

Range Analysis with Widening and Narrowing

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Complete Lattice \mathcal{Z} : Extension of \mathbb{Z} with $-\infty$ and $+\infty$

Let $\mathcal{Z} = \mathbb{Z} \cup \{-\infty, +\infty\}$ and define the partial order \leq on \mathcal{Z} with:

$$-\infty < \dots < -2 < -1 < 0 < 1 < 2 < \dots < +\infty$$

(\mathcal{Z}, \leq) is a **complete lattice**

- ▶ Least element is $-\infty$ and greatest element is $+\infty$.
- ▶ Least upper bound **$\sup X$** and greatest lower bound **$\inf X$** satisfy:

$$\sup X = \begin{cases} -\infty & \text{if } X = \emptyset \\ \max(X) & \text{if it exists} \\ +\infty & \text{otherwise} \end{cases} \quad \inf X = \begin{cases} +\infty & \text{if } X = \emptyset \\ \min(X) & \text{if it exists} \\ -\infty & \text{otherwise} \end{cases}$$

Abstract Lattice of Intervals

$$-\infty < \dots < -2 < -1 < 0 < 1 < 2 < \dots < +\infty$$

$$\text{Int} = \{\perp\} \cup \{(l, u) \in (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{+\infty\}) \mid l \leq u\}$$

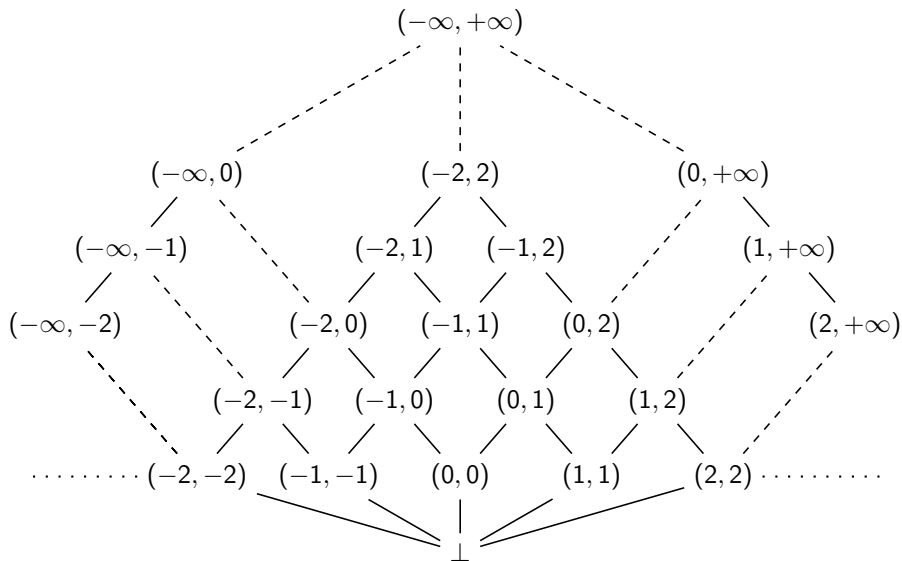
Define the binary relation \sqsubseteq on Int as follows:

$$\begin{aligned} \perp &\sqsubseteq \perp & \perp &\sqsubseteq (l, u) \not\sqsubseteq \perp \\ (l_1, u_1) &\sqsubseteq (l_2, u_2) & \text{if } &l_2 \leq l_1 \leq u_1 \leq u_2 \end{aligned}$$

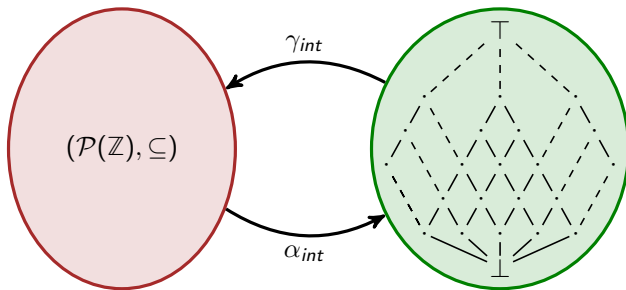
$(\text{Int}, \sqsubseteq)$ is a **complete lattice**

- ▶ Least element is \perp and greatest element is $(-\infty, +\infty)$.
- ▶ $\bigsqcup X$ is either \perp or $(\inf \{l \mid (l, u) \in X\}, \sup \{u \mid (l, u) \in X\})$.
- ▶ $\bigsqcap X$ is either \perp or $(\sup \{l \mid (l, u) \in X\}, \inf \{u \mid (l, u) \in X\})$.

Abstract Lattice of Intervals



Interpretation of Abstract Intervals: Galois Connection

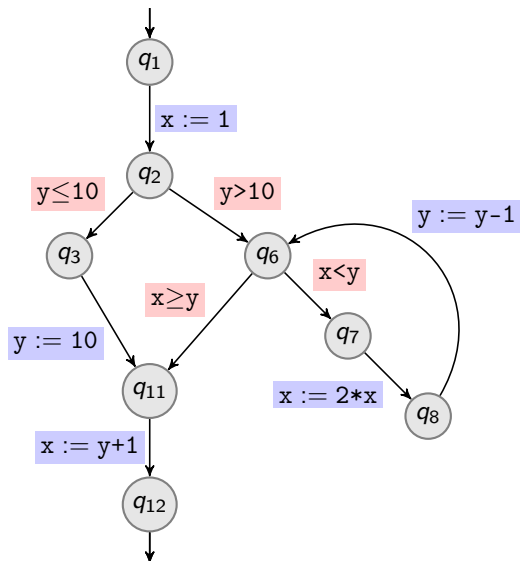


$$\alpha_{int}(S) = \begin{cases} \perp & \text{if } S = \emptyset \\ (\inf S, \sup S) & \text{otherwise} \end{cases}$$

$$\gamma_{int}(\perp) = \emptyset$$

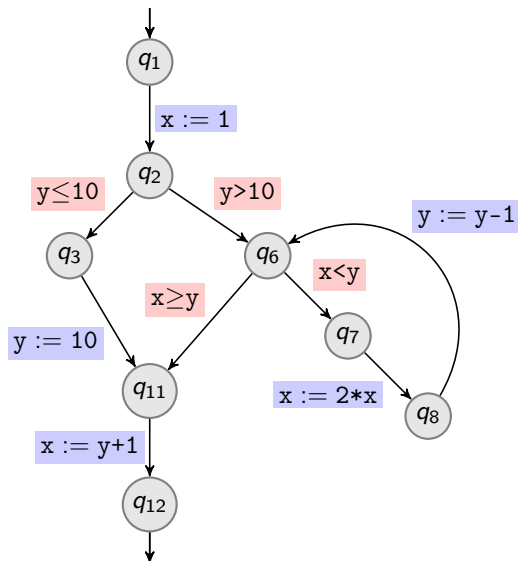
$$\gamma_{int}((l, u)) = \{z \in \mathbb{Z} \mid l \leq z \leq u\}$$

Range Analysis on Running Example (Round-Robin)



	x		y	
q_1	\perp	\perp	\perp	\perp
q_2	\perp	\perp	\perp	\perp
q_3	\perp	\perp	\perp	\perp
q_6	\perp	\perp	\perp	\perp
q_7	\perp	\perp	\perp	\perp
q_8	\perp	\perp	\perp	\perp
q_{11}	\perp	\perp	\perp	\perp
q_{12}	\perp	\perp	\perp	\perp

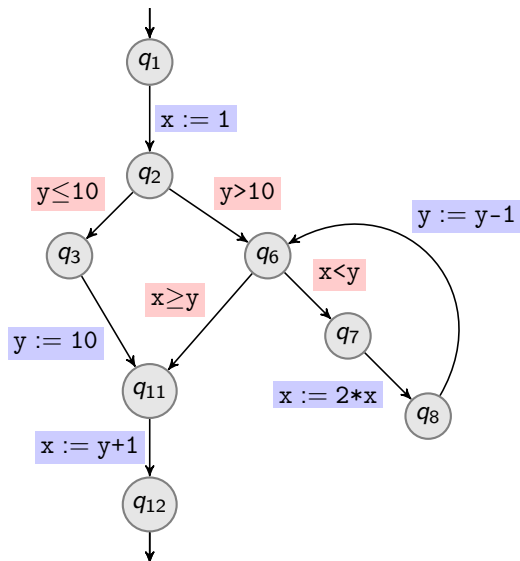
Range Analysis on Running Example (Round-Robin)



	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	\perp	\perp	\perp	\perp
q_3	\perp	\perp	\perp	\perp
q_6	\perp	\perp	\perp	\perp
q_7	\perp	\perp	\perp	\perp
q_8	\perp	\perp	\perp	\perp
q_{11}	\perp	\perp	\perp	\perp
q_{12}	\perp	\perp	\perp	\perp

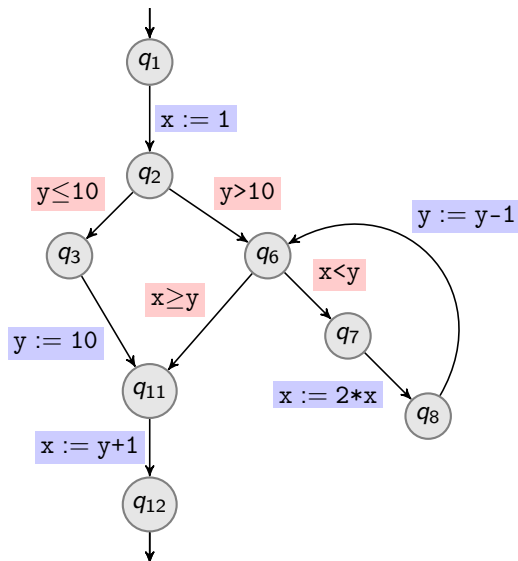
Recall $\top = (-\infty, +\infty)$

Range Analysis on Running Example (Round-Robin)



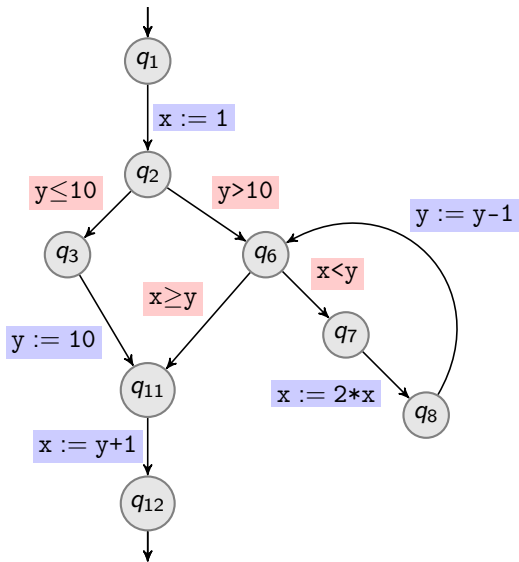
	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	\perp	\perp	\perp	\perp
q_6	\perp	\perp	\perp	\perp
q_7	\perp	\perp	\perp	\perp
q_8	\perp	\perp	\perp	\perp
q_{11}	\perp	\perp	\perp	\perp
q_{12}	\perp	\perp	\perp	\perp

Range Analysis on Running Example (Round-Robin)



	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	\perp	\perp	\perp	\perp
q_6	\perp	\perp	\perp	\perp
q_7	\perp	\perp	\perp	\perp
q_8	\perp	\perp	\perp	\perp
q_{11}	\perp	\perp	\perp	\perp
q_{12}	\perp	\perp	\perp	\perp

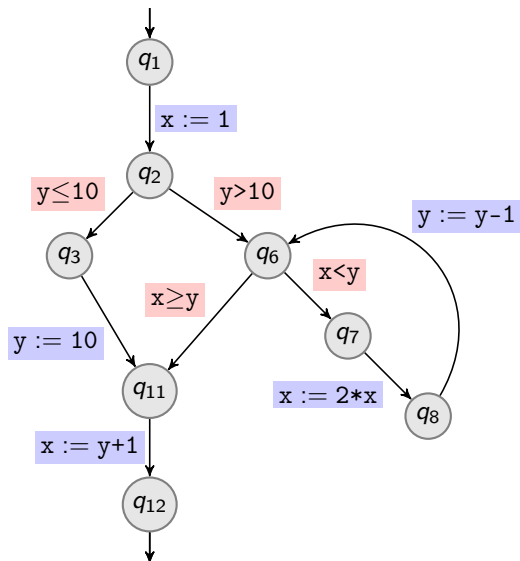
Range Analysis on Running Example (Round-Robin)



	x		y	
q1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q2	1	1	$-\infty$	$+\infty$
q3	1	1	$-\infty$	10
q6	\perp	\perp	\perp	\perp
q7	\perp	\perp	\perp	\perp
q8	\perp	\perp	\perp	\perp
q11	\perp	\perp	\perp	\perp
q12	\perp	\perp	\perp	\perp

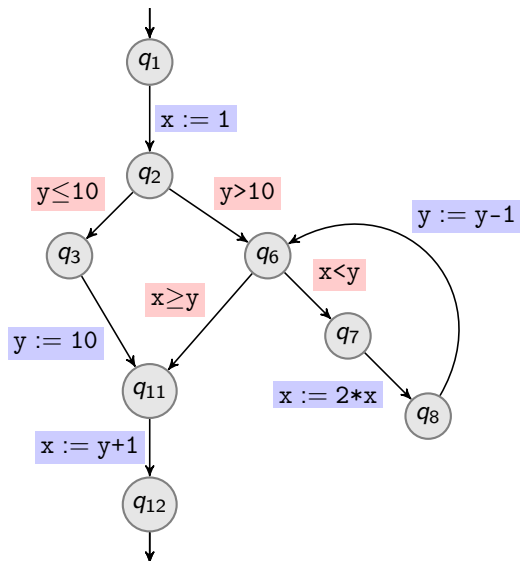
Gained from guard $y \leq 10$

Range Analysis on Running Example (Round-Robin)



	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	\perp	\perp	\perp	\perp
q_7	\perp	\perp	\perp	\perp
q_8	\perp	\perp	\perp	\perp
q_{11}	\perp	\perp	\perp	\perp
q_{12}	\perp	\perp	\perp	\perp

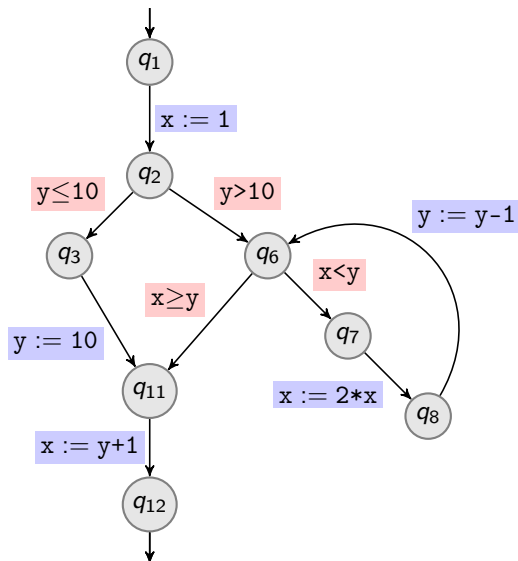
Range Analysis on Running Example (Round-Robin)



	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	\perp	\perp	\perp	\perp
q_8	\perp	\perp	\perp	\perp
q_{11}	\perp	\perp	\perp	\perp
q_{12}	\perp	\perp	\perp	\perp

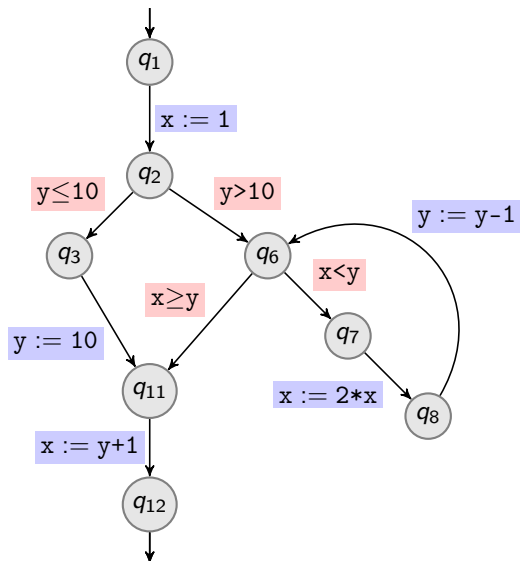
Gained from guard $y > 10$

Range Analysis on Running Example (Round-Robin)



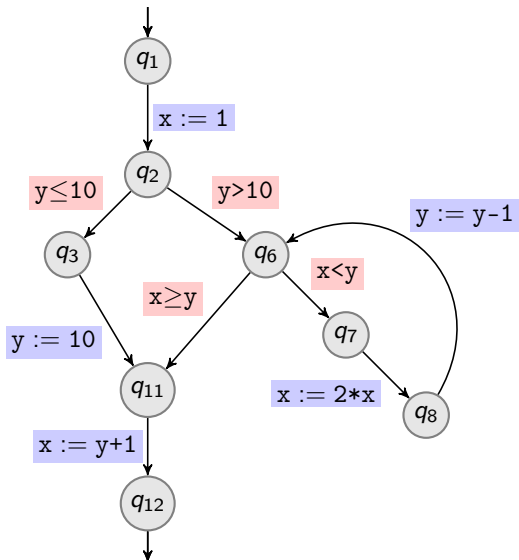
	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	\perp	\perp	\perp	\perp
q_8	\perp	\perp	\perp	\perp
q_{11}	\perp	\perp	\perp	\perp
q_{12}	\perp	\perp	\perp	\perp

Range Analysis on Running Example (Round-Robin)



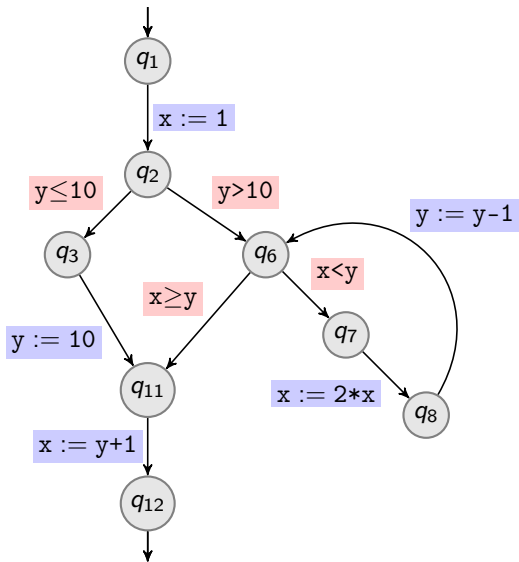
	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	1	1	11	$+\infty$
q_8	\perp	\perp	\perp	\perp
q_{11}	\perp	\perp	\perp	\perp
q_{12}	\perp	\perp	\perp	\perp

Range Analysis on Running Example (Round-Robin)



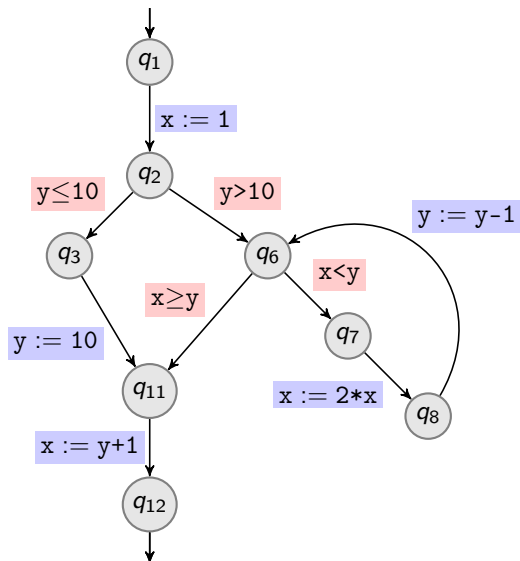
	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	1	1	11	$+\infty$
q_8	\perp	\perp	\perp	\perp
q_{11}	\perp	\perp	\perp	\perp
q_{12}	\perp	\perp	\perp	\perp

