general, we would like to visualize each set  $D_\alpha^e$  as follows

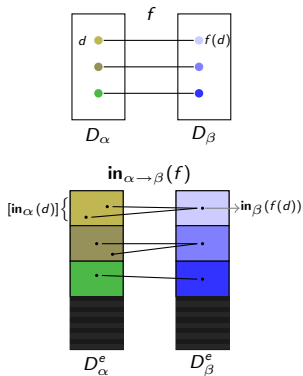


- $\text{in}_\alpha$  represents the injection function, and
- $[d']$  denotes the equivalence class of  $d' \in D_\alpha^e$ .

A null element  $h_0$  is any arbitrary element of  $D_0^e$  different from  $e_{clone}$ . For a type  $\alpha \rightarrow \beta$ , element  $h_{\alpha \rightarrow \beta}$  is the constant function mapping every element to  $h_\beta$ .

## Definition $\mathbf{in}_0$ and $\leftrightarrow_0$

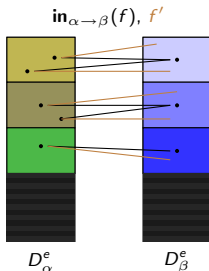
- $\mathbf{in}_0 : D_0 \rightarrow D_0^e$  is the identity.
- $\leftrightarrow_0$  is the smallest equivalence containing  $e \leftrightarrow_0 e_{clone}$ .



## Definition $\mathbf{in}_{\alpha \rightarrow \beta}$

- If  $f \in D_{\alpha \rightarrow \beta}$  then  $\mathbf{in}_{\alpha \rightarrow \beta}(f)$  is  $f' \in D_{\alpha \rightarrow \beta}^e$  such that:

$$f'(d') = \begin{cases} \mathbf{in}_\beta(f(d)) & \text{if } d' \in [\mathbf{in}_\alpha(d)] \\ h_\beta & \text{otherwise} \end{cases}$$



## Equivalence relation

- We say that  $f' \in D_{\alpha \rightarrow \beta}^e$  **simulates**  $f \in D_{\alpha \rightarrow \beta}$  ( $\text{sim}(f', f)$ ) if for all  $d \in D_\alpha$ , for all  $d' \in [\text{in}_\alpha(d)]: f'(d') \leftrightarrow_\beta \text{in}_\beta(f(d))$
- For  $f', g' \in D_{\alpha \rightarrow \beta}^e$ , we have

$$f' \leftrightarrow_{\alpha \rightarrow \beta} g' \quad \text{if for all } h \in D_{\alpha \rightarrow \beta}, \text{sim}(f', h) \Leftrightarrow \text{sim}(g', h).$$

## Observation

For every  $d_1, d_2 \in D_\alpha$ , if  $d_1 \neq d_2$ , then  $\text{in}_\alpha(d_1) \not\leftrightarrow_\alpha \text{in}_\alpha(d_2)$ .

## Definition

A variable assignment  $v'$  on  $\mathcal{D}^e$  *simulates* a variable assignment  $v$  on  $\mathcal{D}$  if for all variables  $x$ :  $\text{sim}(v'(x), v(x))$ .

## Lemma

If  $v'$  *simulates*  $v$  then for every lambda term  $M$ :

$$\text{sim}(\llbracket M \rrbracket_{\mathcal{D}^e}^{v'}, \llbracket M \rrbracket_{\mathcal{D}}^v)$$

where  $\alpha$  is the type of  $M$ .

## Corollary

Every term uniquely determined in  $(\mathcal{D}, v)$  is uniquely determined in  $(\mathcal{D}^e, v')$ .

# Proof of the Singleton Theorem

Consider a lambda term  $\lambda \vec{x}. yM_1 \dots M_k$ , with  $yM_1 \dots M_k$  of type 0.

## Assume

- $M_1, \dots, M_k$  are uniquely determined in a model  $\mathcal{D}$  and a variable assignment  $v$ ,
- $\llbracket yM_1 \dots M_k \rrbracket_{\mathcal{D}}^v = e$ .

Construct the model  $\mathcal{D}^e$  by adding  $e_{clone}$ .

## Variable assignment $v^e$

- 1  $v^e(x) = \mathbf{in}_{\tau(x)}(v(x))$ , if  $x \neq y$ .
- 2 For the variable  $y$ ,

$$v^e(y)(d'_1, \dots, d'_k) = \begin{cases} e_{clone} & \text{if } d'_i \in [\mathbf{in}_{\beta_i}(\llbracket M_i \rrbracket_{\mathcal{D}}^v)], \\ & \text{for } i \in \{1, \dots, k\} \\ \mathbf{in}_{\tau(y)}(v(y))(d'_1, \dots, d'_k) & \text{otherwise} \end{cases}$$

As  $v^e$  simulates  $v$  we have:

- For all lambda terms  $N$ ,  $\text{sim}(\llbracket N \rrbracket_{\mathcal{D}^e}^{v^e}, \llbracket N \rrbracket_{\mathcal{D}}^v)$ , that is,  $\llbracket N \rrbracket_{\mathcal{D}^e}^{v^e} \leftrightarrow_{\beta_i} \mathbf{in}_{\beta_i}(\llbracket N \rrbracket_{\mathcal{D}}^v)$
- So  $M_1, \dots, M_k$  are uniquely determined in  $(\mathcal{D}^e, v^e)$
- Moreover,  $\llbracket yM_1 \dots M_k \rrbracket_{\mathcal{D}^e}^{v^e} = e_{\text{clone}}$ .

## Uniqueness

Let  $\llbracket wN_1 \dots N_p \rrbracket_{\mathcal{D}^e}^{v^e} = e_{\text{clone}}$ .

- $w \neq y$  is not possible.
- when  $w = y$  we get:

$$\begin{aligned} \llbracket N_i \rrbracket_{\mathcal{D}^e}^{v^e} &\in [\mathbf{in}_{\beta_i}(\llbracket M_i \rrbracket_{\mathcal{D}}^v)] \\ &\Rightarrow \mathbf{in}_{\beta_i}(\llbracket N_i \rrbracket_{\mathcal{D}}^v) \leftrightarrow_{\beta_i} \mathbf{in}_{\beta_i}(\llbracket M_i \rrbracket_{\mathcal{D}}^v) \\ &\Rightarrow \llbracket N_i \rrbracket_{\mathcal{D}}^v = \llbracket M_i \rrbracket_{\mathcal{D}}^v \\ &\Rightarrow N_i = M_i \end{aligned}$$

$yM_1 \dots M_k$  uniquely determined implies  $\lambda \vec{x}. yM_1 \dots M_k$  is uniquely determined.

## Base Case

Leaf is a variable  $z$  of type 0.

- Start: trivial model with only one element  $\{\perp\}$  in its atomic set, trivial variable assignment.
- Add an extra element  $\{\perp_{clone}\}$  to type 0.
- New variable assignment assigns  $z$  to  $\perp_{clone}$  and the rest is kept same.

# Conclusions

- In our approach we
  - ▶ define an operation of model extension, and
  - ▶ explain the relation between elements of the initial and extended model.
- We work mostly with semantics, the only syntactic tool is  $\eta$ -long forms (and Böhm trees).

## Related Work:

- ▶ [Statman'82] Finite Completeness Theorem
- ▶ [Statman & Dowek'92]
- ▶ [Salvati'07] Using intersection types