

# Acceleration in Convex Data-Flow Analysis

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Theoretical Computer Science, 2007

## Invariants for Verification

- Verification of safety properties
- Efficient computation of precise enough invariants
- Data-flow analysis, abstract interpretation
- Widenings/narrowings: successful approach, but might lead to invariants too coarse for verification

## Our Objective

Computation of the **exact solution** to data-flow analysis problems

- Meet Over all Paths
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- 1 Introduction
- 2 Convex Data Flow Analysis of Guarded Translation Systems
- 3 Acceleration for Self-Loops
- 4 Acceleration for Cycles
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# Guarded Translation Systems (Syntax)

- We focus on **topologically closed convex** subsets of  $\mathbb{R}^n$
- $\{\vec{x} \mid A\vec{x} \leq \vec{b}\}$  is called a  $\begin{cases} \text{(real) polyhedron} & \text{when } A \in \mathbb{R}^{n \times m} \\ \text{rational polyhedron} & \text{when } A \in \mathbb{Q}^{n \times m} \end{cases}$
- guarded commands of the form **if**  $\vec{x} \in G$  **then**  $\vec{x} := \vec{x} + \vec{d}$

## Definition

An  $n$ -dim **guarded translation system (GTS)** is any pair  $S = (\mathcal{X}, T)$  where:

- $\mathcal{X}$  is a finite set of **variables**
- $T$  is a finite set of **transitions** of the form  $X \xrightarrow{G, \vec{d}} X'$
- Transition  $X \xrightarrow{G, \vec{d}} X'$  represents the assignment  $X' := (X \cap G) + \vec{d}$

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# Guarded Translation Systems (Semantics)

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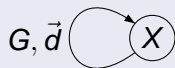
- $\mathcal{X}$  is a finite set of variables
- $T$  is a finite set of transitions of the form  $X \xrightarrow{G, \vec{d}} X'$
- Valuation: function  $\rho$  from  $\mathcal{X}$  to closed convex subsets of  $\mathbb{R}^n$
- Semantics  $\llbracket t \rrbracket$  of transition  $t = X \xrightarrow{G, \vec{d}} X'$  defined by:

$$(\llbracket t \rrbracket(\rho))(Y) = \begin{cases} (\rho(X) \cap G) + \vec{d} & \text{if } Y = X' \\ \rho(Y) & \text{if } Y \neq X' \end{cases}$$

- An  $n$ -dim initialized GTS (IGTS) is any triple  $\mathcal{S} = (\mathcal{X}, T, \rho_0)$

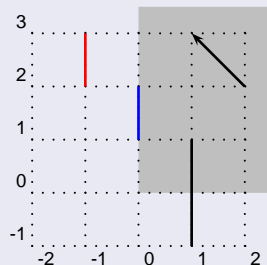
# Example

## IGTS



- $\mathcal{X} = \{X\}$
- $T = \{X \xrightarrow{G, \vec{d}} X\}$  with
$$\begin{cases} G &= \mathbb{R}_+^2 \\ \vec{d} &= (-1, 1) \end{cases}$$
- $\rho_0 = \{X \mapsto 1 \times [-1, 1]\}$

## Semantics



- $\rho_1 = \llbracket t \rrbracket (\rho_0)$
- $\rho_2 = \llbracket t \rrbracket (\rho_1)$
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# Convex Data-Flow Analysis

- We consider the complete lattice of convex closed subsets of  $\mathbb{R}^n$ .
  - partial order is set inclusion  $\subseteq$
  - greatest lower bound is set intersection  $\cap$
  - least upper bound  $\sqcup$  is set union followed by closed convex hull
- Extended to valuations

## Minimum Fix-Point (MFP) Solution

$$\text{MFP}(S) = \bigcap \{ \rho : \text{valuation} \mid \rho_0 \subseteq \rho \text{ and } \llbracket t \rrbracket(\rho) \subseteq \rho \text{ for all } t \in T \}$$

- MFP( $S$ ) is the least fix-point of  $\tau(\rho) = \rho_0 \sqcup \bigsqcup_{t \in T} \llbracket t \rrbracket(\rho)$ .
- Kleene fix-point iteration:  $\bigsqcup_{i \in \mathbb{N}} \tau^i(\perp) \subseteq \text{MFP}(S)$