Range Analysis on Running Example (Round-Robin)



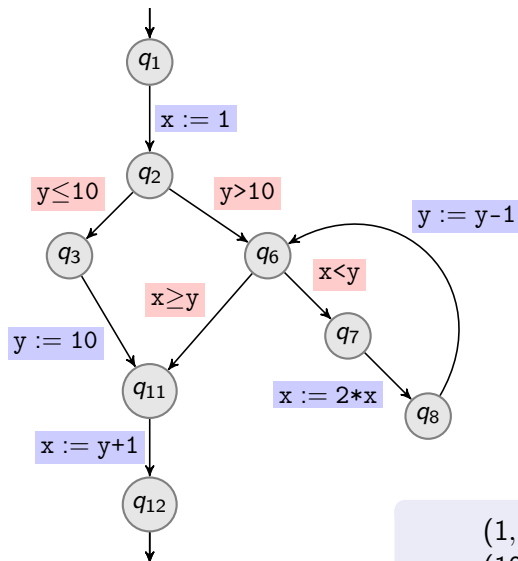
	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	\perp	\perp	\perp	\perp
q_{12}	\perp	\perp	\perp	\perp

Range Analysis on Running Example (Round-Robin)



	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	\perp	\perp	\perp	\perp
q_{12}	\perp	\perp	\perp	\perp

Range Analysis on Running Example (Round-Robin)

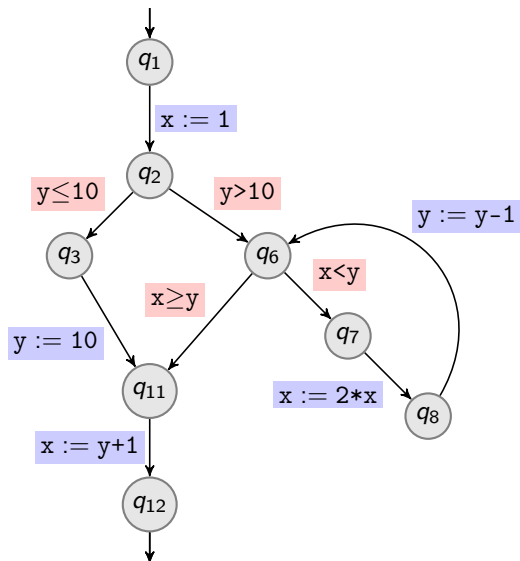


	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	\perp	\perp	\perp	\perp

$$(1, 1) \sqcup \perp = (1, 1)$$

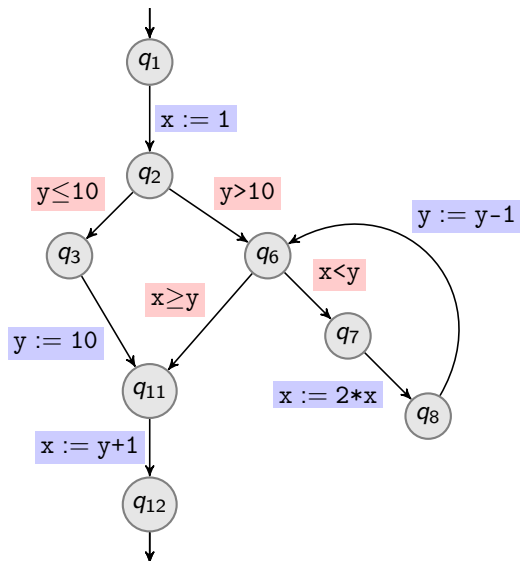
$$(10, 10) \sqcup \perp = (10, 10)$$

Range Analysis on Running Example (Round-Robin)



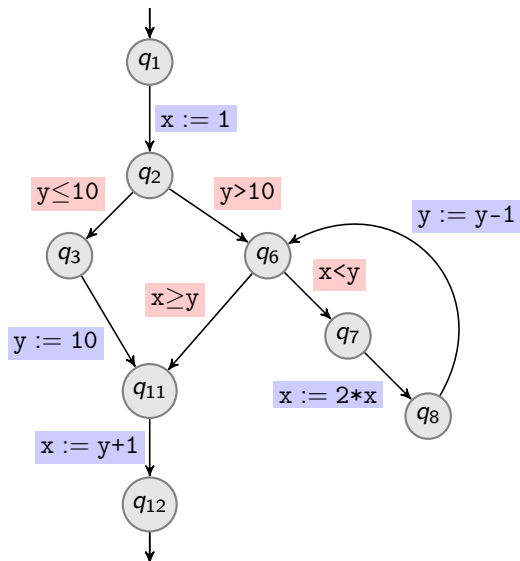
	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	\perp	\perp	\perp	\perp

Range Analysis on Running Example (Round-Robin)



	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

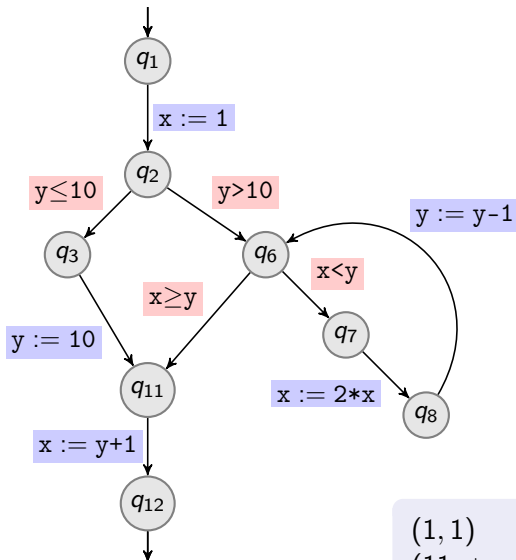
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Running Example (Round-Robin)



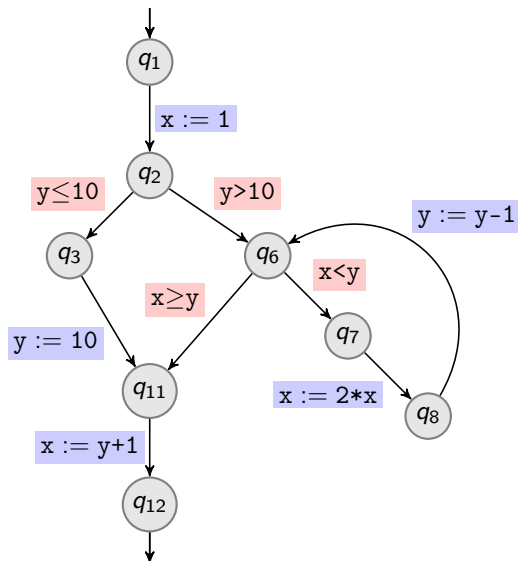
Focus on $\{q_6, q_7, q_8\}$

	x		y	
	min	max	min	max
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2	10	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

$$(1, 1) \sqcup (1, 2) = (1, 2)$$

$$(11, +\infty) \sqcup (10, +\infty) = (10, +\infty)$$

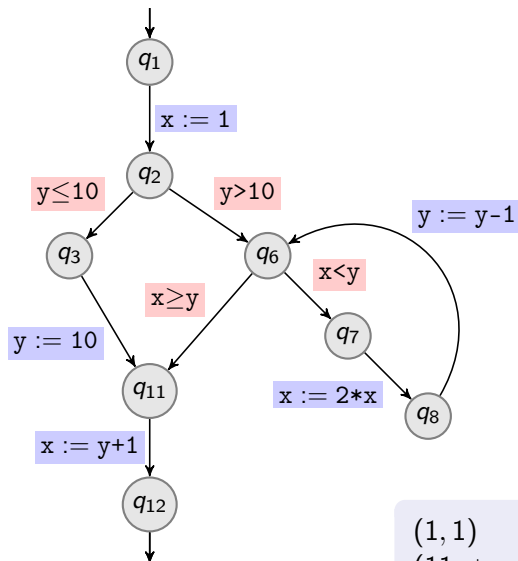
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2	10	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Running Example (Round-Robin)



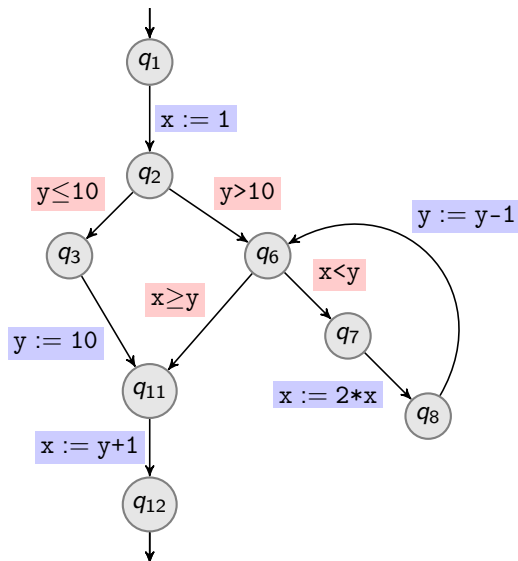
Focus on $\{q_6, q_7, q_8\}$

	x		y	
	min	max	min	max
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2	10	$+\infty$
q_7	1	2	10	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

$$(1, 1) \sqcup (1, 2) = (1, 2)$$

$$(11, +\infty) \sqcup (10, +\infty) = (10, +\infty)$$

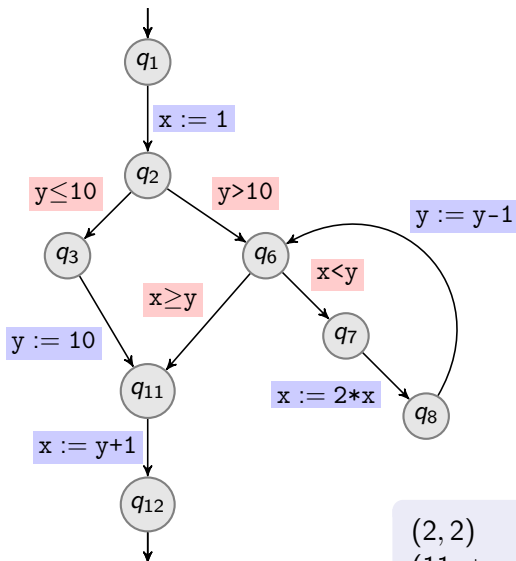
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2	10	$+\infty$
q_7	1	2	10	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Running Example (Round-Robin)



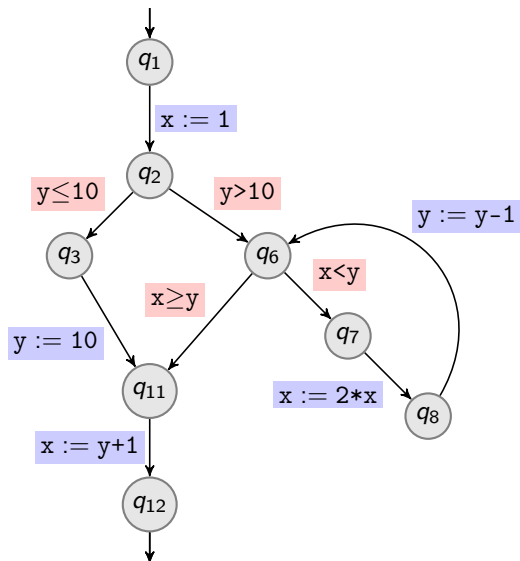
Focus on $\{q_6, q_7, q_8\}$

	x		y	
	min	max	min	max
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2	10	$+\infty$
q_7	1	2	10	$+\infty$
q_8	2	4	10	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

$$(2, 2) \sqcup (2, 4) = (2, 4)$$

$$(11, +\infty) \sqcup (10, +\infty) = (10, +\infty)$$

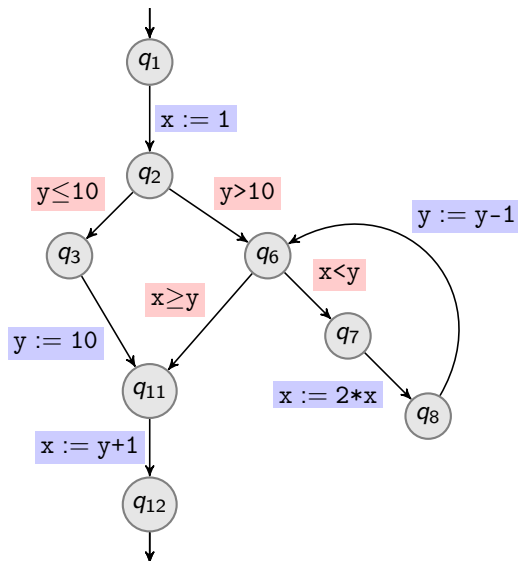
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2	10	$+\infty$
q_7	1	2	10	$+\infty$
q_8	2	4	10	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

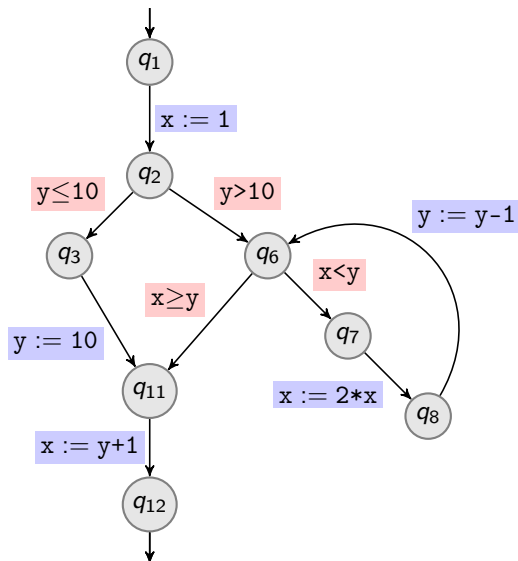
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	4	9	$+\infty$
q_7	1	2	10	$+\infty$
q_8	2	4	10	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

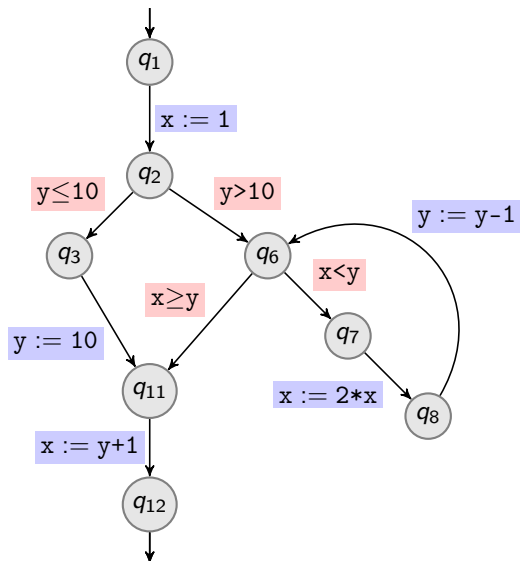
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	4	9	$+\infty$
q_7	1	2	10	$+\infty$
q_8	2	4	10	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

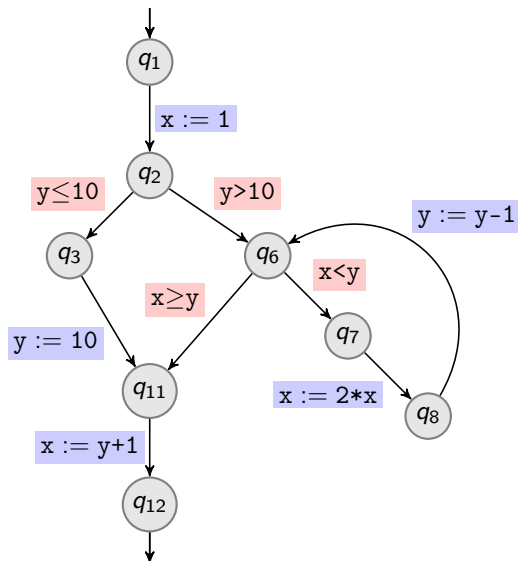
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	4	9	$+\infty$
q_7	1	4	9	$+\infty$
q_8	2	4	10	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

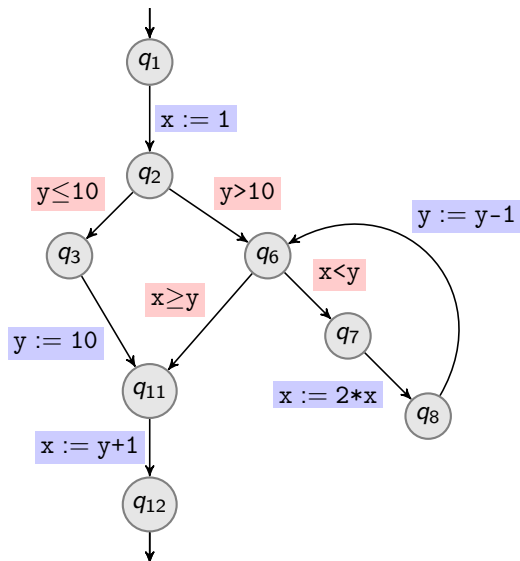
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	4	9	$+\infty$
q_7	1	4	9	$+\infty$
q_8	2	4	10	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

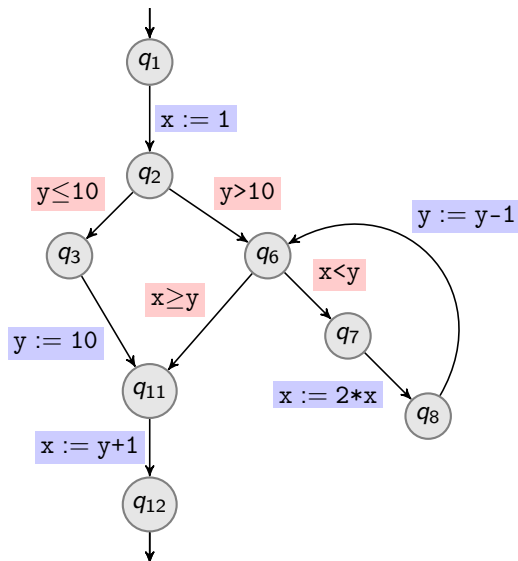
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	4	9	$+\infty$
q_7	1	4	9	$+\infty$
q_8	2	8	9	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

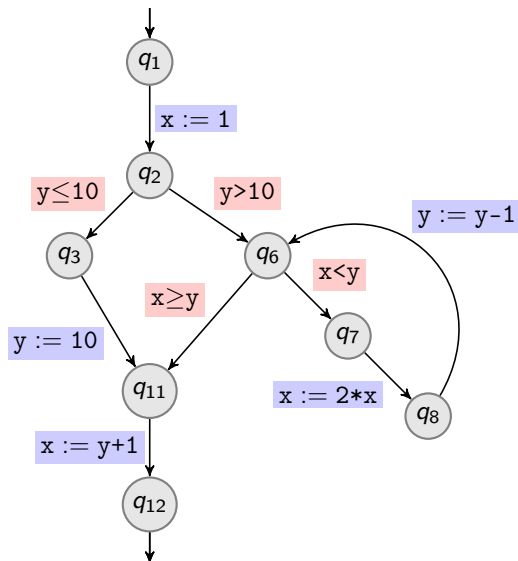
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
	min	max	min	max
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	4	9	$+\infty$
q_7	1	4	9	$+\infty$
q_8	2	8	9	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

