

Interval exchange transformations
(*Intercambios de intervalos*)
Part I: Rotations (*rotaciones*)

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(*Vicente de la Cruz*)

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(*Noviembre 2015*)

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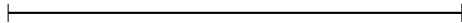
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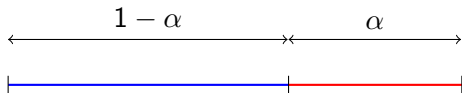


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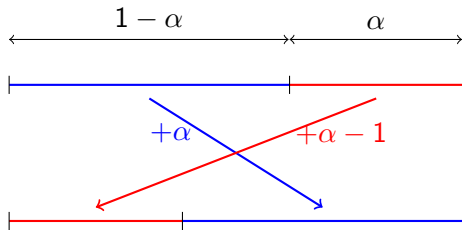


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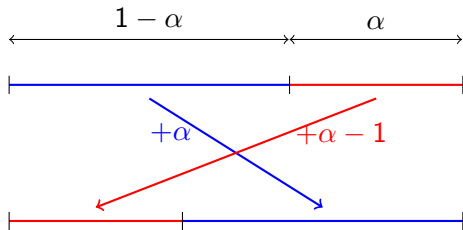


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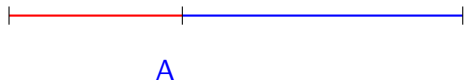
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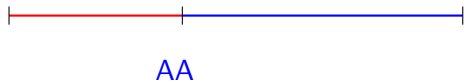
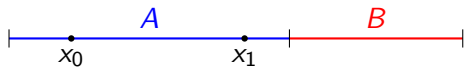
Coding



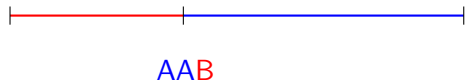
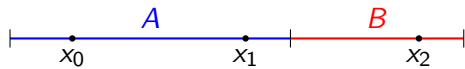
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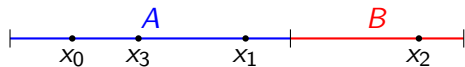
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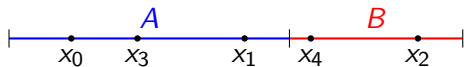


Coding



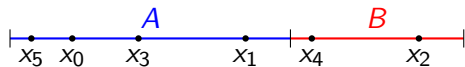
AABA

Coding



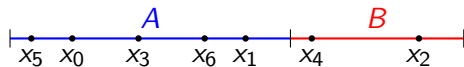
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Coding



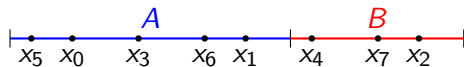
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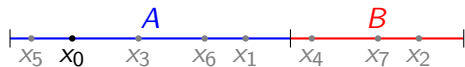
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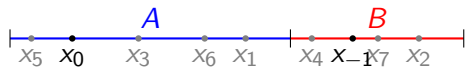
AABABAAB...

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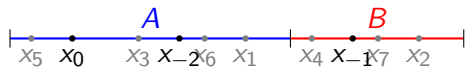
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Coding



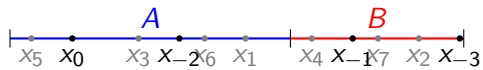
$B.AABABAAB\dots$

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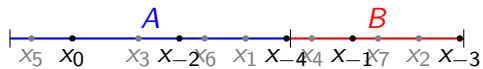
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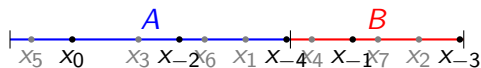
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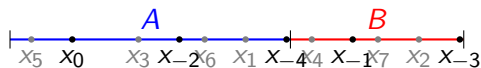
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Taking the closure of the set of codings, we obtain a shift $X_\alpha \subset \{A, B\}^{\mathbb{Z}}$.

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Theorem

If α is irrational, there is a unique continuous map $p : X_\alpha \rightarrow [0, 1]$ so that the coding of $p(w)$ is w . All points have exactly one preimage except the singular orbits that have two.