## Meet Over all Paths (MOP) Solution

$$\text{MOP}(S) = \bigsqcup \{ \llbracket t_1 \rrbracket \circ \dots \circ \llbracket t_k \rrbracket(\rho_0) \mid t_1 \dots t_k \in T^* \text{ is a path} \}$$

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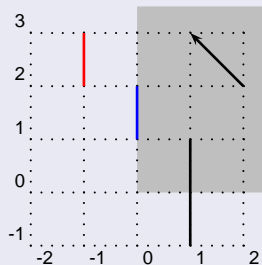
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## MOP Solution



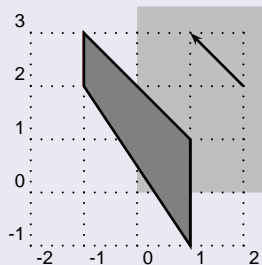
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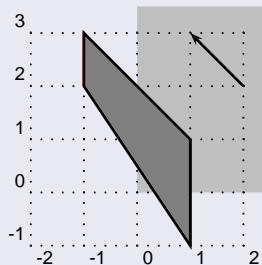
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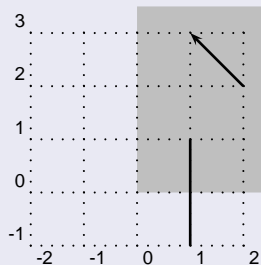
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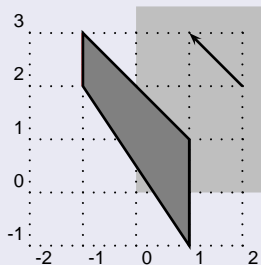
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$$\tau^1(\perp) = \rho_0$$

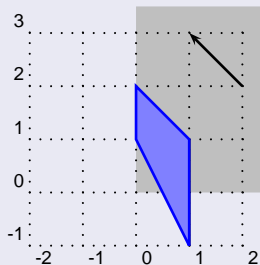
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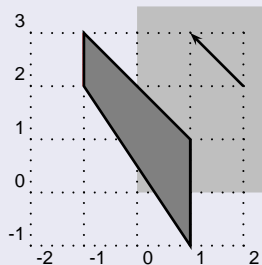
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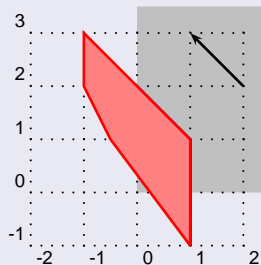
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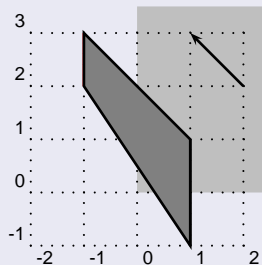
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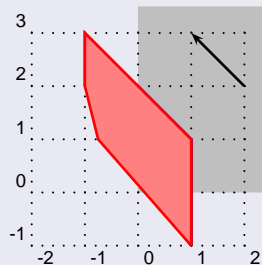
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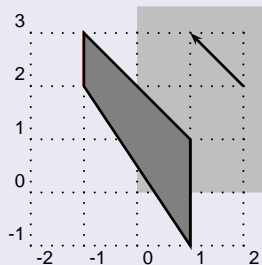
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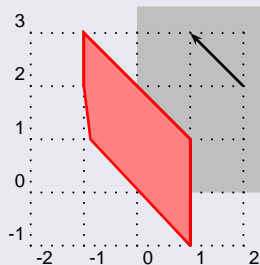
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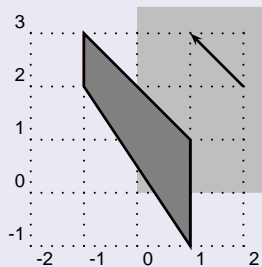
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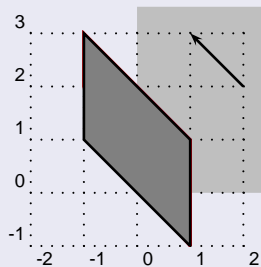
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## MFP Solution