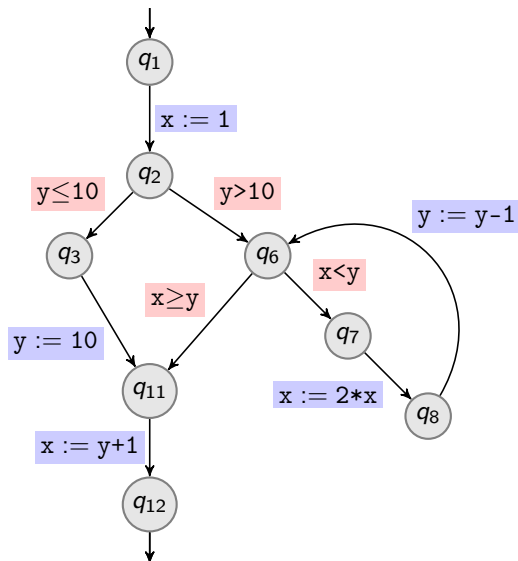
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
	min	max	min	max
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	8	8	$+\infty$
q_7	1	4	9	$+\infty$
q_8	2	8	9	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

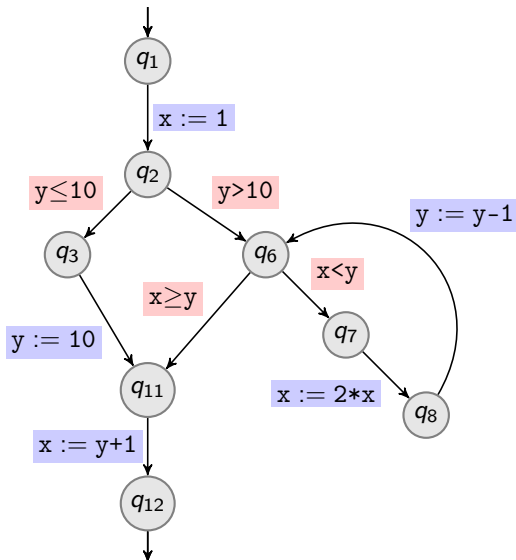
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	8	8	$+\infty$
q_7	1	4	9	$+\infty$
q_8	2	8	9	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Running Example (Round-Robin)

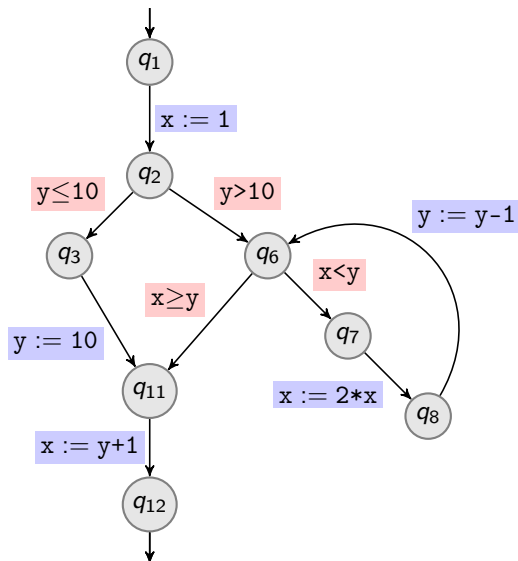


Focus on $\{q_6, q_7, q_8\}$

	x		y	
	min	max	min	max
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	8	8	$+\infty$
q_7	1	8	8	$+\infty$
q_8	2	8	9	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Nothing to be gained from guard $x < y$

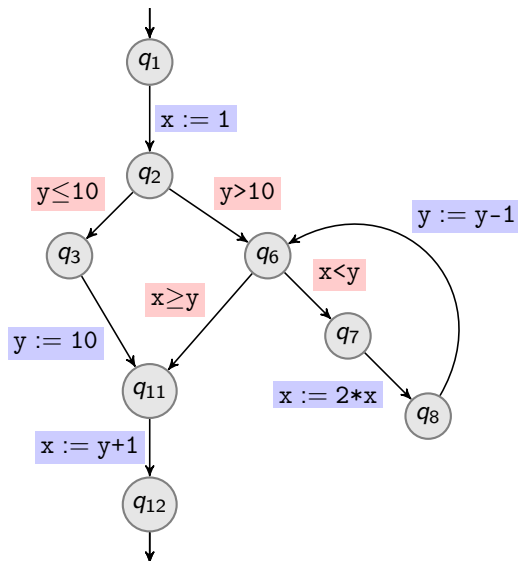
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	8	8	$+\infty$
q_7	1	8	8	$+\infty$
q_8	2	8	9	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

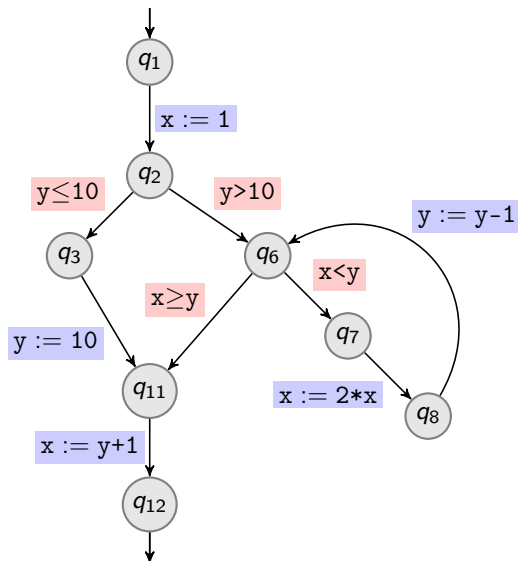
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	8	8	$+\infty$
q_7	1	8	8	$+\infty$
q_8	2	16	8	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

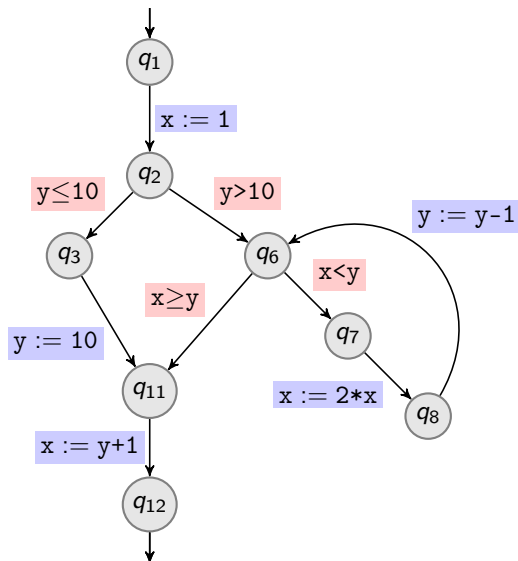
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	8	8	$+\infty$
q_7	1	8	8	$+\infty$
q_8	2	16	8	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

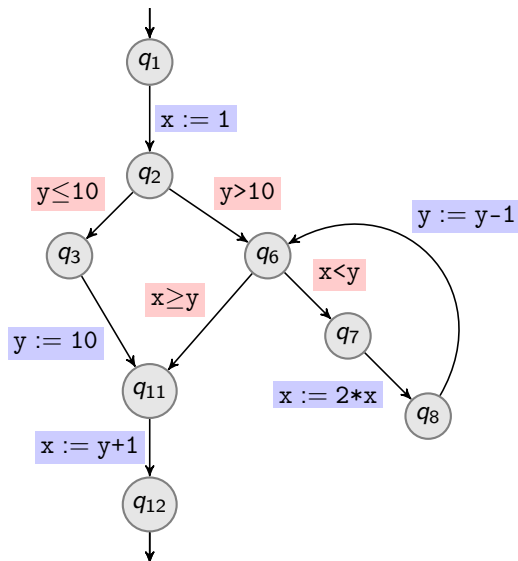
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	16	7	$+\infty$
q_7	1	16	7	$+\infty$
q_8	2	32	7	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

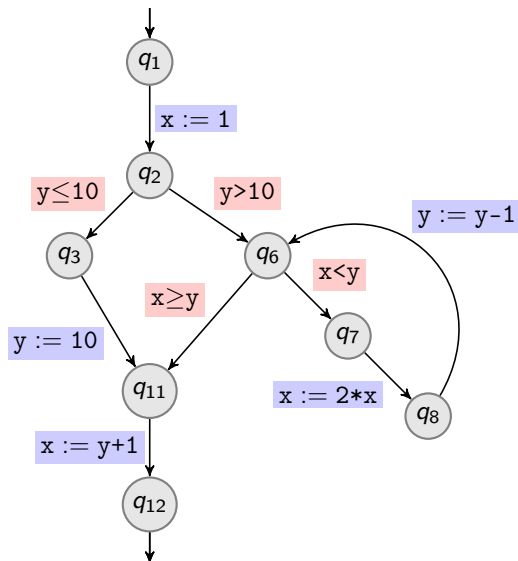
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	32	6	$+\infty$
q_7	1	32	6	$+\infty$
q_8	2	2^6	6	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

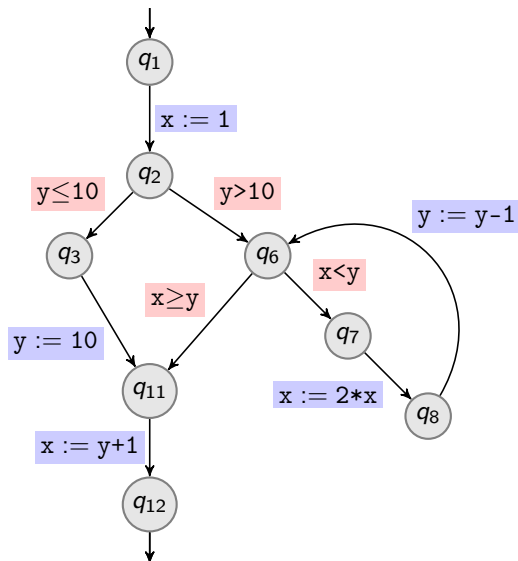
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^6	5	$+\infty$
q_7	1	2^6	5	$+\infty$
q_8	2	2^7	5	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

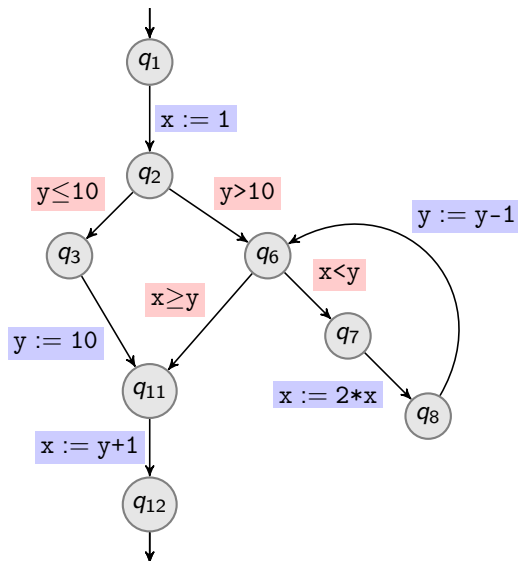
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^7	4	$+\infty$
q_7	1	2^7	4	$+\infty$
q_8	2	2^8	4	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

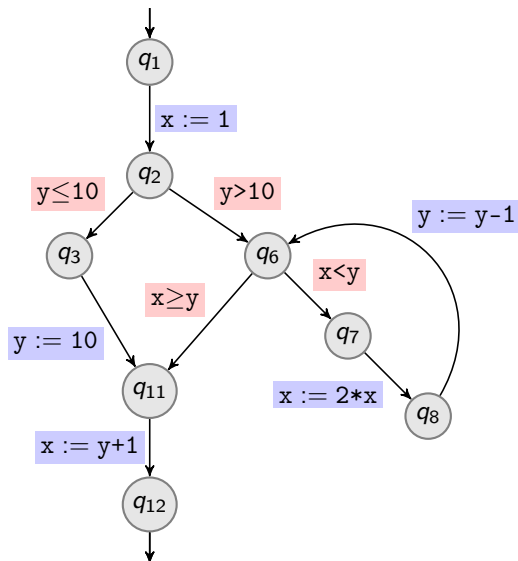
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^8	3	$+\infty$
q_7	1	2^8	3	$+\infty$
q_8	2	2^9	3	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

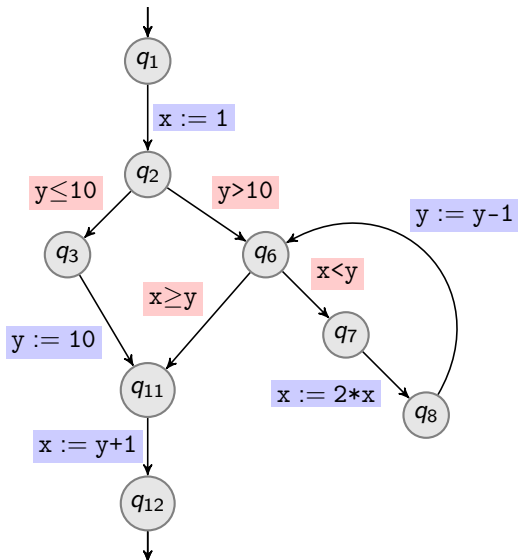
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
	min	max	min	max
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^9	2	$+\infty$
q_7	1	2^9	2	$+\infty$
q_8	2	2^{10}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

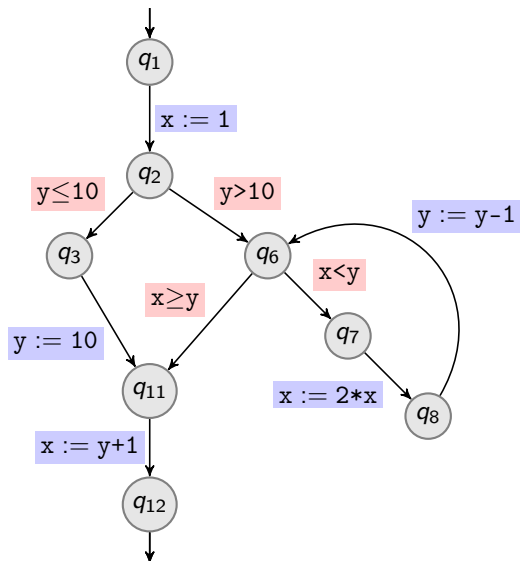
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^9	2	$+\infty$
q_7	1	2^9	2	$+\infty$
q_8	2	2^{10}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

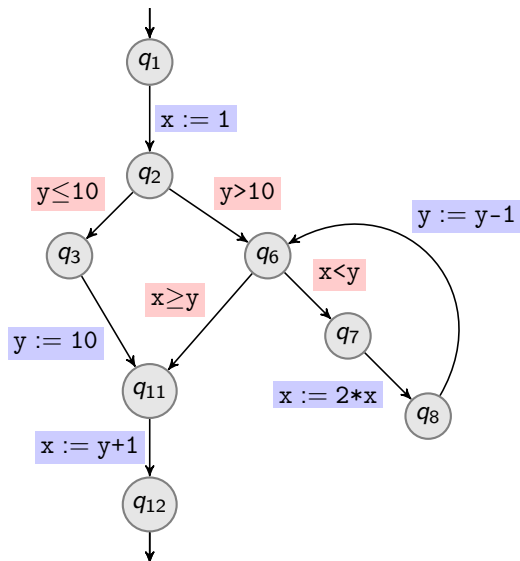
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{10}	1	$+\infty$
q_7	1	2^9	2	$+\infty$
q_8	2	2^{10}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

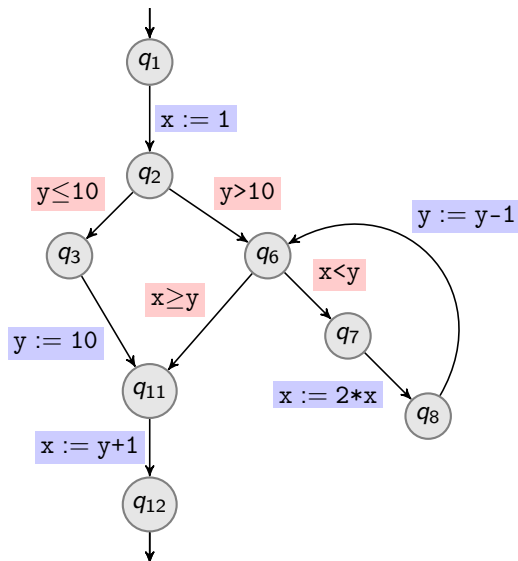
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
	min	max	min	max
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{10}	1	$+\infty$
q_7	1	2^9	2	$+\infty$
q_8	2	2^{10}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Running Example (Round-Robin)

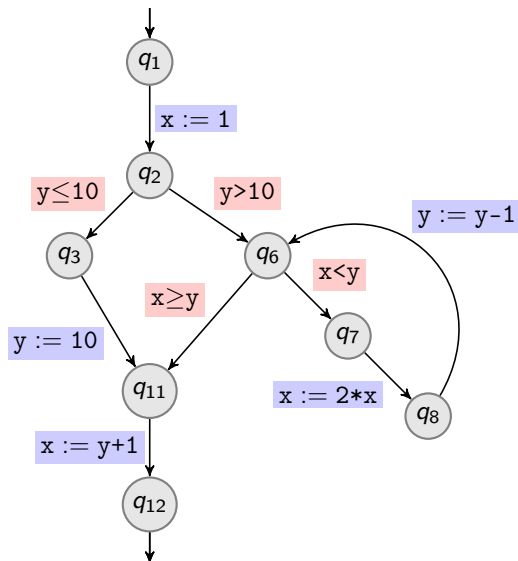


Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{10}	1	$+\infty$
q_7	1	2^{10}	2	$+\infty$
q_8	2	2^{10}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Gained from guard $x < y$

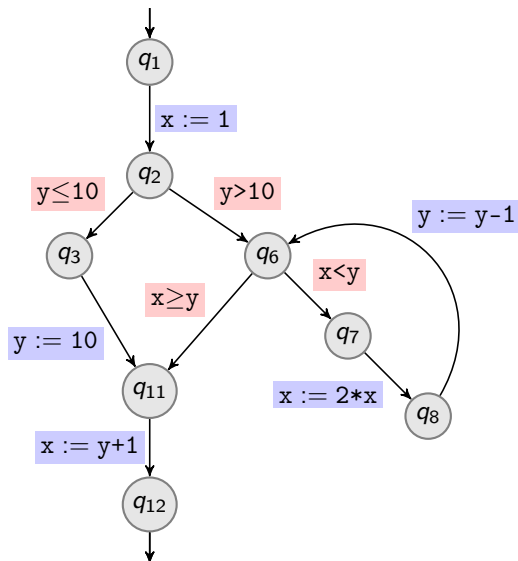
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{10}	1	$+\infty$
q_7	1	2^{10}	2	$+\infty$
q_8	2	2^{10}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

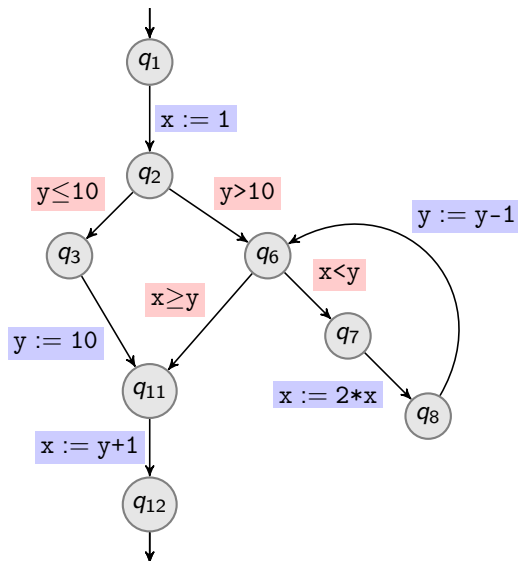
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{10}	1	$+\infty$
q_7	1	2^{10}	2	$+\infty$
q_8	2	2^{11}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