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The points in X_α that project to the same point then correspond to the orbit of $\omega_+AB\omega_-$ and $\omega_+BA\omega_-$.

Dynamical results

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Let α be irrational, and X_α be the Sturmian shift associated to the rotation T_α . Then:

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remark: for the clopens $Y = [A]$ or $Y = [B]$ we can pick $C_Y = 1$ (1-balancedness).

Complexity and minimality

Proof.

Irrationality!



Rauzy induction and continued fractions

For a pair of positive real numbers $\lambda = (\lambda_A, \lambda_B)$ we consider the map $T_\lambda : [0, |\lambda|] \rightarrow [0, |\lambda|]$ given by

$$T_\lambda(x) = x \mapsto (x + \lambda_B) \bmod (\lambda_A + \lambda_B).$$

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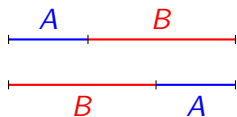
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The *Rauzy induction* is the procedure which associates to the map T_λ the induced map on $[0, \min(\lambda_A, \lambda_B)]$.

Rauzy induction and continued fractions

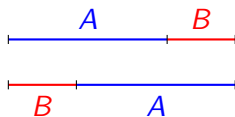
top induction

case $\lambda_B > \lambda_A$



bot induction

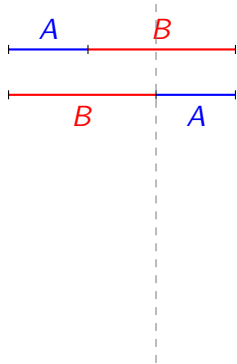
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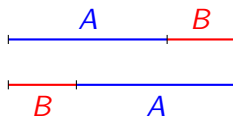
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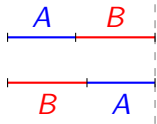
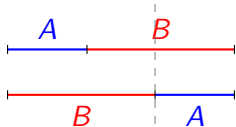
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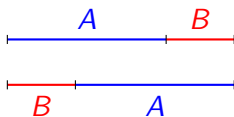
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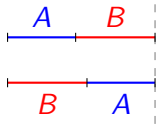
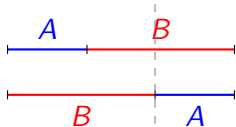
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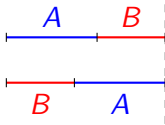
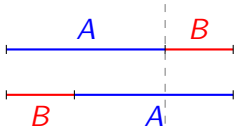
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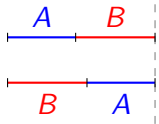
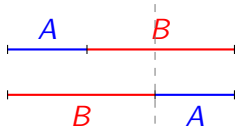
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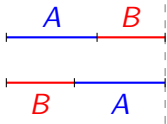
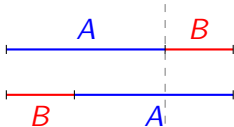
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$$(\lambda_A, \lambda_B) \mapsto (\lambda_A, \lambda_B - \lambda_A)$$

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Rauzy induction and Rohlin towers

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Hecke-Ostrowski theorem

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- ▶ (Hecke (1922), Ostrowski (1922)) any clopen $Y \subset X_\alpha$ has bounded remainder: there exists μ_Y and C_Y so that

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In particular, the shift X_α is **uniquely ergodic**.

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- ▶ Letters have uniform frequencies,
- ▶ From letters to all cylinders (... Sage ...).



Rauzy induction and continued fractions

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$$A(\lambda) = \begin{cases} A^{bot} & \text{if } \lambda_A > \lambda_B, \\ A^{top} & \text{if } \lambda_A < \lambda_B. \end{cases}$$

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$R^n(\lambda) = (A_n(\lambda))^{-1}\lambda$ where

$$A_n(\lambda) = A(\lambda)A(R\lambda)\dots A(R^{n-1}\lambda).$$

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From this description it is possible to show that (λ_A, λ_B) can be written as

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(...exercise ...)