## Remark

Kleene fix-point iteration  
does not stabilize

## Objectives

- Speed up Kleene fix-point iteration
- Don't lose precision

## Minimum Fix-Point Computation with Acceleration

- 1 do  $\rho \leftarrow \rho \sqcup \llbracket t \rrbracket(\rho)$  for some transition  $t = X \xrightarrow{G, \vec{d}} X'$
- 2 or select a cycle in  $\mathcal{S}$  and:
  - 1 let  $\mathcal{S}'$  denote the cyclic sub-IGTS, initialized with  $\rho$  (restricted to  $\mathcal{X}'$ )
  - 2  $\rho \leftarrow \rho \sqcup \text{MFP}(\mathcal{S}')$  (or  $\rho \leftarrow \rho \sqcup \text{MOP}(\mathcal{S}')$ )

## Acceleration Problem

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# MFP Solution for Self-Loop IGTS

## Theorem

For any  $n$ -dim self-loop IGTS  $(\{X\}, \{X \xrightarrow{G, \vec{d}} X\}, \rho_0)$ , the MFP solution is the valuation:

$$X \mapsto \begin{cases} \rho_0(X) & \text{if } G \cap \rho_0(X) = \emptyset \\ \rho_0(X) \sqcup ((G \cap (\rho_0(X) + \mathbb{R}_+ \vec{d})) + \vec{d}) & \text{otherwise} \end{cases}$$

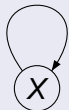
## Proof Ideas

-  The given expression is a post-fix-point of  $\left[ X \xrightarrow{G, \vec{d}} X \right]$ .
-  Proof by contradiction, using topological and convexity properties of both the guard and MFP solution.

# Comparison with Standard Widening on Polyhedra [Cousot & Halbwachs, POPL'78]

## IGTS

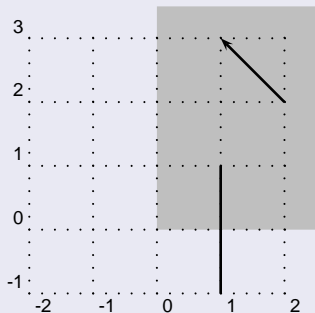
$\mathbb{R}_+^2, (-1, 1)$



$$\rho_0 = \{X \mapsto 1 \times [-1, 1]\}$$

- Application of widening
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## Iteration with Widening



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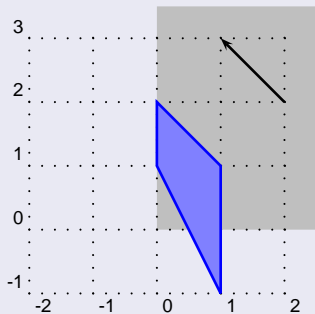
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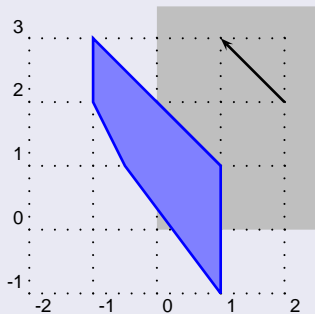
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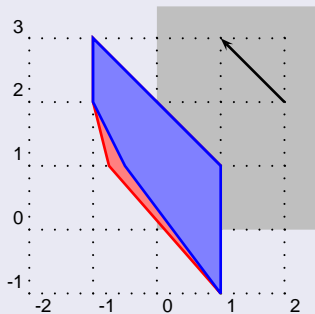
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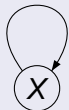
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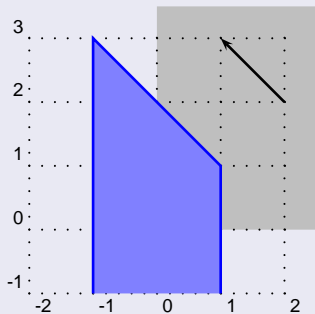
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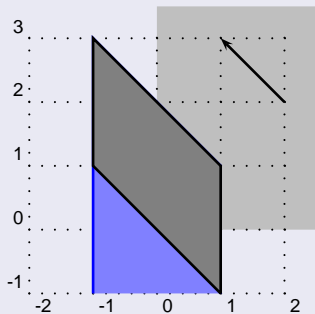
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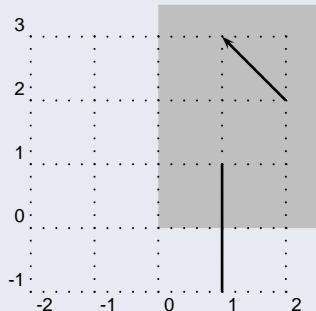
# Comparison with Polyhedral Abstract Acceleration [Gonnord & Halbwachs, SAS'06]