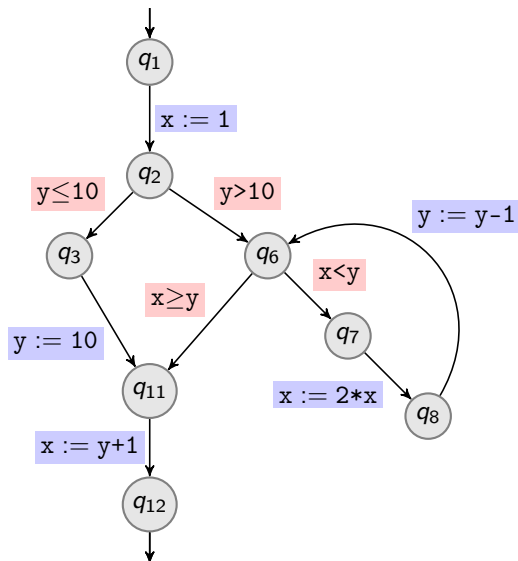
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{10}	1	$+\infty$
q_7	1	2^{10}	2	$+\infty$
q_8	2	2^{11}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

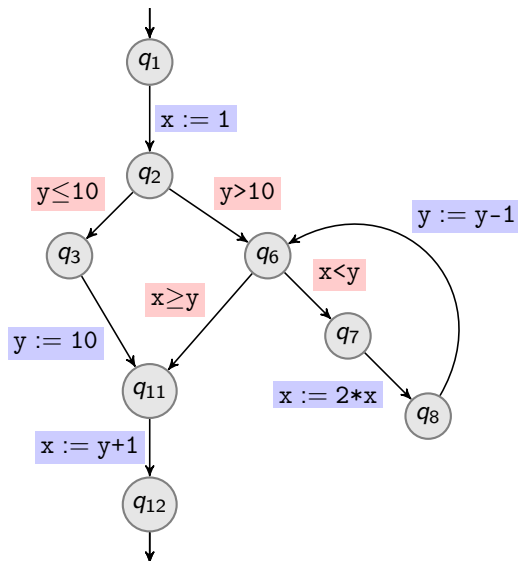
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
	min	max	min	max
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{11}	1	$+\infty$
q_7	1	2^{11}	2	$+\infty$
q_8	2	2^{12}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

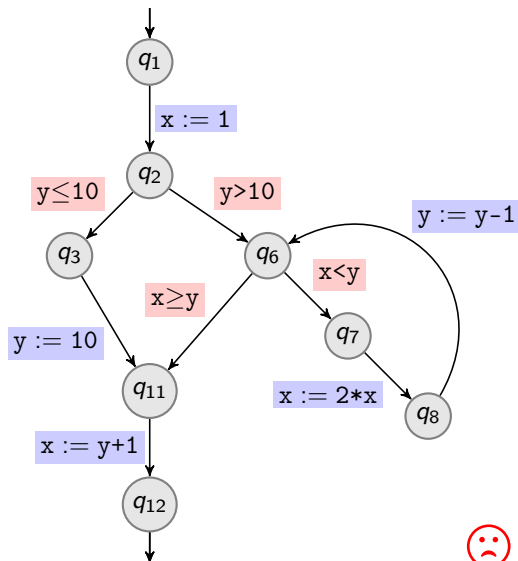
Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{12}	1	$+\infty$
q_7	1	2^{12}	2	$+\infty$
q_8	2	2^{13}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Running Example (Round-Robin)



Focus on $\{q_6, q_7, q_8\}$

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	...	1	$+\infty$
q_7	1	...	2	$+\infty$
q_8	2	...	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10



Does not converge!

Dynamic Approximation: Widening Operators

Consider a complete lattice (L, \sqsubseteq) .

Objective of Widening Operators

Soundly extrapolate “limits” of ascending chains

Definition

A **widening** operator for (L, \sqsubseteq) is a function $\nabla : (L \times L) \rightarrow L$ such that:

1. $x \sqcup y \sqsubseteq x \nabla y$ (for all $x, y \in L$)
2. for any ascending chain $x_0 \sqsubseteq x_1 \sqsubseteq \dots$ of elements of L , the ascending chain $y_0 \sqsubseteq y_1 \sqsubseteq \dots$ defined by

$$\begin{cases} y_0 = x_0 \\ y_{i+1} = y_i \nabla x_{i+1} \quad \text{for all } i \in \mathbb{N} \end{cases}$$

is not strictly increasing (i.e. $y_{i+1} = y_i$ for some $i \in \mathbb{N}$).

Correctness of Kleene Iteration with Widening

Consider a complete lattice (L, \sqsubseteq) and a monotonic function $f : L \rightarrow L$.

Theorem

If $\nabla : (L \times L) \rightarrow L$ is a widening operator then the ascending chain $y_0 \sqsubseteq y_1 \sqsubseteq \dots$ defined by

$$y_0 = \perp$$
$$y_{i+1} = \begin{cases} y_i & \text{if } f(y_i) \sqsubseteq y_i \\ y_i \nabla f(y_i) & \text{otherwise} \end{cases}$$

is eventually stationary, and its limit satisfies $\bigsqcup \{y_i \mid i \in \mathbb{N}\} \sqsupseteq \text{lfp}(f)$.

Correctness of Kleene Iteration with Widening

Consider a complete lattice (L, \sqsubseteq) and a monotonic function $f : L \rightarrow L$.

Theorem

If $\nabla : (L \times L) \rightarrow L$ is a widening operator then the ascending chain $y_0 \sqsubseteq y_1 \sqsubseteq \dots$ defined by

$$y_0 = \perp$$
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is eventually stationary, and its limit satisfies $\sqcup \{y_i \mid i \in \mathbb{N}\} \sqsupseteq \text{lfp}(f)$.

Application to MFP Approximation

Replacing \sqcup with ∇ in Kleene / round-robin / worklist algorithms

- ▶ guarantees **termination**, but
- ▶ at the expense of **precision**.

Round-Robin Iteration with Widening

Consider a Galois connection $(\mathcal{P}(X \rightarrow \mathbb{Z}), \sqsubseteq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$, and a widening ∇ for (A, \sqsubseteq) .

Pre-condition:

$$a(q) \sqsubseteq \text{MFP}(q) \quad \text{for all } q \in Q$$

```
do
  change ← false
  foreach  $q \in Q$ 
    new ←  $I_q^\# \sqcup \bigsqcup_{p \xrightarrow{\text{op}} q} \text{post}_{\text{op}}^\#(a[p])$ 
    if new  $\not\sqsubseteq$  a[q]
      a[q] ← a[q]  $\nabla$  new
      change ← true
while change
return a
```

- ▶ this algorithm **terminates**
- ▶ the returned $a \in Q \rightarrow L$ satisfies:

$$\text{MFP}(q) \sqsubseteq a(q)$$

for every $q \in Q$.

Widening Operator for Range Analysis: Intuition

Objective of Widening Operators

Soundly extrapolate “limits” of ascending chains

Put ∞ when the bound is moving towards ∞

Examples

$$\dots, (1, 2), (1, 3), (1, 4) \longrightarrow (1, +\infty)$$

$$\dots, (1, 2), (-1, 2), (-6, 2) \longrightarrow (-\infty, 2)$$

$$\dots, (1, 2), (-9, 3), (-19, 4) \longrightarrow (-\infty, +\infty)$$

$$\dots, (1, +\infty), (-9, +\infty), (-19, +\infty) \longrightarrow (-\infty, +\infty)$$

∇ only looks at the last two elements of the sequence

Widening Operator for Range Analysis

Widening Operator on the Complete Lattice (Int, \sqsubseteq) of Intervals

$$\perp \nabla \perp = \perp \qquad \perp \nabla (l, u) = (l, u) \nabla \perp = (l, u)$$

$$(l_1, u_1) \nabla (l_2, u_2) = (l_{\nabla}, u_{\nabla}) \quad \text{where} \quad \begin{cases} l_{\nabla} = \begin{cases} -\infty & \text{if } l_2 < l_1 \\ l_1 & \text{otherwise} \end{cases} \\ u_{\nabla} = \begin{cases} +\infty & \text{if } u_2 > u_1 \\ u_1 & \text{otherwise} \end{cases} \end{cases}$$

Widening Operator for Range Analysis

Widening Operator on the Complete Lattice (Int, \sqsubseteq) of Intervals

$$\perp \nabla \perp = \perp \qquad \perp \nabla (l, u) = (l, u) \nabla \perp = (l, u)$$

$$(l_1, u_1) \nabla (l_2, u_2) = (l_{\nabla}, u_{\nabla}) \quad \text{where} \quad \begin{cases} l_{\nabla} = \begin{cases} -\infty & \text{if } l_2 < l_1 \\ l_1 & \text{otherwise} \end{cases} \\ u_{\nabla} = \begin{cases} +\infty & \text{if } u_2 > u_1 \\ u_1 & \text{otherwise} \end{cases} \end{cases}$$

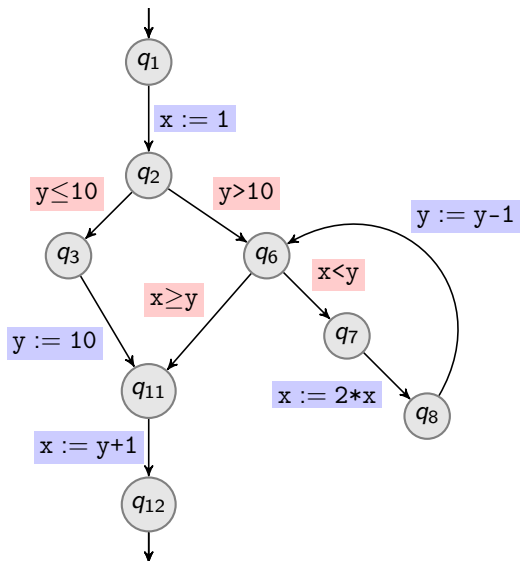
Widening Operator on the Complete Lattice $(X \rightarrow Int, \sqsubseteq)$

Component-wise extension ∇ of the above widening ∇ , defined by:

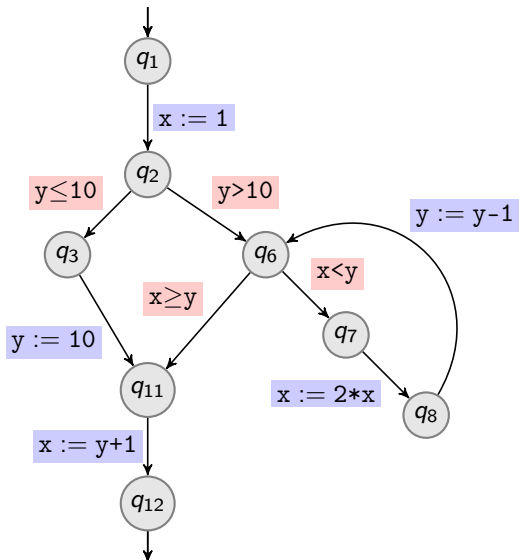
$$(a_1 \nabla a_2)(x) = a_1(x) \nabla a_2(x)$$

Range Analysis on Example with Widening

Show that $x > 0$ at q_{12}



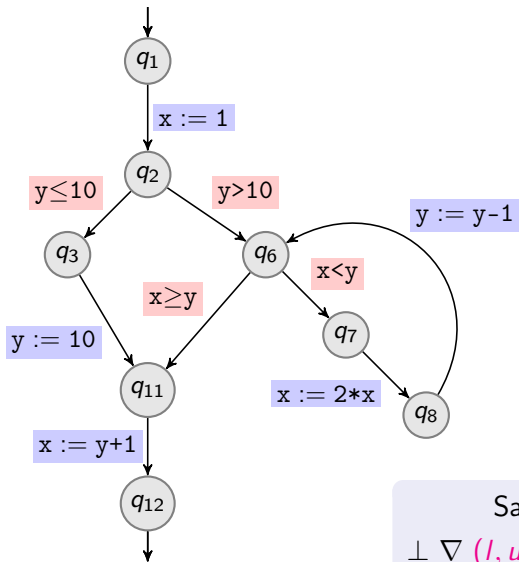
Range Analysis on Example with Widening



Show that $x > 0$ at q_{12}

	x		y	
q_1	\perp	\perp	\perp	\perp
q_2	\perp	\perp	\perp	\perp
q_3	\perp	\perp	\perp	\perp
q_6	\perp	\perp	\perp	\perp
q_7	\perp	\perp	\perp	\perp
q_8	\perp	\perp	\perp	\perp
q_{11}	\perp	\perp	\perp	\perp
q_{12}	\perp	\perp	\perp	\perp

Range Analysis on Example with Widening



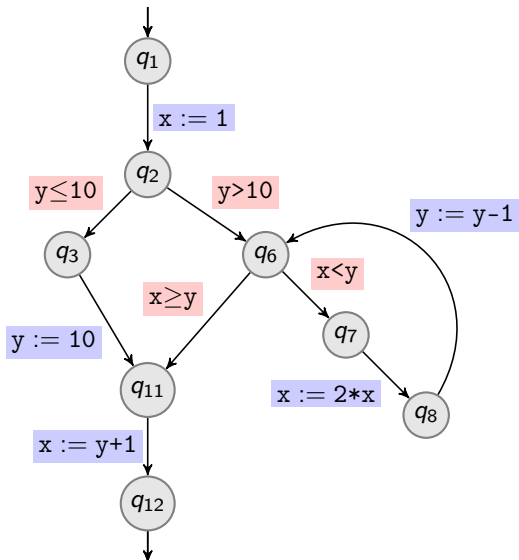
Show that $x > 0$ at q_{12}

	x		y	
	l	u	l	u
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Same as without ∇ since

$$\perp \nabla (l, u) = \perp \sqcup (l, u) = (l, u)$$

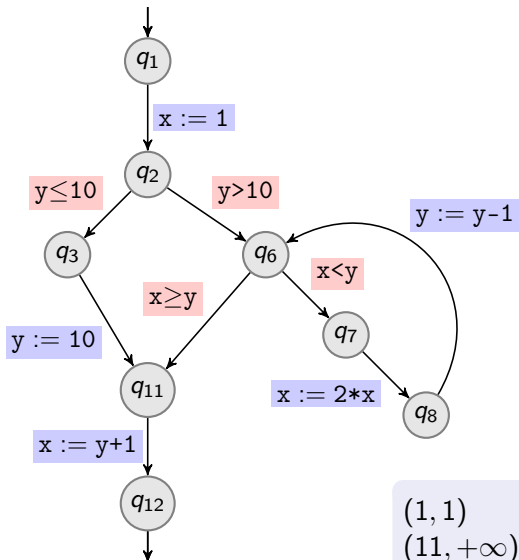
Range Analysis on Example with Widening



Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Example with Widening

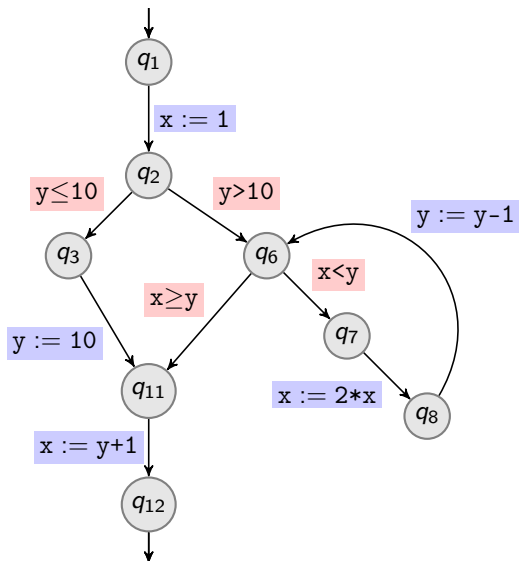


Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

$$\begin{aligned}
 (1, 1) &\quad \nabla \quad (1, 2) &= (1, +\infty) \\
 (11, +\infty) &\quad \nabla \quad (10, +\infty) &= (-\infty, +\infty)
 \end{aligned}$$

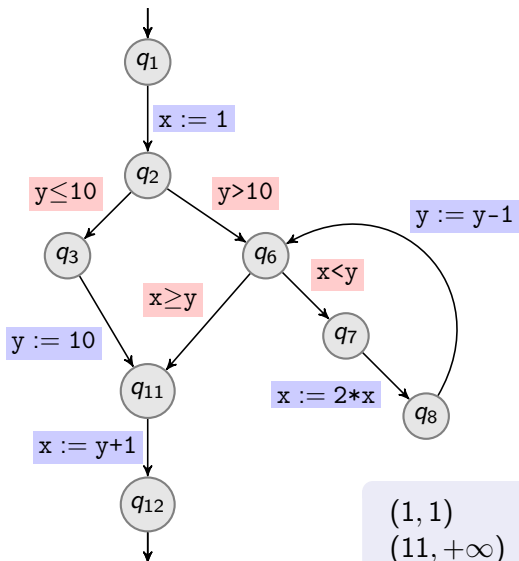
Range Analysis on Example with Widening



Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Example with Widening



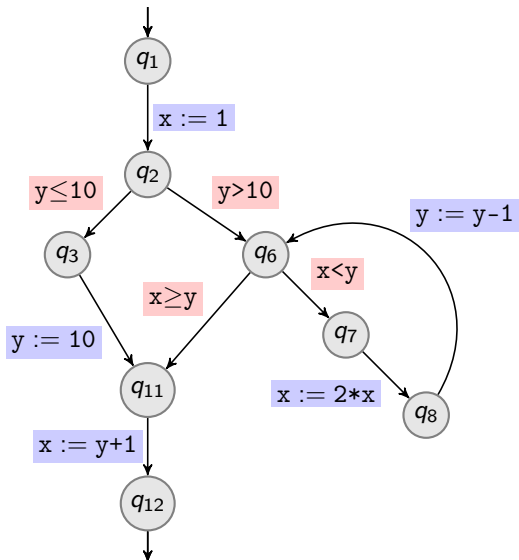
Show that $x > 0$ at q_{12}

	x		y	
	min	max	min	max
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	$-\infty$	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

$$(1, 1) \quad \nabla \quad (1, +\infty) = (1, +\infty)$$

$$(11, +\infty) \quad \nabla \quad (2, +\infty) = (-\infty, +\infty)$$

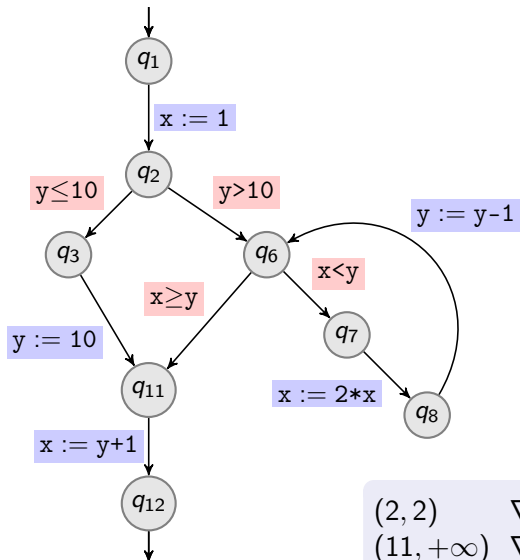
Range Analysis on Example with Widening



Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	$-\infty$	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Example with Widening

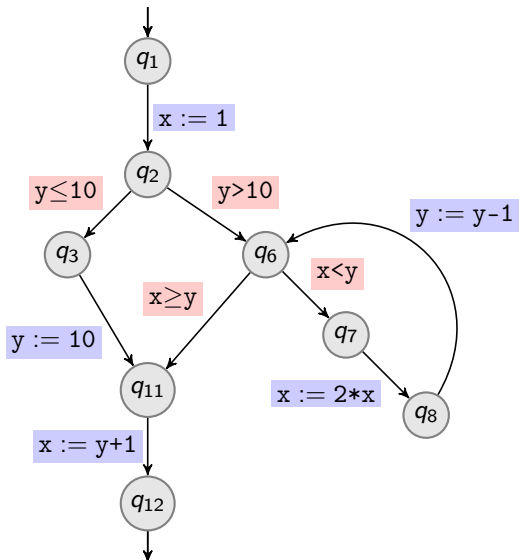


Show that $x > 0$ at q_{12}

	x		y	
	min	max	min	max
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	$-\infty$	$+\infty$
q_8	2	$+\infty$	$-\infty$	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

$$\begin{aligned}
 (2, 2) & \nabla (2, +\infty) = (2, +\infty) \\
 (11, +\infty) & \nabla (-\infty, +\infty) = (-\infty, +\infty)
 \end{aligned}$$

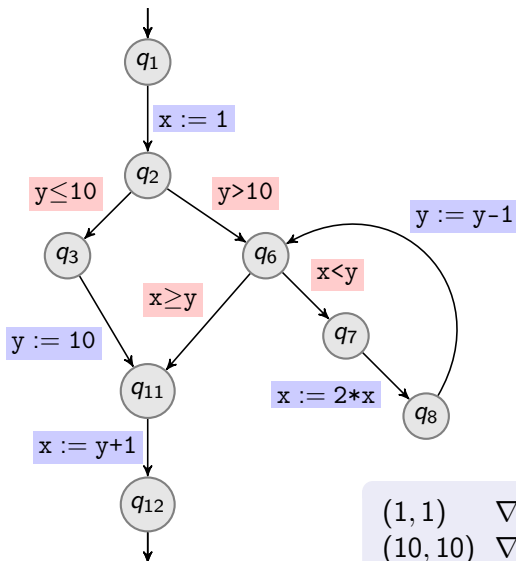
Range Analysis on Example with Widening



Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	$-\infty$	$+\infty$
q_8	2	$+\infty$	$-\infty$	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Example with Widening



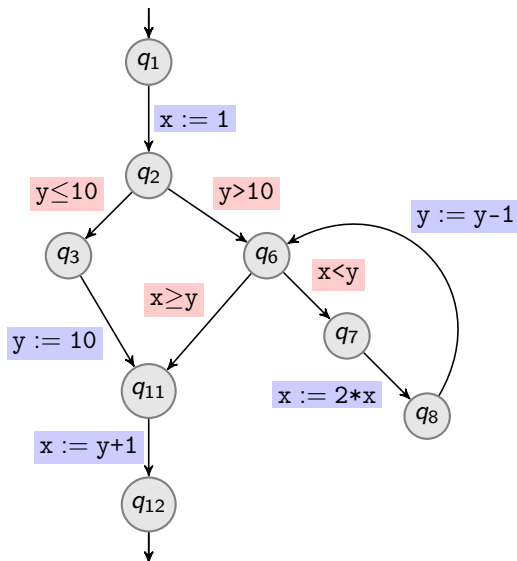
Show that $x > 0$ at q_{12}

	x		y	
	min	max	min	max
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	$-\infty$	$+\infty$
q_8	2	$+\infty$	$-\infty$	$+\infty$
q_{11}	1	$+\infty$	$-\infty$	$+\infty$
q_{12}	11	11	10	10

$$(1, 1) \nabla (1, +\infty) = (1, +\infty)$$

$$(10, 10) \nabla (-\infty, +\infty) = (-\infty, +\infty)$$

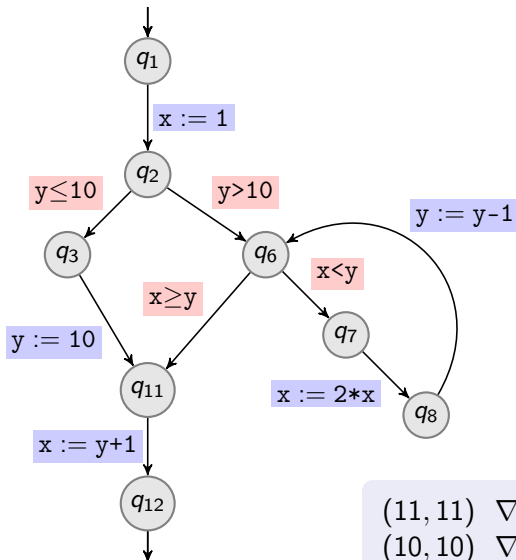
Range Analysis on Example with Widening



Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	$-\infty$	$+\infty$
q_8	2	$+\infty$	$-\infty$	$+\infty$
q_{11}	1	$+\infty$	$-\infty$	$+\infty$
q_{12}	11	11	10	10

Range Analysis on Example with Widening



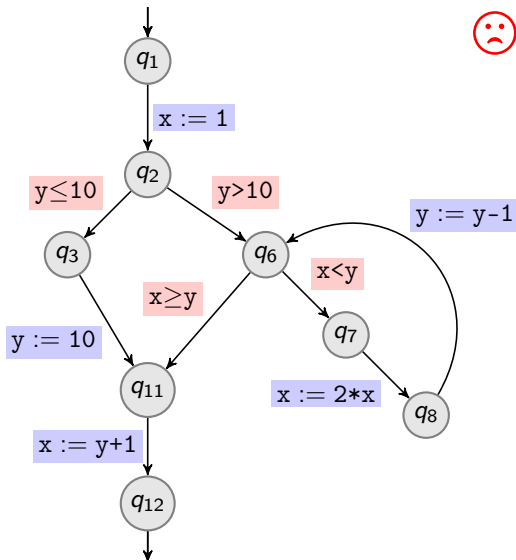
Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	$-\infty$	$+\infty$
q_8	2	$+\infty$	$-\infty$	$+\infty$
q_{11}	1	$+\infty$	$-\infty$	$+\infty$
q_{12}	$-\infty$	$+\infty$	$-\infty$	$+\infty$

$$(11, 11) \nabla (-\infty, +\infty) = (-\infty, +\infty)$$

$$(10, 10) \nabla (-\infty, +\infty) = (-\infty, +\infty)$$

Range Analysis on Example with Widening

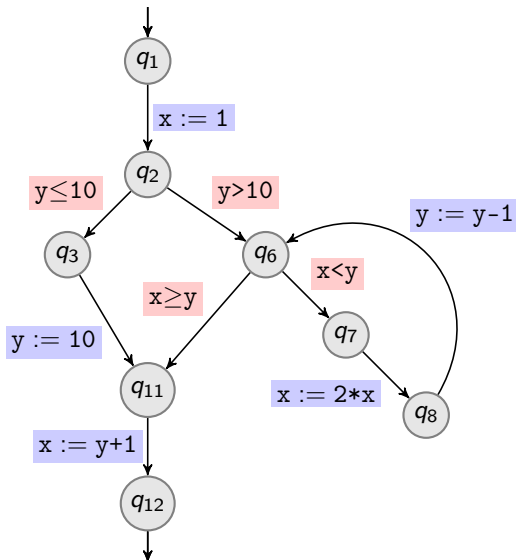


Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	$-\infty$	$+\infty$
q_8	2	$+\infty$	$-\infty$	$+\infty$
q_{11}	1	$+\infty$	$-\infty$	$+\infty$
q_{12}	$-\infty$	$+\infty$	$-\infty$	$+\infty$

Too coarse!

Range Analysis on Example with Delayed Widening

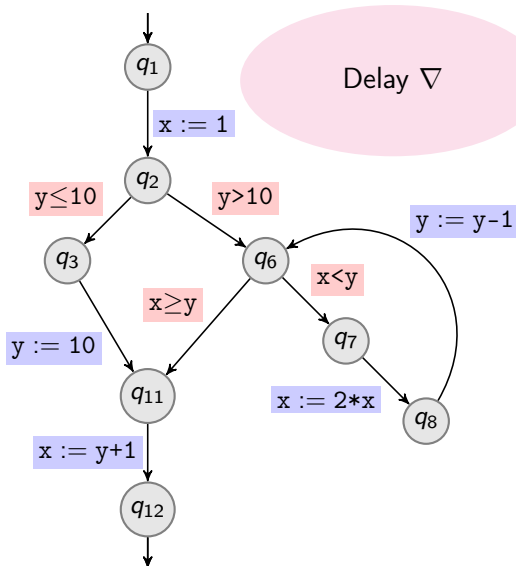


Show that $x > 0$ at q_{12}

Delayed Widening

1. Keep \sqcup for the first iterations
2. Track number of “updates” for each location
3. Switch to ∇ after a **suitable** “delay”

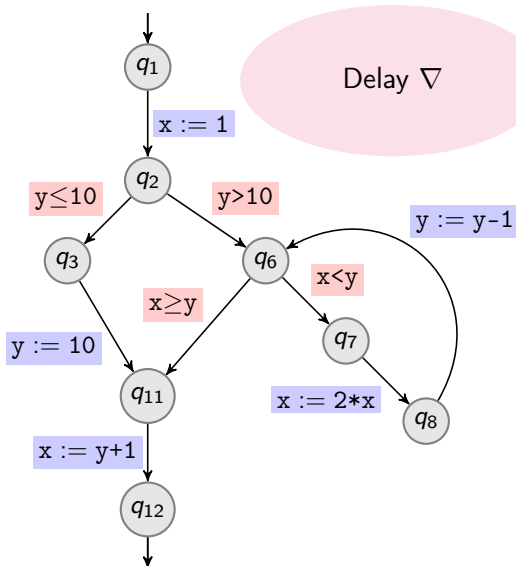
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
q_1	\perp	\perp	\perp	\perp
q_2	\perp	\perp	\perp	\perp
q_3	\perp	\perp	\perp	\perp
q_6	\perp	\perp	\perp	\perp
q_7	\perp	\perp	\perp	\perp
q_8	\perp	\perp	\perp	\perp
q_{11}	\perp	\perp	\perp	\perp
q_{12}	\perp	\perp	\perp	\perp

Range Analysis on Example with Delayed Widening

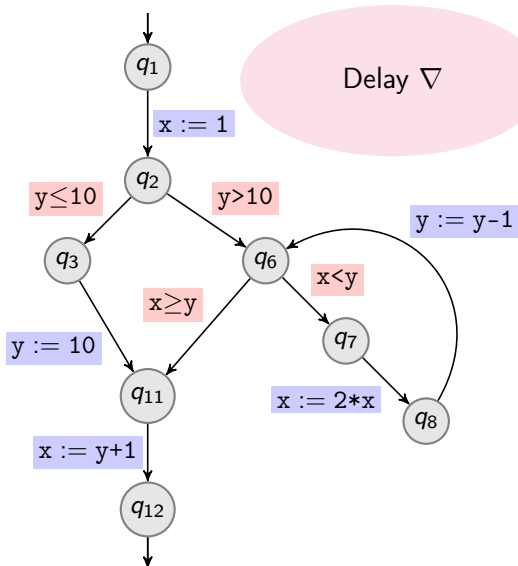


Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Same as without ∇

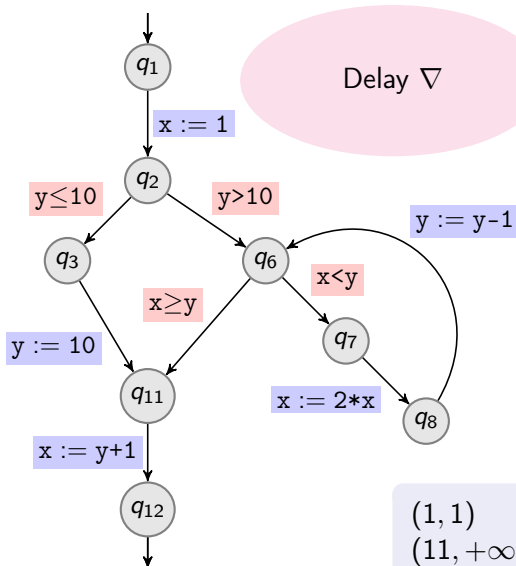
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Example with Delayed Widening



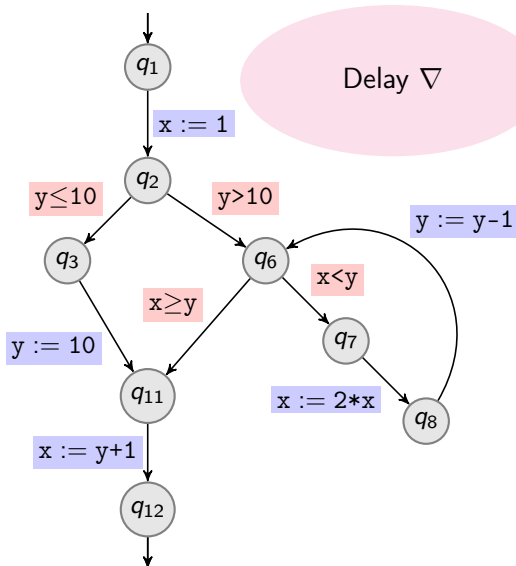
Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2	10	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

$$(1, 1) \sqcup (1, 2) = (1, 2)$$

$$(11, +\infty) \sqcup (10, +\infty) = (10, +\infty)$$

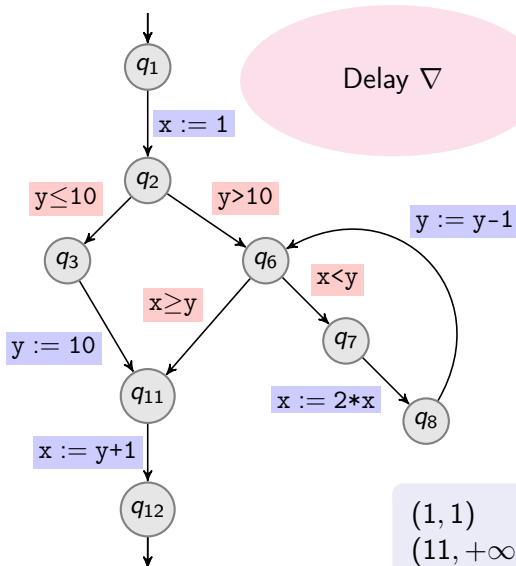
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2	10	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Example with Delayed Widening



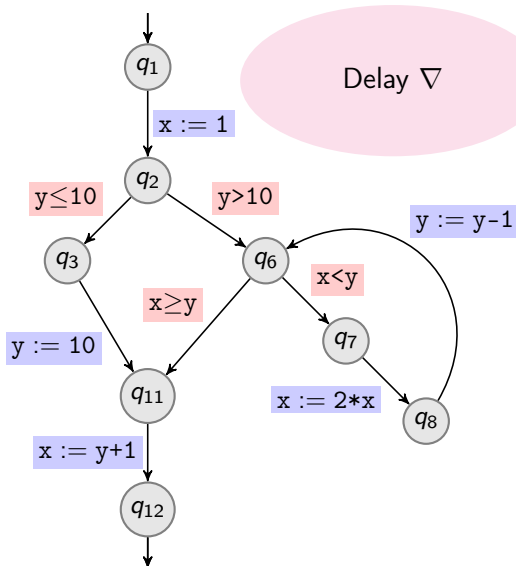
Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2	10	$+\infty$
q_7	1	2	10	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

$$(1, 1) \sqcup (1, 2) = (1, 2)$$

$$(11, +\infty) \sqcup (10, +\infty) = (10, +\infty)$$

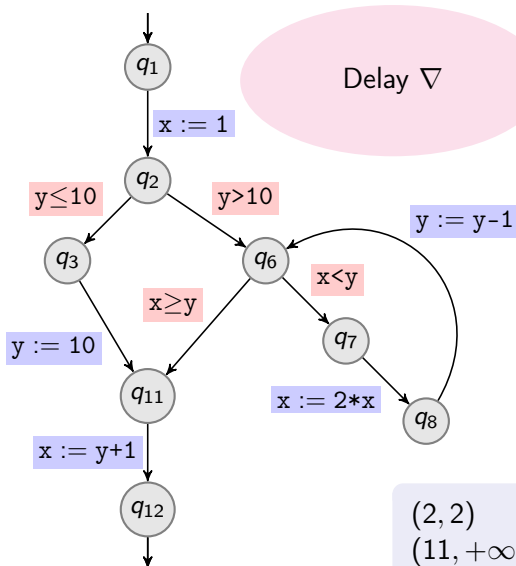
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2	10	$+\infty$
q_7	1	2	10	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Example with Delayed Widening



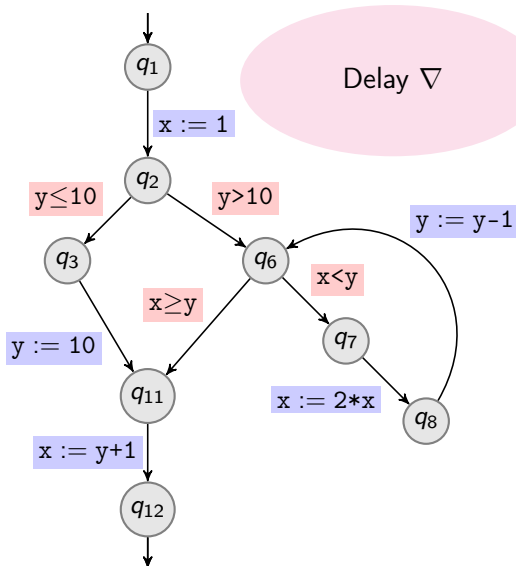
Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2	10	$+\infty$
q_7	1	2	10	$+\infty$
q_8	2	4	10	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

$$(2, 2) \sqcup (2, 4) = (2, 4)$$

$$(11, +\infty) \sqcup (10, +\infty) = (10, +\infty)$$

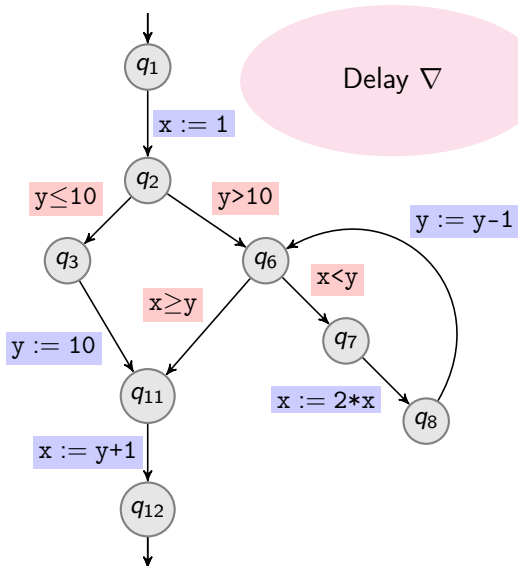
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2	10	$+\infty$
q_7	1	2	10	$+\infty$
q_8	2	4	10	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

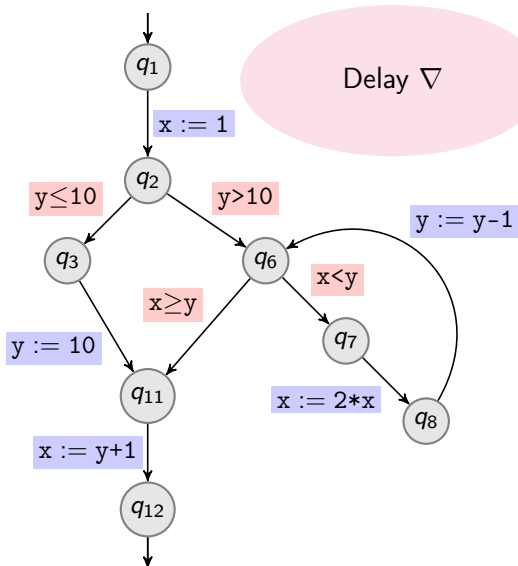
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	4	9	$+\infty$
q_7	1	4	9	$+\infty$
q_8	2	8	9	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