- Consider an IGTS  $\mathcal{S} = (\{X\}, \{X \xrightarrow{G, \vec{d}} X\}, \rho_0)$

## Abstract Acceleration

$\text{AbAc}(\mathcal{S}) = \rho_0(X) \sqcup \text{MFP}(\mathcal{S}')$   
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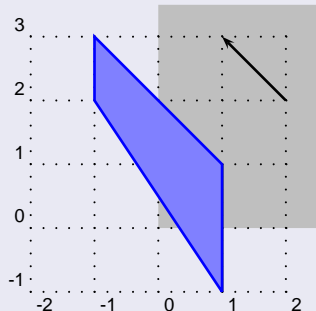
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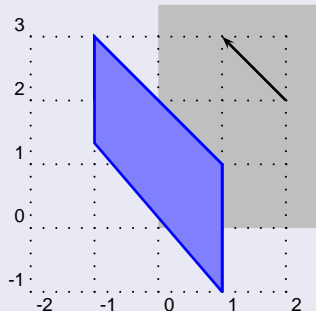
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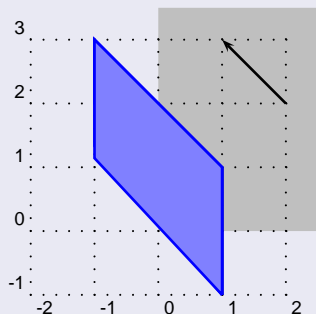
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## Remark

Iteration does not terminate!

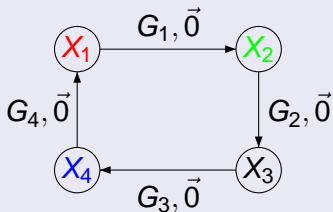
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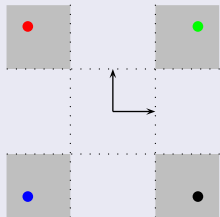
# 2-dim Example

## GTS



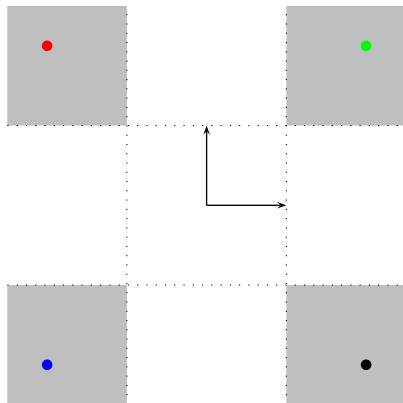
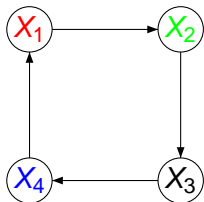
$$\begin{aligned}G_1 &= ]-\infty, -1] \times [1, +\infty[ \\G_2 &= [1, +\infty[ \times [1, +\infty[ \\G_3 &= [1, +\infty[ \times ]-\infty, -1] \\G_4 &= ]-\infty, -1] \times ]-\infty, -1]\end{aligned}$$

## Initial Valuation

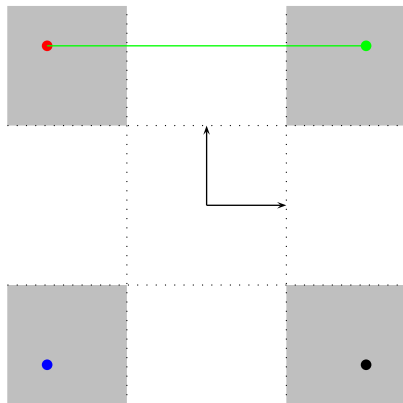
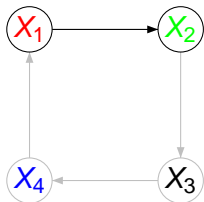


$$\begin{aligned}X_1 &\mapsto \{(-2, 2)\} \\X_2 &\mapsto \{(2, 2)\} \\X_3 &\mapsto \{(2, -2)\} \\X_4 &\mapsto \{(-2, -2)\}\end{aligned}$$