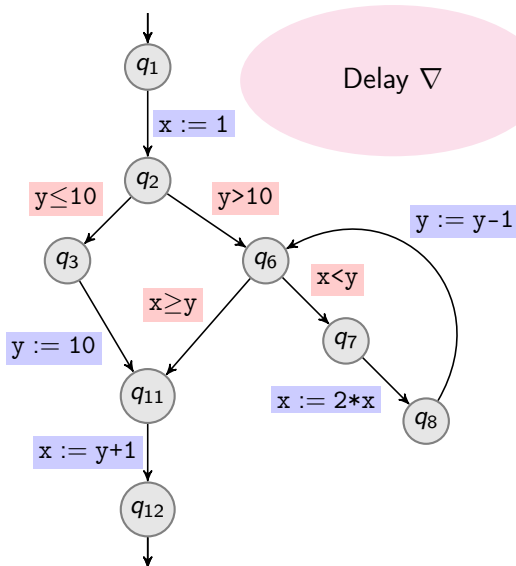
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	8	8	$+\infty$
q_7	1	8	8	$+\infty$
q_8	2	16	8	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

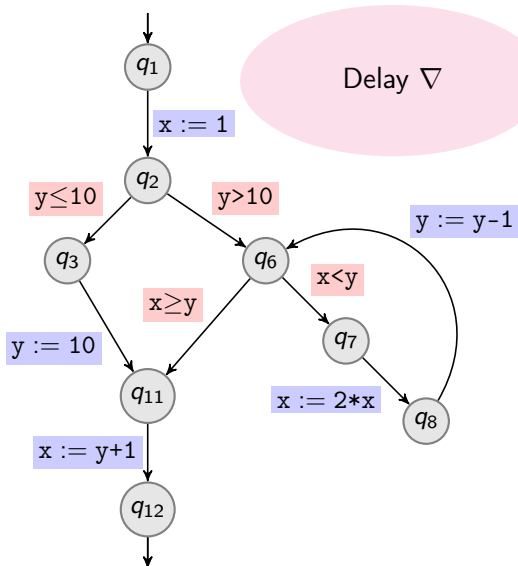
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	16	7	$+\infty$
q_7	1	16	7	$+\infty$
q_8	2	32	7	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

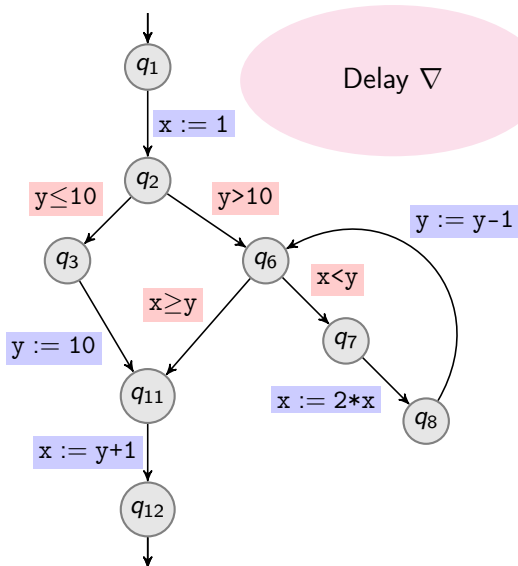
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	32	6	$+\infty$
q_7	1	32	6	$+\infty$
q_8	2	2^6	6	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

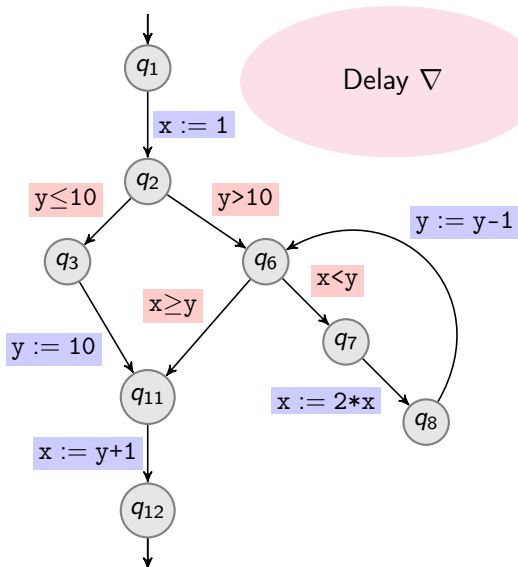
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^6	5	$+\infty$
q_7	1	2^6	5	$+\infty$
q_8	2	2^7	5	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

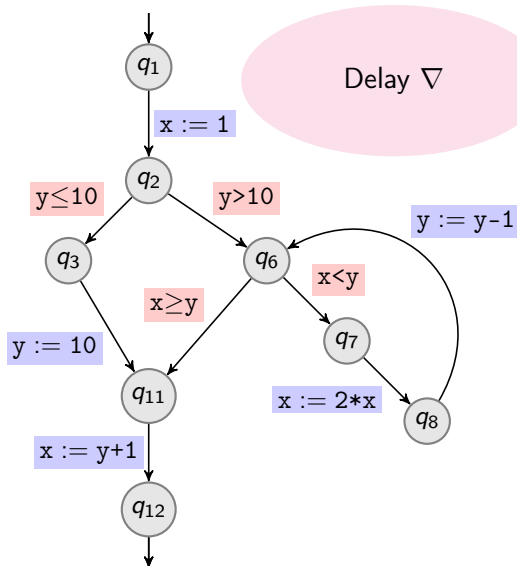
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^7	4	$+\infty$
q_7	1	2^7	4	$+\infty$
q_8	2	2^8	4	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

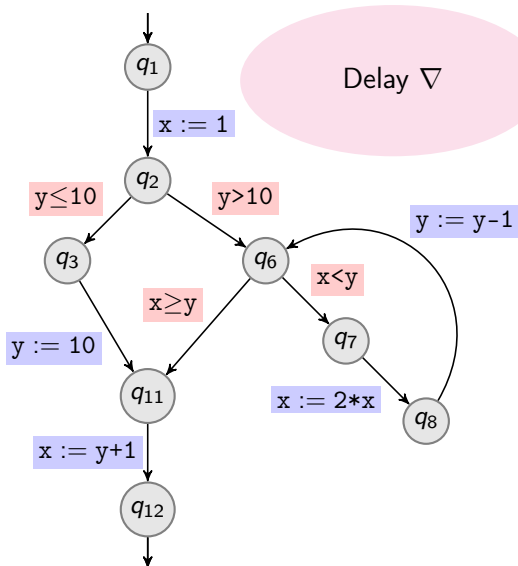
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^8	3	$+\infty$
q_7	1	2^8	3	$+\infty$
q_8	2	2^9	3	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

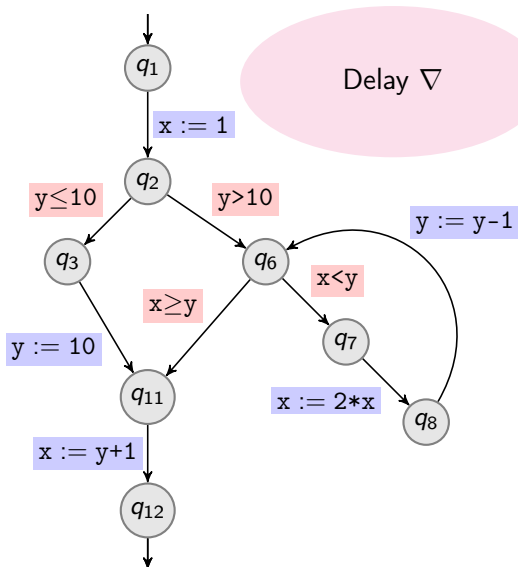
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^9	2	$+\infty$
q_7	1	2^9	2	$+\infty$
q_8	2	2^{10}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

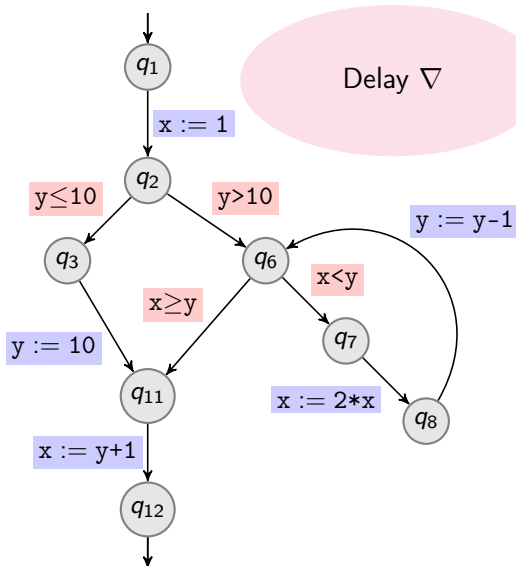
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^9	2	$+\infty$
q_7	1	2^9	2	$+\infty$
q_8	2	2^{10}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

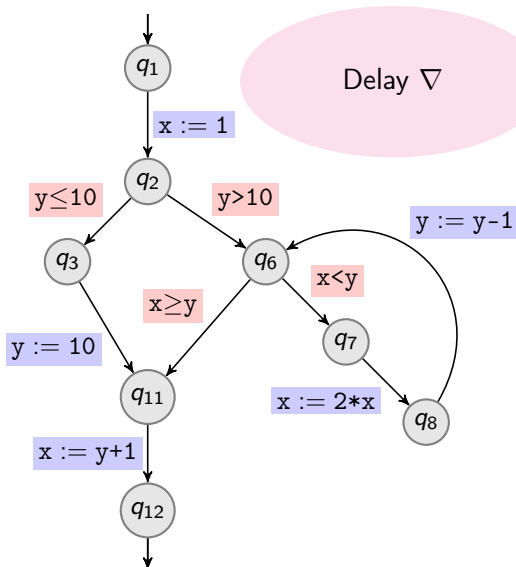
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{10}	1	$+\infty$
q_7	1	2^9	2	$+\infty$
q_8	2	2^{10}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

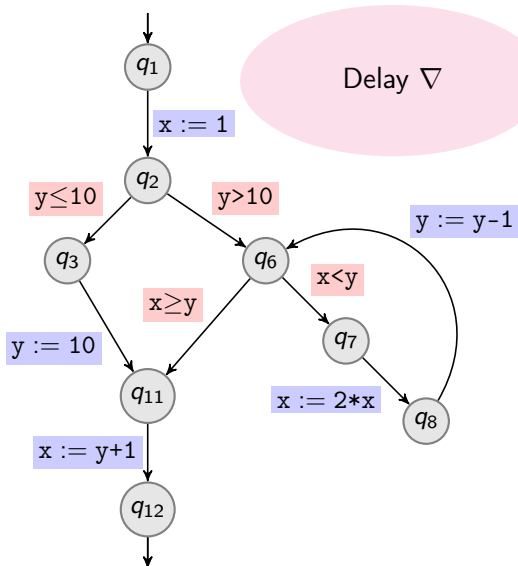
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{10}	1	$+\infty$
q_7	1	2^9	2	$+\infty$
q_8	2	2^{10}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Example with Delayed Widening

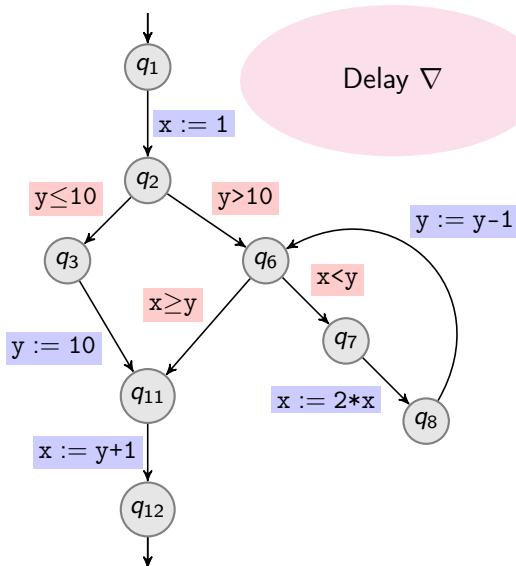


Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{10}	1	$+\infty$
q_7	1	2^{10}	2	$+\infty$
q_8	2	2^{10}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Gained from guard $x < y$

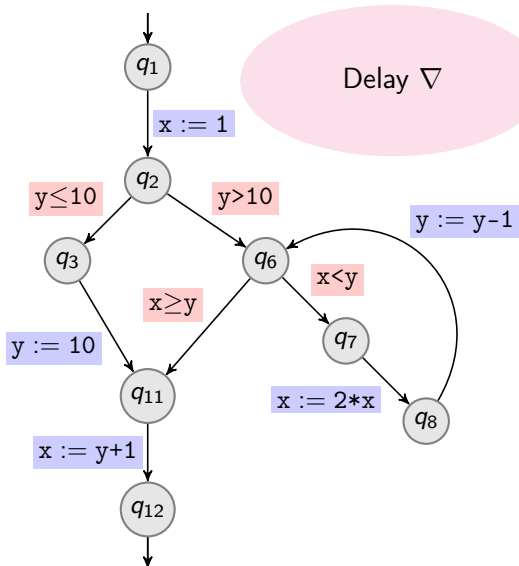
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{10}	1	$+\infty$
q_7	1	2^{10}	2	$+\infty$
q_8	2	2^{10}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

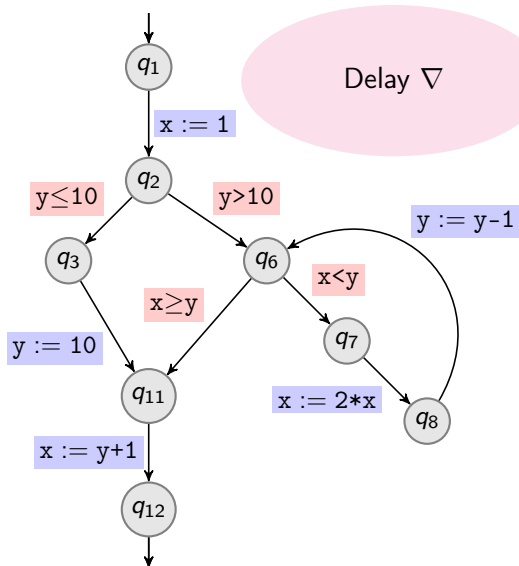
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{10}	1	$+\infty$
q_7	1	2^{10}	2	$+\infty$
q_8	2	2^{11}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

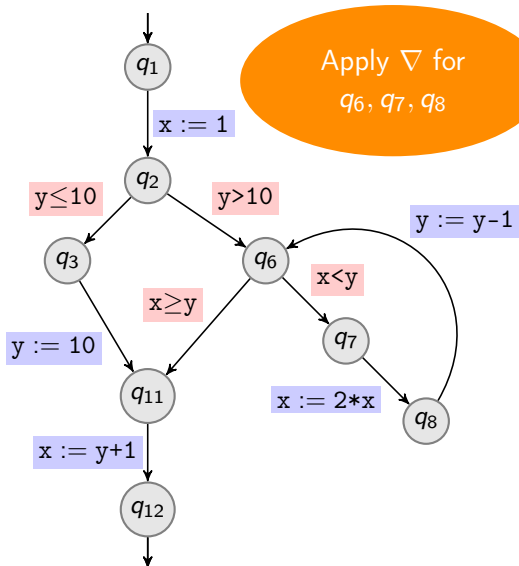
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{10}	1	$+\infty$
q_7	1	2^{10}	2	$+\infty$
q_8	2	2^{11}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

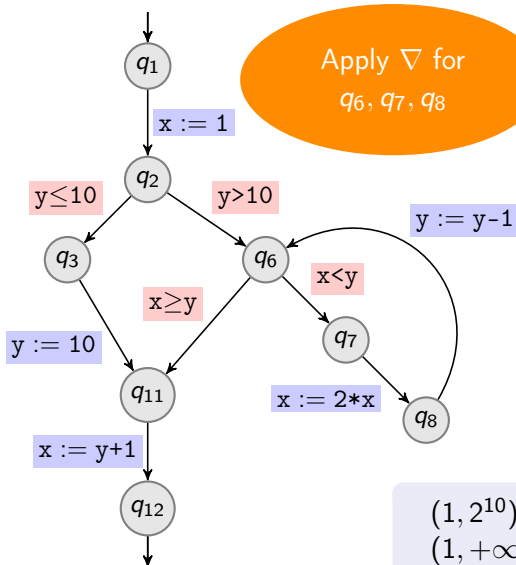
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	2^{10}	1	$+\infty$
q_7	1	2^{10}	2	$+\infty$
q_8	2	2^{11}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Example with Delayed Widening



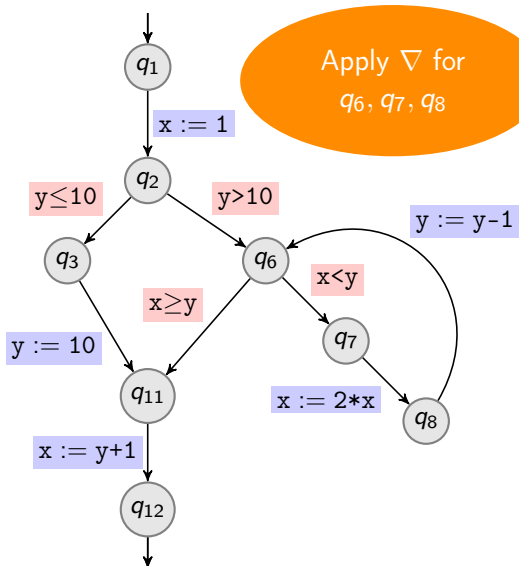
Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	2^{10}	2	$+\infty$
q_8	2	2^{11}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

$$(1, 2^{10}) \nabla (2, 2^{11}) = (1, +\infty)$$

$$(1, +\infty) \nabla (1, +\infty) = (1, +\infty)$$

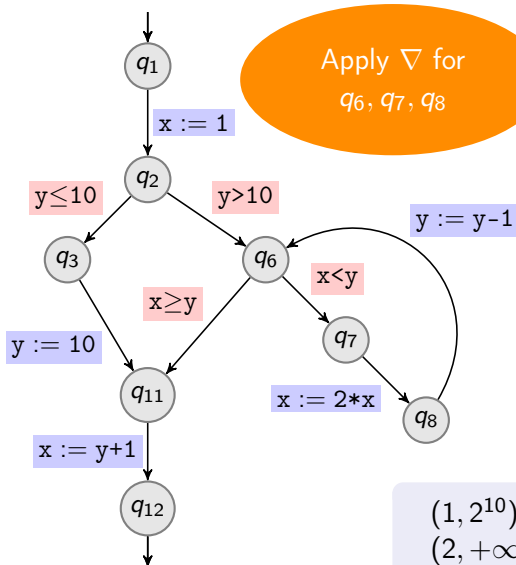
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	2^{10}	2	$+\infty$
q_8	2	2^{11}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Example with Delayed Widening



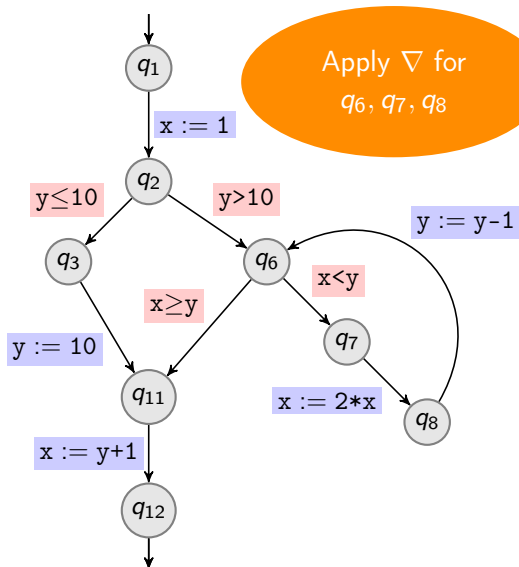
Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	2^{11}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

$$(1, 2^{10}) \quad \nabla \quad (1, +\infty) = (1, +\infty)$$

$$(2, +\infty) \quad \nabla \quad (2, +\infty) = (2, +\infty)$$

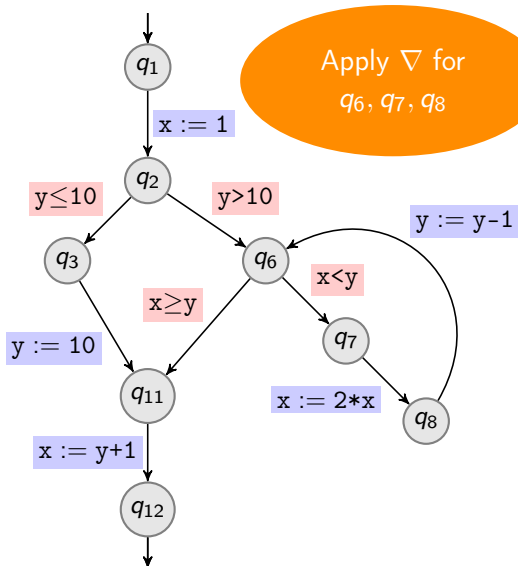
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	2^{11}	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