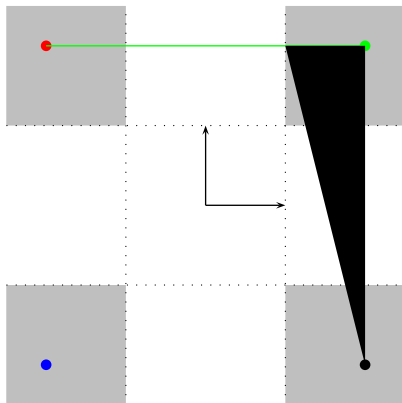
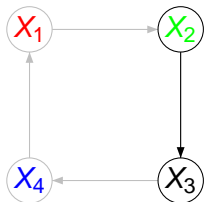
# Kleene iteration on 2-dim Example



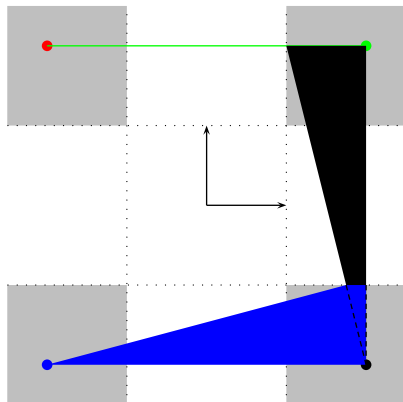
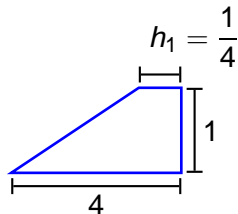
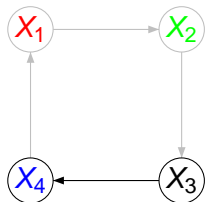
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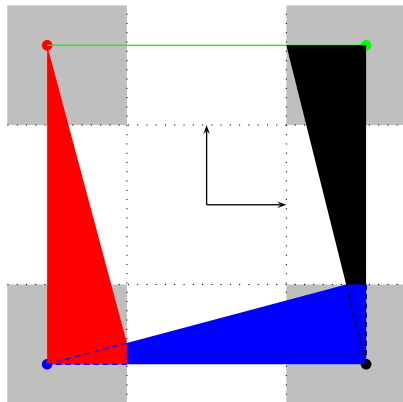
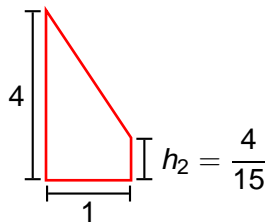
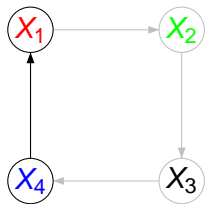
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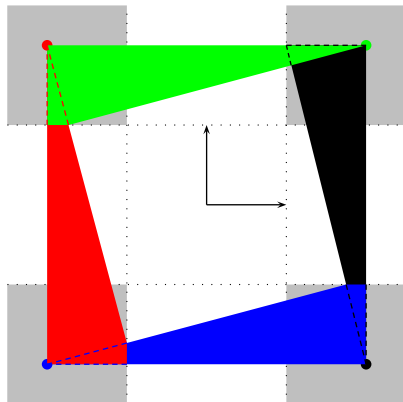
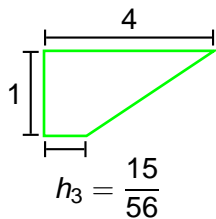
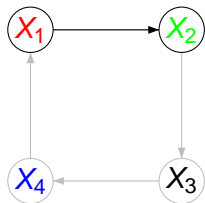
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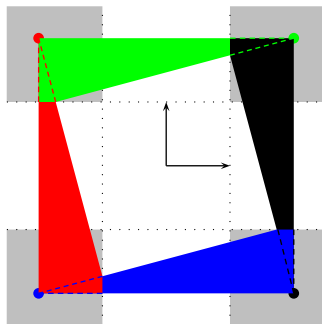
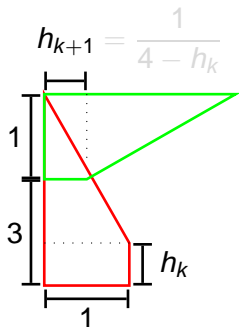
# Kleene iteration on 2-dim Example