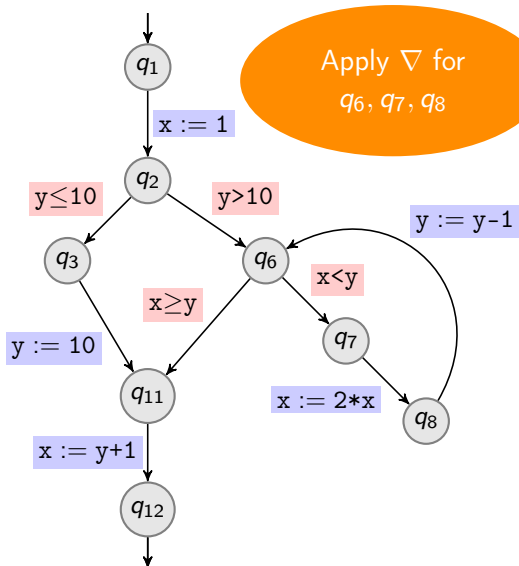
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

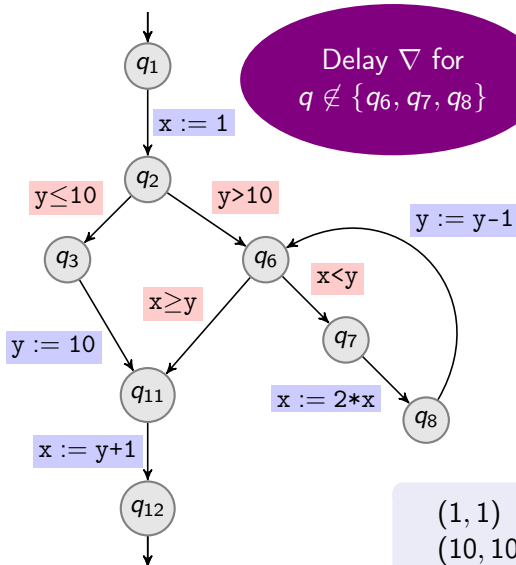
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Range Analysis on Example with Delayed Widening



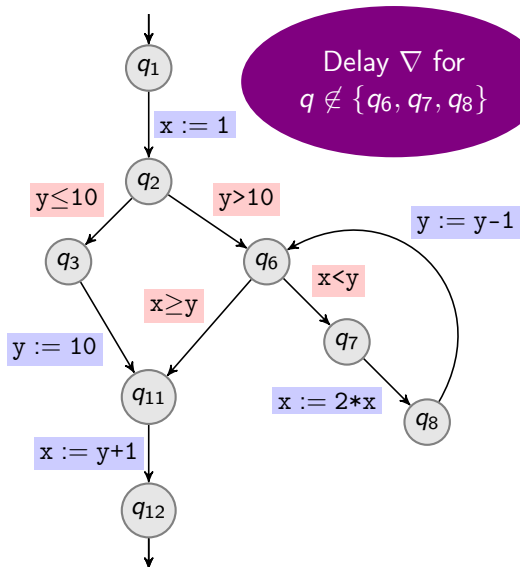
Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
q_{11}	1	$+\infty$	1	$+\infty$
q_{12}	11	11	10	10

$$(1, 1) \sqcup (1, +\infty) = (1, +\infty)$$

$$(10, 10) \sqcup (1, +\infty) = (1, +\infty)$$

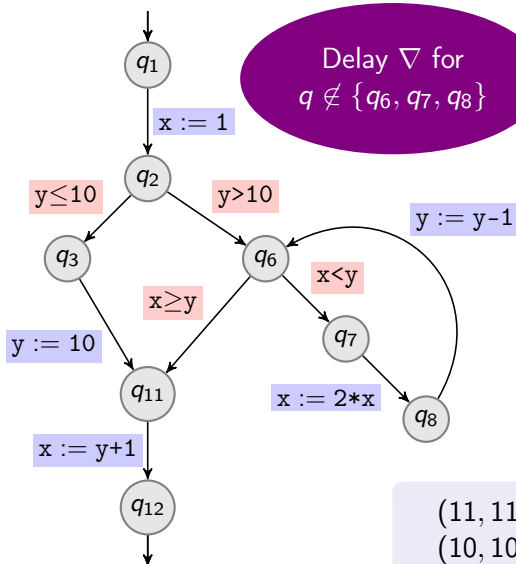
Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
q_{11}	1	$+\infty$	1	$+\infty$
q_{12}	11	11	10	10

Range Analysis on Example with Delayed Widening



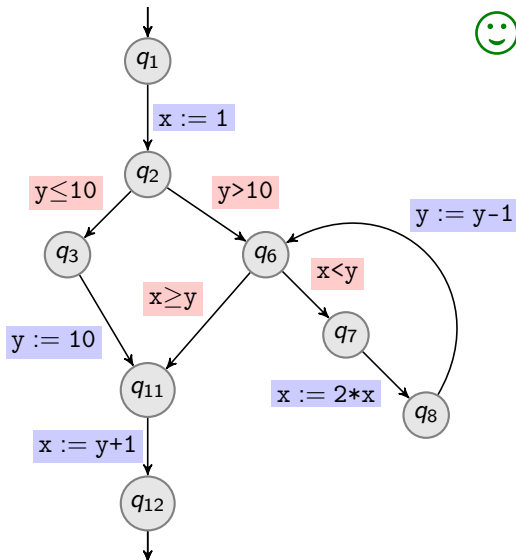
Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
q_{11}	1	$+\infty$	1	$+\infty$
q_{12}	2	$+\infty$	1	$+\infty$

$$(11, 11) \sqcup (2, +\infty) = (2, +\infty)$$

$$(10, 10) \sqcup (1, +\infty) = (1, +\infty)$$

Range Analysis on Example with Delayed Widening



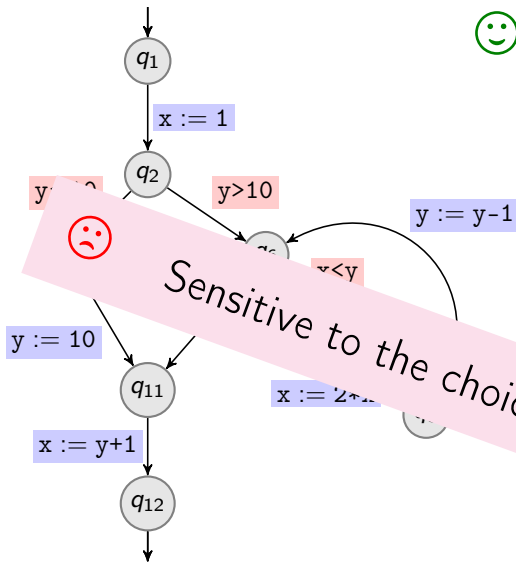
Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
q_{11}	1	$+\infty$	1	$+\infty$
q_{12}	2	$+\infty$	1	$+\infty$

Range Analysis on Example with Delayed Widening



Show that $x > 0$ at q_{12}



	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
	1	$+\infty$	1	$+\infty$
	$+\infty$	$+\infty$	1	$+\infty$

Sensitive to the choice of delay



Precision Improvement with Narrowing

Consider a complete lattice (L, \sqsubseteq) .

Objective of Narrowing Operators

Soundly improve the precision of an approximation obtained with ∇

Definition

A **narrowing** operator for (L, \sqsubseteq) is a function $\Delta : (L \times L) \rightarrow L$ such that:

1. $x \sqsupseteq y \implies x \sqsupseteq (x \Delta y) \sqsupseteq y$ (for all $x, y \in L$)
2. for any descending chain $x_0 \sqsupseteq x_1 \sqsupseteq \dots$ of elements of L , the descending chain $y_0 \sqsupseteq y_1 \sqsupseteq \dots$ defined by

$$\begin{cases} y_0 = x_0 \\ y_{i+1} = y_i \Delta x_{i+1} \quad \text{for all } i \in \mathbb{N} \end{cases}$$

is not strictly decreasing (i.e. $y_{i+1} = y_i$ for some $i \in \mathbb{N}$).

Correctness of Decreasing Iteration with Narrowing

Consider a complete lattice (L, \sqsubseteq) and a monotonic function $f : L \rightarrow L$.

A **post-fixpoint** of f is any element $a \in L$ satisfying $a \sqsupseteq f(a)$.

Theorem

If $\Delta : (L \times L) \rightarrow L$ is a narrowing operator then for any post-fixpoint a of f , the descending chain $y_0 \sqsupseteq y_1 \sqsupseteq \dots$ defined by

$$\begin{aligned}y_0 &= a \\ y_{i+1} &= y_i \Delta f(y_i)\end{aligned}$$

is eventually stationary, and its limit satisfies $\bigsqcap \{y_i \mid i \in \mathbb{N}\} \sqsupseteq \text{lfp}(f)$.

Correctness of Decreasing Iteration with Narrowing

Consider a complete lattice (L, \sqsubseteq) and a monotonic function $f : L \rightarrow L$.
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Theorem

If $\Delta : (L \times L) \rightarrow L$ is a narrowing operator then for any post-fixpoint a of f , the descending chain $y_0 \sqsupseteq y_1 \sqsupseteq \dots$ defined by

$$\begin{aligned}y_0 &= a \\ y_{i+1} &= y_i \Delta f(y_i)\end{aligned}$$

is eventually stationary, and its limit satisfies $\bigsqcap \{y_i \mid i \in \mathbb{N}\} \sqsupseteq \text{lfp}(f)$.

Application to Precision Improvement of MFP Approximations

1. Compute an approximation of *MFP* by Kleene iteration with ∇ .
2. Then perform a decreasing iteration with Δ to regain precision.

Round-Robin Iteration with Narrowing

Consider a Galois connection $(\mathcal{P}(X \rightarrow \mathbb{Z}), \sqsubseteq) \xleftrightarrow[\alpha]{\gamma} (A, \sqsubseteq)$, and a narrowing Δ for (A, \sqsubseteq) .

Pre-condition:

$$a(q) \sqsupseteq I_q^\# \sqcup \bigsqcup_{p \xrightarrow{\text{op}} q} \text{post}_{\text{op}}^\#(a(p)) \quad \text{for all } q \in Q$$

```
do
  change ← false
  foreach  $q \in Q$ 
    new ←  $I_q^\# \sqcup \bigsqcup_{p \xrightarrow{\text{op}} q} \text{post}_{\text{op}}^\#(a[p])$ 
    if new  $\not\sqsubseteq$  a[q]
      a[q] ← a[q]  $\Delta$  new
      change ← true
while change
return a
```

- ▶ this algorithm **terminates**
- ▶ the returned $a \in Q \rightarrow L$ satisfies:

$$\text{MFP}(q) \sqsubseteq a(q)$$

for every $q \in Q$.

Narrowing Operator for Range Analysis: Intuition

Objective of Narrowing Operators

Soundly improve the precision of an approximation obtained with ∇

∇ may have introduced infinite bounds to accelerate convergence.

Improve infinite bounds when possible (leave the non-infinite ones)

Examples

$$(1, +\infty) \Delta (1, 4) = (1, 4)$$

$$(1, 10) \Delta (1, 4) = (1, 10)$$

$$(-\infty, 10) \Delta (1, 4) = (1, 10)$$

Narrowing Operator for Range Analysis

Narrowing Operator on the Complete Lattice (Int, \sqsubseteq) of Intervals

$$\perp \Delta \perp = \perp$$

$$\perp \Delta (l, u) = (l, u) \Delta \perp = \perp$$

$$(l_1, u_1) \Delta (l_2, u_2) = (l_\Delta, u_\Delta) \quad \text{where} \quad \begin{cases} l_\Delta = \begin{cases} l_2 & \text{if } l_1 = -\infty \\ l_1 & \text{otherwise} \end{cases} \\ u_\Delta = \begin{cases} u_2 & \text{if } u_1 = +\infty \\ u_1 & \text{otherwise} \end{cases} \end{cases}$$

Narrowing Operator for Range Analysis

Narrowing Operator on the Complete Lattice (Int, \sqsubseteq) of Intervals

$$\perp \Delta \perp = \perp \quad \perp \Delta (l, u) = (l, u) \Delta \perp = \perp$$

$$(l_1, u_1) \Delta (l_2, u_2) = (l_\Delta, u_\Delta) \quad \text{where} \quad \begin{cases} l_\Delta = \begin{cases} l_2 & \text{if } l_1 = -\infty \\ l_1 & \text{otherwise} \end{cases} \\ u_\Delta = \begin{cases} u_2 & \text{if } u_1 = +\infty \\ u_1 & \text{otherwise} \end{cases} \end{cases}$$

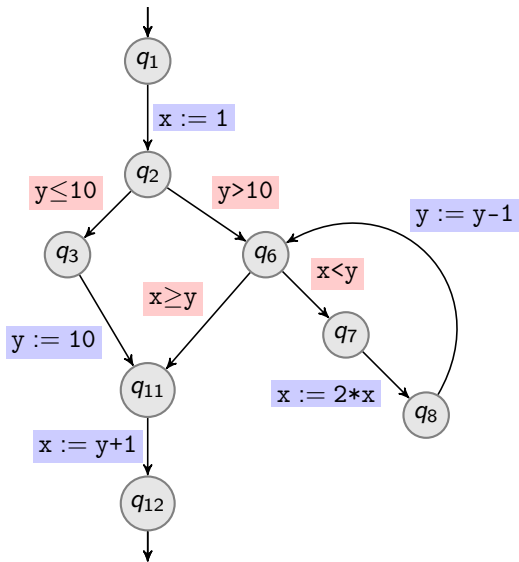
Narrowing Operator on the Complete Lattice $(X \rightarrow Int, \sqsubseteq)$

Component-wise extension Δ of the above narrowing Δ , defined by:

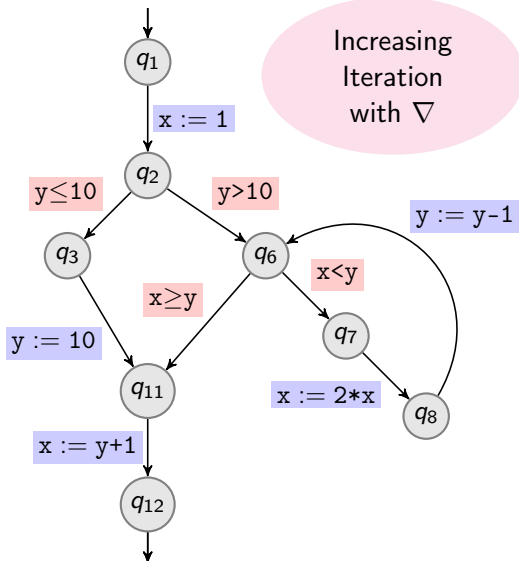
$$(a_1 \Delta a_2)(x) = a_1(x) \Delta a_2(x)$$

Range Analysis on Example with ∇ and Δ

Show that $x > 0$ at q_{12}



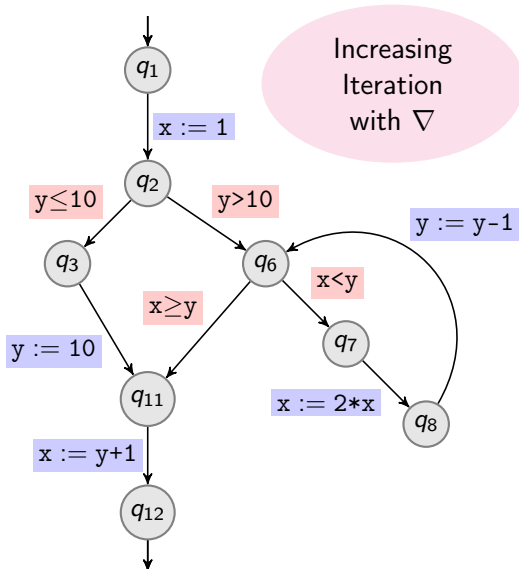
Range Analysis on Example with ∇ and Δ



Show that $x > 0$ at q_{12}

	x		y	
q_1	\perp	\perp	\perp	\perp
q_2	\perp	\perp	\perp	\perp
q_3	\perp	\perp	\perp	\perp
q_6	\perp	\perp	\perp	\perp
q_7	\perp	\perp	\perp	\perp
q_8	\perp	\perp	\perp	\perp
q_{11}	\perp	\perp	\perp	\perp
q_{12}	\perp	\perp	\perp	\perp

Range Analysis on Example with ∇ and Δ

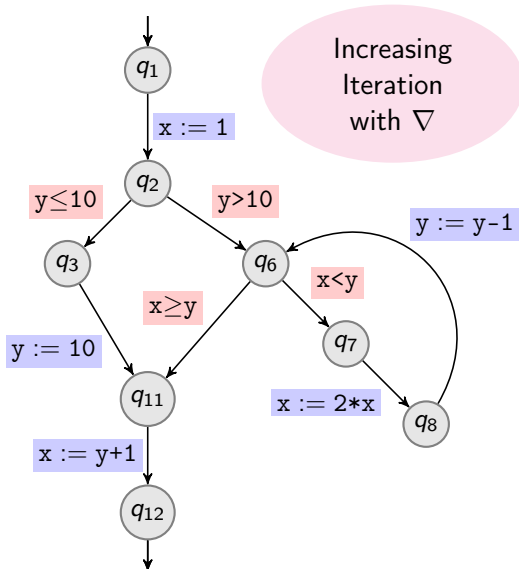


Show that $x > 0$ at q_{12}

	x		y	
	∇	Δ	∇	Δ
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Same as with non-delayed widening

Range Analysis on Example with ∇ and Δ

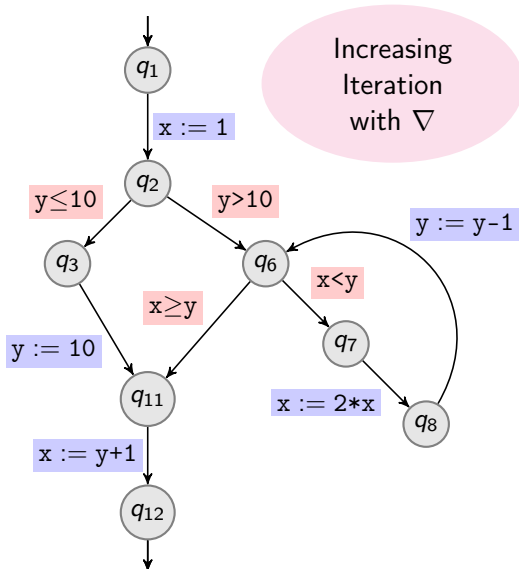


Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	1	11	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Same as with non-delayed widening

Range Analysis on Example with ∇ and Δ

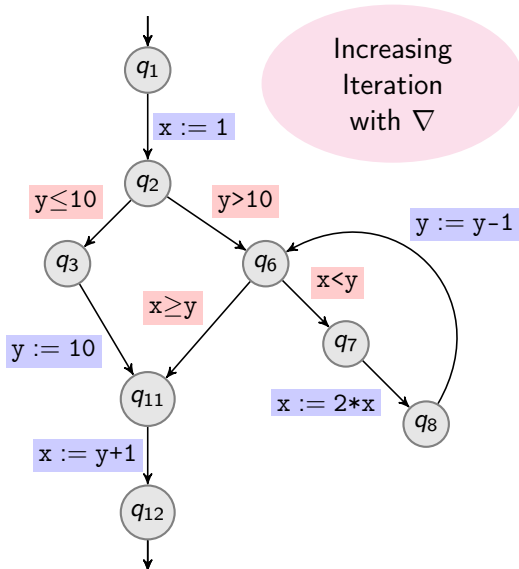


Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	1	11	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
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Same as with non-delayed widening

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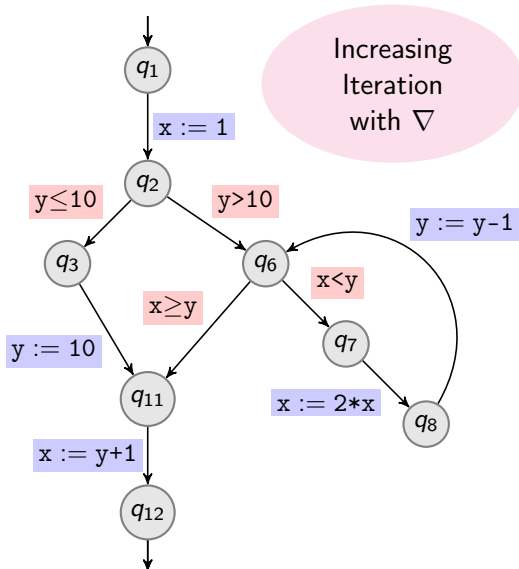


Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	$-\infty$	$+\infty$
q_8	2	2	11	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Same as with non-delayed widening

Range Analysis on Example with ∇ and Δ

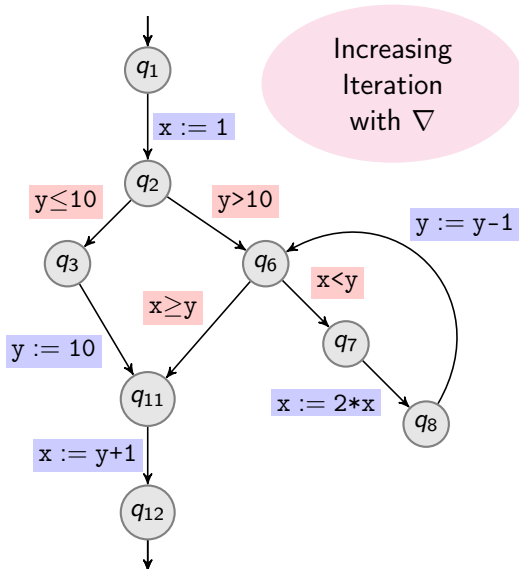


Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	$-\infty$	$+\infty$
q_8	2	$+\infty$	$-\infty$	$+\infty$
q_{11}	1	1	10	10
q_{12}	11	11	10	10

Same as with non-delayed widening

Range Analysis on Example with ∇ and Δ

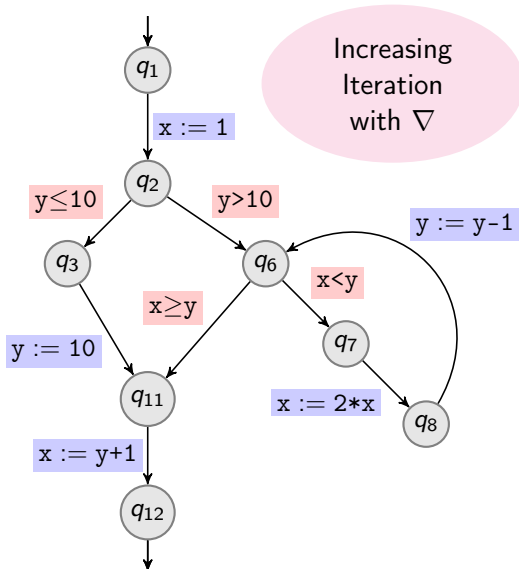


Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	$-\infty$	$+\infty$
q_8	2	$+\infty$	$-\infty$	$+\infty$
q_{11}	1	$+\infty$	$-\infty$	$+\infty$
q_{12}	11	11	10	10

Same as with non-delayed widening

Range Analysis on Example with ∇ and Δ

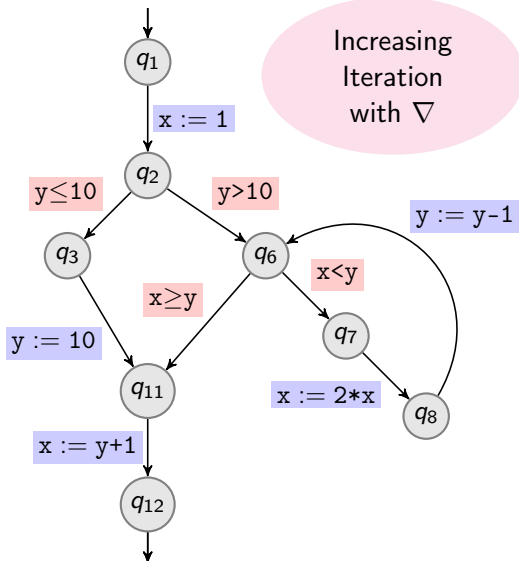


Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	$-\infty$	$+\infty$
q_8	2	$+\infty$	$-\infty$	$+\infty$
q_{11}	1	$+\infty$	$-\infty$	$+\infty$
q_{12}	$-\infty$	$+\infty$	$-\infty$	$+\infty$

Same as with non-delayed widening

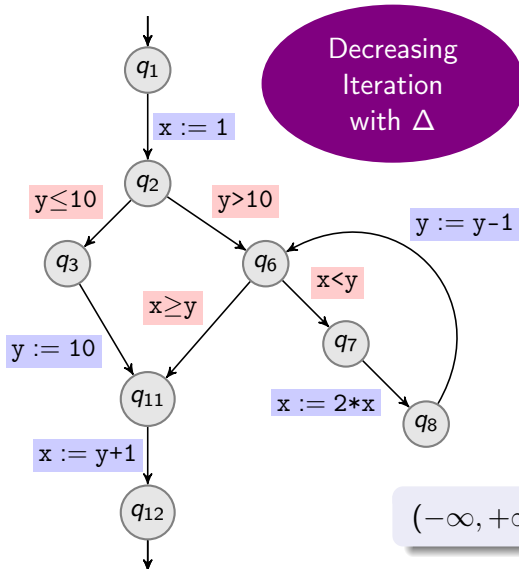
Range Analysis on Example with ∇ and Δ



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	$-\infty$	$+\infty$
q_8	2	$+\infty$	$-\infty$	$+\infty$
q_{11}	1	$+\infty$	$-\infty$	$+\infty$
q_{12}	$-\infty$	$+\infty$	$-\infty$	$+\infty$

Range Analysis on Example with ∇ and Δ

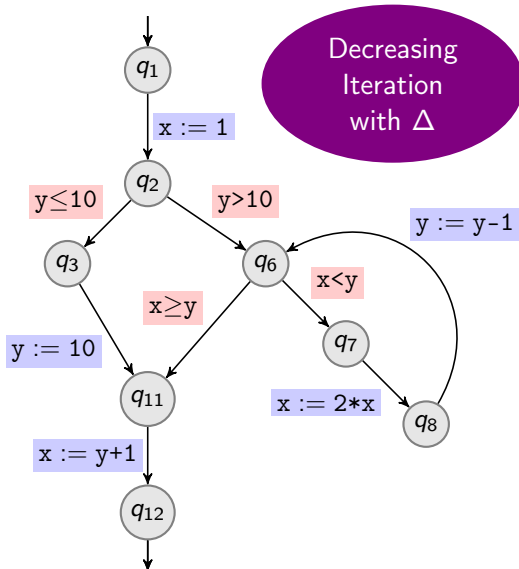


Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	$-\infty$	$+\infty$
q_{11}	1	$+\infty$	$-\infty$	$+\infty$
q_{12}	$-\infty$	$+\infty$	$-\infty$	$+\infty$

$$(-\infty, +\infty) \Delta (2, +\infty) = (2, +\infty)$$

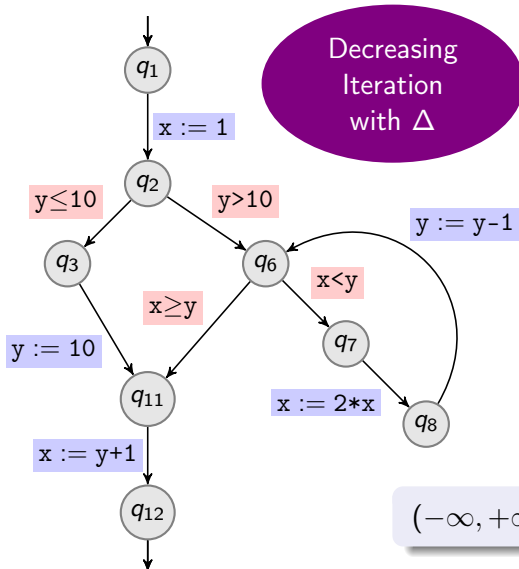
Range Analysis on Example with ∇ and Δ



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	$-\infty$	$+\infty$
q_{11}	1	$+\infty$	$-\infty$	$+\infty$
q_{12}	$-\infty$	$+\infty$	$-\infty$	$+\infty$

Range Analysis on Example with ∇ and Δ

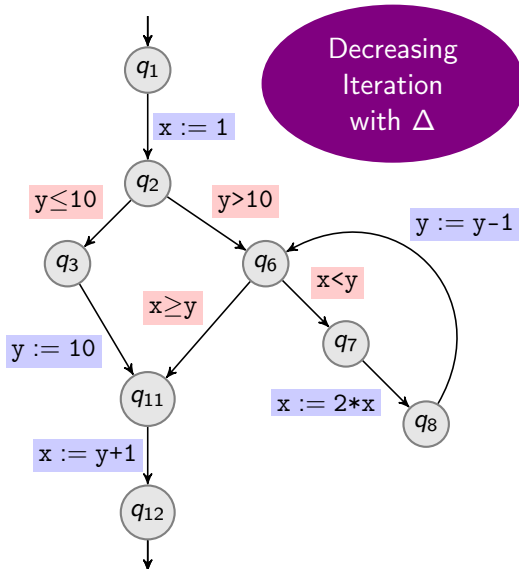


Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
q_{11}	1	$+\infty$	$-\infty$	$+\infty$
q_{12}	$-\infty$	$+\infty$	$-\infty$	$+\infty$

$$(-\infty, +\infty) \Delta (2, +\infty) = (2, +\infty)$$

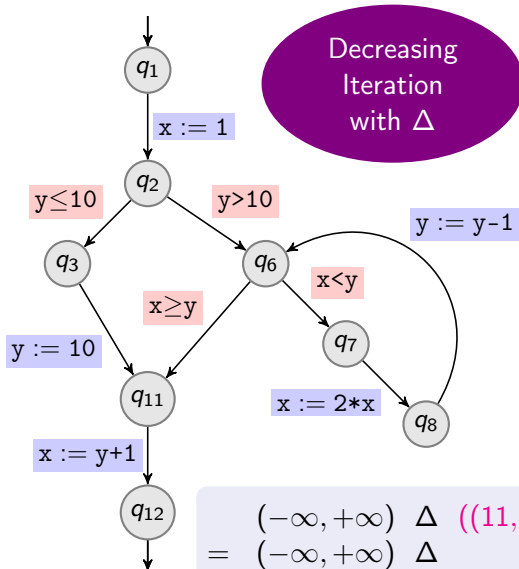
Range Analysis on Example with ∇ and Δ



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	$-\infty$	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
q_{11}	1	$+\infty$	$-\infty$	$+\infty$
q_{12}	$-\infty$	$+\infty$	$-\infty$	$+\infty$

Range Analysis on Example with ∇ and Δ

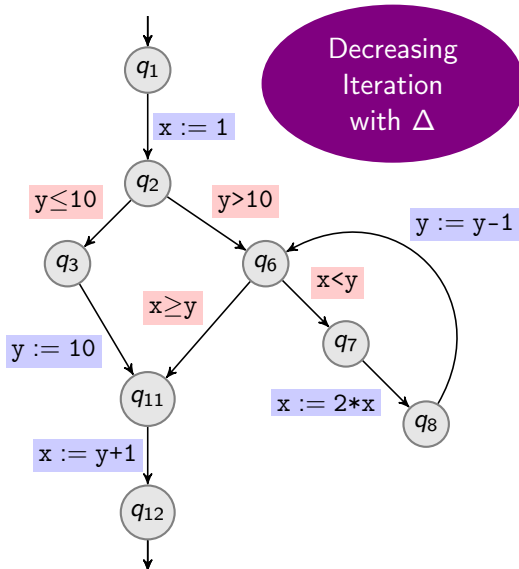


Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
q_{11}	1	$+\infty$	$-\infty$	$+\infty$
q_{12}	$-\infty$	$+\infty$	$-\infty$	$+\infty$

$$\begin{aligned}
 & (-\infty, +\infty) \Delta ((11, +\infty) \sqcup (1, +\infty)) \\
 = & (-\infty, +\infty) \Delta (1, +\infty) = (1, +\infty)
 \end{aligned}$$

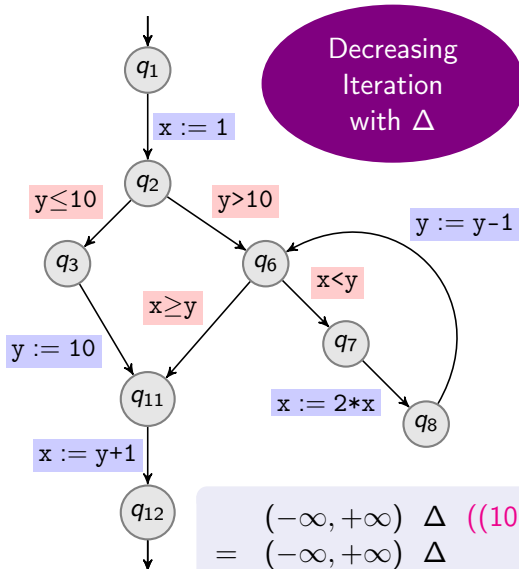
Range Analysis on Example with ∇ and Δ



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
q_{11}	1	$+\infty$	$-\infty$	$+\infty$
q_{12}	$-\infty$	$+\infty$	$-\infty$	$+\infty$

Range Analysis on Example with ∇ and Δ

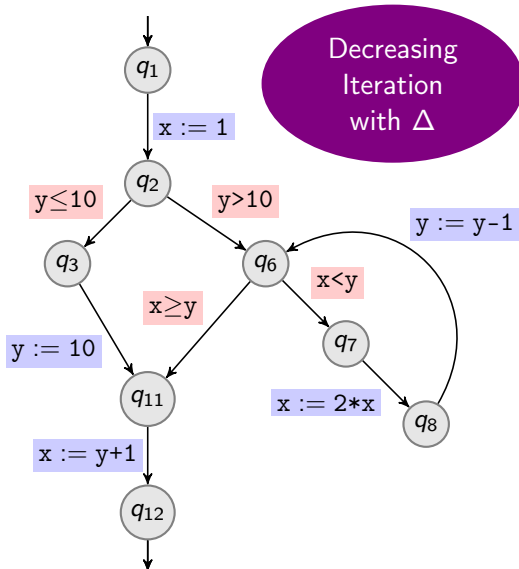


Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
q_{11}	1	$+\infty$	1	$+\infty$
q_{12}	$-\infty$	$+\infty$	$-\infty$	$+\infty$

$$\begin{aligned}
 & (-\infty, +\infty) \Delta ((10, 10) \sqcup (1, +\infty)) \\
 = & (-\infty, +\infty) \Delta (1, +\infty) = (1, +\infty)
 \end{aligned}$$

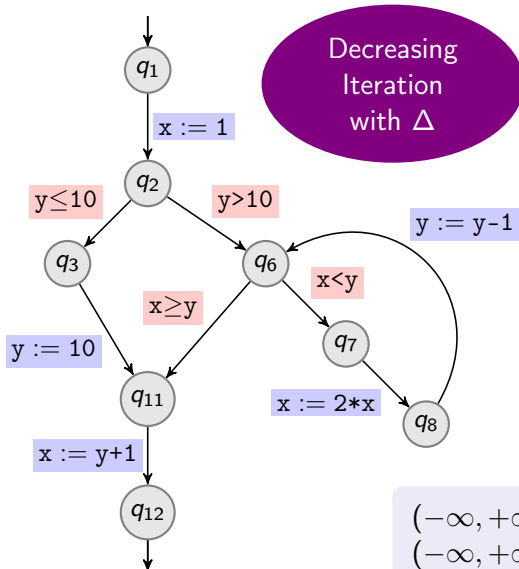
Range Analysis on Example with ∇ and Δ



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
q_{11}	1	$+\infty$	1	$+\infty$
q_{12}	$-\infty$	$+\infty$	$-\infty$	$+\infty$

Range Analysis on Example with ∇ and Δ



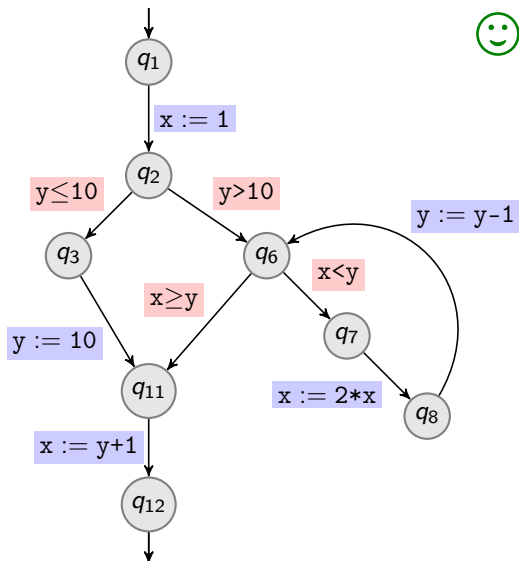
Show that $x > 0$ at q_{12}

	x		y	
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
q_{11}	1	$+\infty$	1	$+\infty$
q_{12}	2	$+\infty$	1	$+\infty$

$$(-\infty, +\infty) \Delta (2, +\infty) = (2, +\infty)$$

$$(-\infty, +\infty) \Delta (1, +\infty) = (1, +\infty)$$

Range Analysis on Example with ∇ and Δ



Show that $x > 0$ at q_{12}

	x		y	
	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_1	$-\infty$	$+\infty$	$-\infty$	$+\infty$
q_2	1	1	$-\infty$	$+\infty$
q_3	1	1	$-\infty$	10
q_6	1	$+\infty$	1	$+\infty$
q_7	1	$+\infty$	2	$+\infty$
q_8	2	$+\infty$	2	$+\infty$
q_{11}	1	$+\infty$	1	$+\infty$
q_{12}	2	$+\infty$	1	$+\infty$

Selective Application of Widening

Widening introduces imprecision that often cannot be regained by narrowing.

To ensure convergence it is **enough** to only apply widening at **cut points**

Cut points: set of locations that cut each loop (in the control flow automaton's graph)

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Widening introduces imprecision that often cannot be regained by narrowing.

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Other Methods to Reduce Precision Loss of Widening

- ▶ Delayed widening
- ▶ Widening “up to”

Given a finite set $M \subseteq L$, use $(x \nabla y) \sqcap \bigsqcap \{m \in M \mid x \sqcup y \sqsubseteq m\}$.

- ▶ Look-ahead widening
- ▶ ...