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# MFP Solution for 2-dim Example

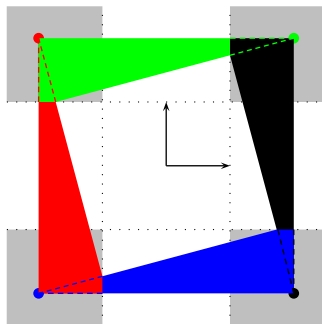
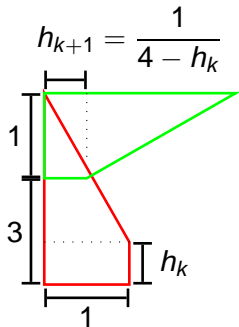


- $(h_k)_{k \in \mathbb{N}}$  is nondecreasing, and  $\lim_{k \rightarrow \infty} h_k = 2 - \sqrt{3}$

## Remark

The MFP solution of this 2-dim cyclic IGTS is not rational polyhedral

# MFP Solution for 2-dim Example

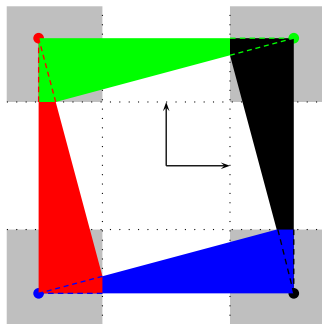
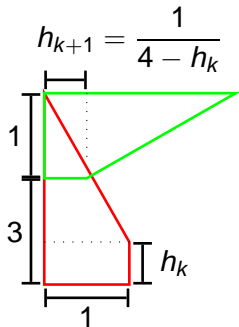


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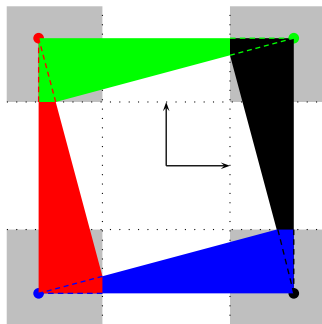
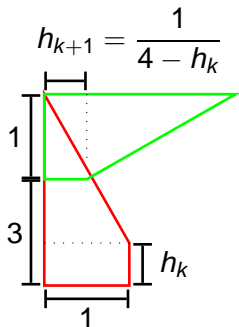


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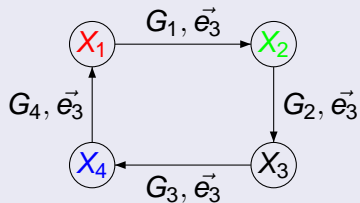
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# 3-dim Example

## GTS



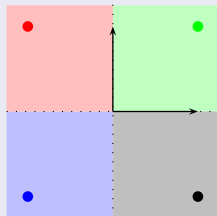
$$G_1 = \mathbb{R}_- \times \mathbb{R}^+ \times \mathbb{R}$$

$$G_2 = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$$

$$G_3 = \mathbb{R}^+ \times \mathbb{R}_- \times \mathbb{R}$$

$$G_4 = \mathbb{R}^- \times \mathbb{R}^- \times \mathbb{R}$$

## Initial Valuation



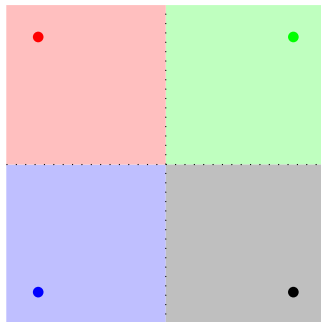
$$X_1 \mapsto \{(-1, 1)\} \times \mathbb{R}^+$$

$$X_2 \mapsto \{(1, 1)\} \times \mathbb{R}^+$$

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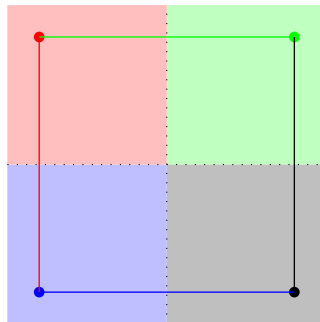
# Kleene iteration on 3-dim Example



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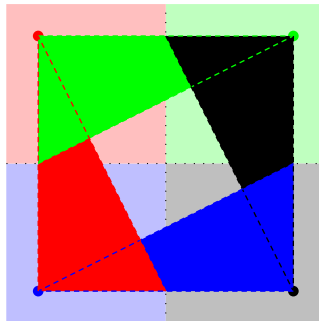
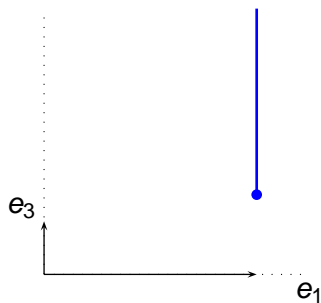
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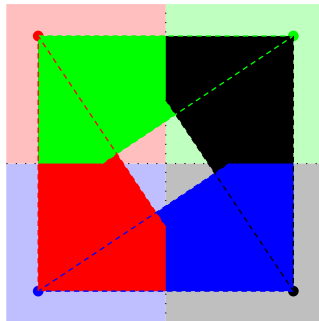
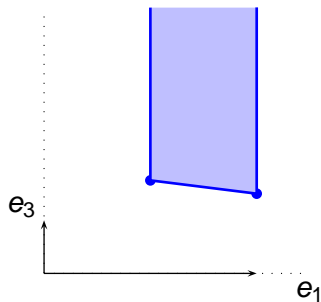
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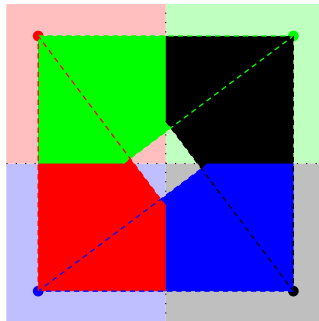
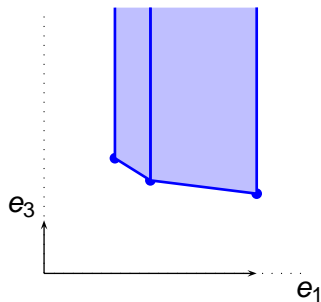
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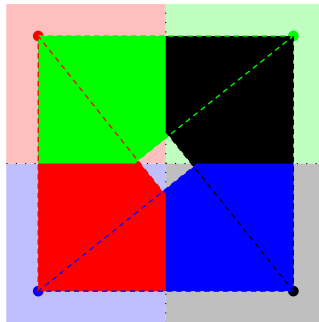
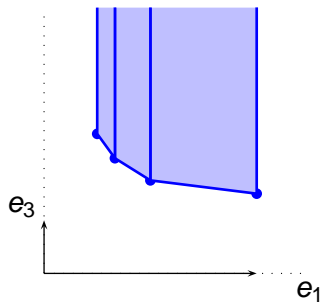
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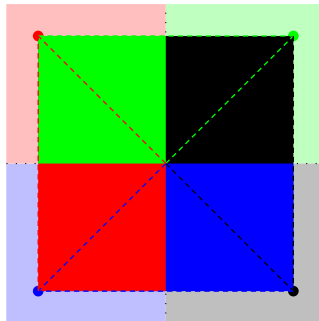
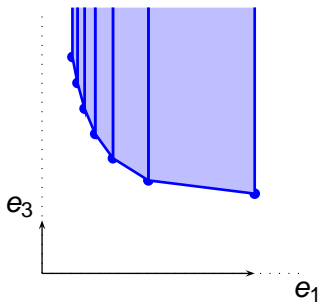
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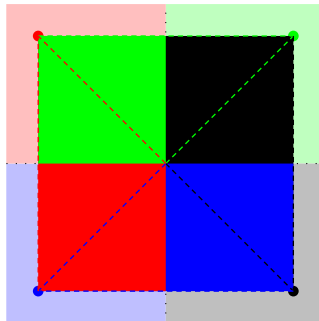
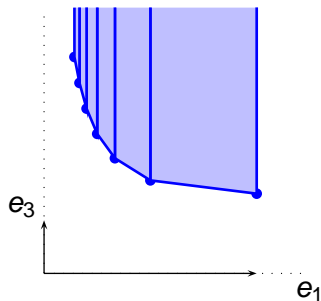
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# Acceleration Results for Cycles

- 2-dim cyclic example with a real (non rational) polyhedral MFP solution
- 3-dim cyclic example with a non-polyhedral MFP solution

## Question

Is the MFP polyhedral for all 2-dim cyclic IGTS?

## Theorem

*The MFP solution of any 2-dim IGTS is an algebraic polyhedron.*

- An algebraic number is any real number definable in  $\langle \mathbb{R}, +, \cdot, \leq \rangle$
- Algebraic polyhedrality is required even for cyclic 2-dim IGTS

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## MFP Solution Expression

$$\text{MFP}(X) = \bigsqcup_{\substack{X_0 \in \mathcal{X} \\ t_1 \cdots t_k \in L_{X_0, X}}} \llbracket t_1 \rrbracket \circ \cdots \circ \llbracket t_k \rrbracket (\Delta(X_0)) + \boxed{0^+ \text{MFP}(X)}$$

where:

$$\Delta(X) = \rho_0(X) \sqcup \bigsqcup_{X \xrightarrow{G, \vec{d}} X'} \boxed{\text{bd}(G) \cap \text{MFP}(X)}$$

- $\text{bd}(G)$  is the topological boundary of  $G$
- $L_{X_0, X}$  is the set of simple paths from  $X_0$  to  $X$
- $0^+ C = \{\vec{d} \mid C + \mathbb{R}_+ \vec{d} \subseteq C\}$

## Proof (2)

Observe that  $0^+ \text{MFP}(X)$  is a **cone** in dimension 2.

### $0^+ \text{MFP}(X)$

There exists  $\vec{d}_1, \vec{d}_2, \vec{d}_3 \in \mathbb{R}^2$  such that:

$$0^+ \text{MFP}(X) = \mathbb{R}_+ \vec{d}_1 + \mathbb{R}_+ \vec{d}_2 + \mathbb{R}_+ \vec{d}_3$$

Reduce to the case  $G$  is an half-space.

$\implies$   $\text{bd}(G)$  is a **line**.

### $\text{bd}(G) \cap \text{MFP}(X)$

There exists two half-spaces  $H_1, H_2$  such that:

$$\text{bd}(G) \cap \text{MFP}(X) = \text{bd}(G) \cap H_1 \cap H_2$$

Therefore the MFP solution is definable by a formula in  $\langle \mathbb{R}, +, \cdot, \leq \rangle$ .

- 1 Introduction
- 2 Convex Data Flow Analysis of Guarded Translation Systems
- 3 Acceleration for Self-Loops
- 4 Acceleration for Cycles
- 5 Conclusion**

# Summary

		Guarded Translation Systems		
		Self-loops	Cyclic	General
MOP	$n \geq 1$	Rational Poly.	Rational Poly.	Not Polyhedral
MFP	1	Rational Poly.	Rational Poly.	Rational Poly.
	2	Rational Poly.	Algebraic Poly.	Algebraic Poly.
	$n \geq 3$	Rational Poly.	Not Polyhedral	Not Polyhedral

- Polyhedra are computable for **Rational Poly.** and **Algebraic Poly.**
- Results on self-loops carry over to singly initialized cycles

## Related Work

- Interval analysis [Su & Wagner, TACAS'04], [Seidl & Gawlitza, ESOP'07], [L. & S., SAS'07]
- Abstract acceleration for convex polyhedra [Gonnord & Halbwachs, SAS'06]
  - Acceleration technique for two self-loops, operations include reset
  - Incomplete for single self-loops

## Future Work

- Multiple self-loops
- Other abstract lattices
  - octogons [Miné, AST'01]
  - templates [Sankaranarayanan *et al.*, VMCAI'05]
  - two variables per linear inequality [Simon *et al.*, LOPSTR'02]

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